THE NECESSARY AND SUFFICIENT CONDITIONS FOR THE GLOBAL STABILITY OF TYPE-K LOTKA–VOLTERRA SYSTEM

TU CAIFENG AND JIANG JIFA

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Abstract. This paper provides necessary and sufficient conditions for the type-K Lotka-Volterra system to have a globally asymptotically stable positive steady state. The generalization of such a result is given.

1. Introduction

Consider the Lotka-Volterra system

\[ \dot{x} = \text{diag}(x)(r + Mx), \quad x \in \mathbb{R}^n_+, \quad r \in \mathbb{R}^n, \]

where

\[ M = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}, \]

A is a \( k \times k \) matrix with nonnegative off-diagonal elements, \( D \) is an \( (n-k) \times (n-k) \) matrix with the same property and \( B \geq 0, C \geq 0 \). We call such matrix \( M \) at y p e-K matrix and system (S) a type-K monotone system. There are many papers about the global behavior of monotone systems (S). Some interesting results are due to Takeuchi and Adachi. Let \( N = \{1, 2, \ldots, n\} \) and let \( Q \) be a subset of \( N \). In [1], Takeuchi and Adachi proved that if \( M \) is stable, then (S) has a unique nonnegative steady state \( \bar{x} \) with \( \bar{x}_q = 0 \) for \( q \in Q \) and \( \bar{x}_r > 0 \) for \( r \in N \setminus Q \), which attracts all solutions with initial conditions in \( \{x \in \mathbb{R}^n_+ : x_r > 0 \text{ for } r \in N \setminus Q\} \). They also proved that \( \bar{x} \) is globally asymptotically stable relative to \( \{x \in \mathbb{R}^n_+ : x_r > 0 \text{ for } r \in N \setminus Q\} \) if and only if

\[ r_q + \sum_{j=1}^{n} m_{qj} \bar{x}_j \leq 0 \quad \text{for all } q \in Q. \]

Concerning system (S), as pointed out by Smith [2, p. 872], one of the most interesting problems is whether or not the groups \( I = \{1, 2, \ldots, k\} \) and \( J = \{k+1, \ldots, n\} \) can coexist. The results of Hirsch [3], [4] suggest that the coexistence must take the form of a positive steady state which should be asymptotically stable. Therefore, providing conditions to guarantee that (S) has a globally asymptotically stable steady state is of great interest. Smith [2, Theorem 4.1] gave one set of

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sufficient conditions for the existence of such a steady state which can only be applied to nonobligate systems \((r > 0)\). The present authors generalized Smith’s result to obligate systems in [5].

The goal of the present paper is to provide the necessary and sufficient conditions for \((S)\) to have a globally asymptotically stable steady state in the positive orthant. Obviously, the essential condition for \((S)\) to have a globally asymptotically stable steady state is that \(M\) is stable. Actually, in this paper, we shall consider the more general system

\[(S^*) \quad \dot{x} = \text{diag}(x)f(x), \quad x \in \mathbb{R}^n_+,
\]

where \(Df(x)\) is a type-\(K\) matrix as in (T). We call such system \((S^*)\) a general type-\(K\) monotone system. Under the suitable condition

\[(C) \quad Df(x) \leq K M, \quad \text{for all} \quad x \in \mathbb{R}^n_+,
\]

where \(M\) is a type-\(K\) matrix as in (T) and stable, we shall present necessary and sufficient conditions for \((S^*)\) to have a globally asymptotically stable steady state. “\(\leq_K\)” will be defined in the next section. Let \(s(M) = \max \text{ Re } \lambda\), where \(\lambda\) runs through the eigenvalues of \(M\). \(M\) is stable, that is, \(s(M) < 0\) if and only if the principal minors of \(M^+\) alternate in sign as follows:

\[
(-1)^k \begin{vmatrix}
    a_{i1}^+ & \cdots & a_{ik}^+ \\
    \vdots & \ddots & \vdots \\
    a_{k1}^+ & \cdots & a_{kk}^+
\end{vmatrix} > 0, \quad 1 \leq k \leq n,
\]

where \(a_{ij}^+ = |a_{ij}|\) for \(i \neq j\) and \(a_{ii}^+ = a_{ii}\).

### 2. The Main Result

In this section, we will agree on some notation and establish some conventions. Then, the main result will be stated.

Let \(\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}\) denote the nonnegative orthant and \(\text{Int } \mathbb{R}_+^n = \{x \in \mathbb{R}_+^n : x_i > 0, 1 \leq i \leq n\}\) denote its interior. \(x > 0\) means \(x_i > 0\) for all \(i\).

In this paper, \(K\) is a proper cone in \(\mathbb{R}^n\) which is a nonempty closed convex subset of \(\mathbb{R}^n\) with the property \(K \cap (-K) = \{0\}\). If the partial order relation is generated by a cone \(K\), we write \(x \leq_K y\) whenever \(y - x \in K\). Then \(K = R^n_+ \times (-R^{n-k}_+)^\ast\) is a cone. Let \(x^1 \in \mathbb{R}^k, x^2 \in \mathbb{R}^{n-k}\); it is convenient to write \(x = (x^1, x^2) \in \mathbb{R}^n\). Then \(x \leq_K y\), where \(x = (x^1, x^2), y = (y^1, y^2)\) implies \(x^1 \leq y^1, x^2 \geq y^2\). For two \(n \times n\) type-\(K\) matrices \(M_1\) and \(M_2\), \(M_1 \geq_K M_2\) if and only if \(A_1 \geq A_2, B_1 \geq B_2, C_1 \geq C_2, D_1 \geq D_2\).

Let \(N = \{1, 2, \cdots, n\}, I = \{1, 2, \cdots, k\}\) and \(J = \{k + 1, \cdots, n\}\), where \(n\) is the dimension of our Euclidean space \(\mathbb{R}^n\). If \(L, P\) are nonempty sets of \(N\) such that \(L \supset I\) and \(P \supset J\), then \(\bar{L} = N \setminus L\) and \(\bar{P} = N \setminus P\) denote their complementary sets in \(N\). \(u\) will always represent vectors in \(\mathbb{R}^L\)-dimensional Euclidean space \(\mathbb{R}^L_+\), \(v\) in \(\mathbb{R}^L\)-dimensional Euclidean space \(\mathbb{R}^L_+\), \(w\) in \(\mathbb{R}^P_+\) and \(z\) in \(\mathbb{R}^P_+\). \(\sharp L\) represents the cardinality of \(L\). \(\sharp \bar{L}, \sharp \bar{P}\) and \(\sharp \bar{P}\) have the same meanings respectively.

Without loss of generality, we may assume \(L = \{1, 2, \cdots, k, k + 1, \cdots, l\}\), \(k < l < n\), and \(P = \{n - p + 1, \cdots, k + 1, \cdots n\}\), \(p > n - k\). Let \(x = (u, v)\) and
\[ f(x) = (f_L(u,v), f_L(u,v)). \] Then we can rewrite the system \((S^*)\) as

\[
(S_1) \quad \begin{cases} 
\dot{u} = \text{diag}(u) f_L(u,v), \\
\dot{v} = \text{diag}(v) f_L(u,v),
\end{cases} \quad u \in R^2_+, \quad v \in R^2_+.
\]

Similarly, let \(x = (z,w)\) and \(f(x) = (f_P(z,w), f_P(z,w))\). Then we can rewrite the system \((S^*)\) as

\[
(S_2) \quad \begin{cases} 
\dot{z} = \text{diag}(z) f_P(z,w), \\
\dot{w} = \text{diag}(w) f_P(z,w),
\end{cases} \quad z \in R^2_+, \quad w \in R^2_+.
\]

Setting \(v = 0\) and \(z = 0\) in \((S_1)\) and \((S_2)\) respectively, we obtain two subsystems

\[
(S_L) \quad \dot{u} = \text{diag}(u) f_L(u,0), \quad u \in R^2_+,
\]

and

\[
(S_P) \quad \dot{w} = \text{diag}(w) f_P(0, w), \quad w \in R^2_+.
\]

Because \(Df(u,v)\) and \(Df(z,w)\) are type-\(K\) matrices, \(Df_L(u,0)\) is a type-\(K_1\) submatrix of \(Df(u,0)\) and \(Df_P(0,w)\) is a type-\(K_2\) submatrix of \(Df(0,w)\), where \(K_1 = R^2_+ \times (-R^{l-k}_+)\) and \(K_2 = R^{l-k}_+ \times (-R^{l-k}_+)\).

We write \(\phi_i(x)\) for the unique solution \(x(t)\) of \((S^*)\) satisfying \(x(0) = x; \phi_i^0(u), \phi_i^0(w)\) are the unique solutions \(u(t)\) of \((S_L)\) and \(w(t)\) of \((S_P)\) respectively, satisfying \(u(0) = u\) and \(w(0) = w\). \(\{\phi_i^0(u)\}_i\) consists of components \(\{\phi_i^0(u)\}_i\), of \(\phi_i^0(u)\) for all \(i \in I\), and \((\phi_i^0(w))_j\) is defined similarly.

Similar to the result of Takeuchi and Adachi, we have proved the following theorem [7, Theorem 4.1] for the type-\(K\) system \((S^*)\).

**Theorem 2.1.** Assume that the system \((S^*)\) is type-\(K\) monotone and the condition \((C)\) holds. If there exists a nonnegative steady state \(c \in R_+^n\) with \(c_Q > 0\), \(c_Q = 0\) where we agree on \(c = 0\) if \(Q = \emptyset\), then \(c\) attracts all solutions with initial conditions in \(\{x \in R_+^n : x_Q > 0\}\) if and only if \(f(c) \leq 0\).

Our main result is as follows.

**Theorem A.** Assume that the condition \((C)\) holds. Then the type-\(K\) monotone system \((S^*)\) has a unique positive steady state which is globally asymptotically stable relative to \(\text{Int} R_+^n\) if and only if \((S^*)\) satisfies the following conditions:

1. There exists \(L \subset N\) with \(L \supset I\) such that \((S_L)\) has a positive steady state \(w^0\) and \(f_L(w_0^0,0) > 0\).
2. There exists \(P \subset N\) with \(P \supset J\) such that \((S_P)\) has a positive steady state \(w^0\) and \(f_P(0,w^0) > 0\).

3. **The proof of the main result**

It is essential to establish some preliminary results before giving the proof of Theorem A.

**Theorem 3.1** (Kamke Theorem). Assume that the system \((S^*)\) is type-\(K\) monotone, and \(x(t), y(t)\) are the solutions of \((S^*)\) defined on \(a \leq t \leq b\) with \(x(a) \leq_K y(a)\). Then \(x(t) \leq_K y(t)\) for all \(t \in [a, b]\).

**Theorem 3.2.** Let the system \((S^*)\) be a type-\(K\) monotone system and let \(f(x) \geq_K 0\) for some \(x \in R^n_+\). Then \(\{\phi_i(x)\}_i\) is nondecreasing if \(i \in I\) and \(\{\phi_i(x)\}_j\) is nonincreasing if \(j \in J\) for all \(t \geq 0\) for which the solution exists. A similar result holds if \(f(x) \leq_K 0\).
Theorem 3.1 is extended in a natural way from cooperative systems to type-\(K\) monotone systems (see [2, Theorem 2.4]). Theorem 3.2 is a criterion for the monotonicity of every component of a solution which is generalized from a cooperative system given by Selgrade [6] to a type-\(K\) monotone system ([2, p.862]).

The proof of Theorem A (Necessity). Since the condition (C) holds, every solution of the type-\(K\) monotone system (\(S^*\)) is bounded by Proposition 4.4 in [7]. Let \(p = (p_1, p_2)\) be a positive steady state of (\(S^*\)) which is globally asymptotically stable in Int \(R^p_+\). Then the subsystems (\(S_I\)) and (\(S_J\)) have positive steady states \(x_1^0\) and \(x_2^0\), respectively, and \(x_1^0 \geq p_1, x_2^0 \geq p_2\) by Proposition 3.3 in [2].

We claim that \(J_1 = \{ j \in J : f_j(x_1^0, 0) > 0 \} \neq \emptyset\). Otherwise, \(f_j(x_1^0, 0) \leq 0\), which implies \(f(x_1^0, 0) \leq 0\). Then \((x_1^0, 0)\) and \(p\) are both globally asymptotically stable relative to Int \(R^p_+\) by Theorem 2.1 and the assumption. This is a contradiction and proves our claim.

Set \(I_1 = I \cup J_1\). We consider the subsystem

\[ (S_{I_1}) \]
\[ \dot{x}_{I_1} = \text{diag}(x_{I_1}) f_{I_1}(x_{I_1}, 0). \]

Obviously, \(f_{I_1}(x_1^0, 0) \leq K_1, 0\) where \(K_1 = R^k_+ \times (-R^p_{\#J_1})\). Then there exists sufficiently small \(v \in \text{Int} \bigcap \{ x \in R^k_+ : f_I(x, 0) \geq 0 \}\) because of \(\partial f_I/\partial x_{J_1} \leq 0 \) and \(f_I(x_1^0, 0) > 0 \) because of the continuity of \(f\). It follows that \(\phi^{I_1}_{I_1}(x_1^0, v) \) is type-\(K_1\) nonincreasing from Theorem 3.2. Choose a small \(v\) such that \((x_1^0, v) \geq K_1 (p_1, p_{J_1})\). Then \(\phi^{I_1}_{I_1}(x_1^0, v) \geq \phi^{I_1}_{I_1}(p_{J_1})\) by Theorem 3.1, that is, \(x_1^0 \geq \{ \phi^{I_1}_{I_1}(x_1^0, v) \}_{I_1} \geq \{ \phi^{I_1}_{I_1}(p_{J_1}) \}_{I_1} \) and \(v \leq \{ \phi^{I_1}_{I_1}(x_1^0, v) \}_{I_1} \leq \{ \phi^{I_1}_{I_1}(p_{J_1}) \}_{I_1} \) for all \(t > 0\).

It is easy to prove that \(f_{I_1}(p_{J_1}, 0) \geq K_1, 0\). In fact, \(f_I(p_{J_1}, 0) \geq f_{I_1}(p_{J_1}, p_{J_1}) = 0\) because of \(\partial f_{I_1}/\partial x_{J_1} \leq 0 \) and \(f_{I_1}(p_{J_1}, p_{J_1}) \) is type-\(K_1\) nondecreasing from Theorem 3.2. Then \(\phi^{I_1}_{I_1}(p_{J_1}) \) is type-\(K_1\) nondecreasing from Theorem 3.2. Then \(\{ \phi^{I_1}_{I_1}(p_{J_1}) \} \geq \phi_{I_1}^{I_1}(x_1^0, v) \), \(v \leq \{ \phi^{I_1}_{I_1}(x_1^0, v) \}_{I_1} \leq p_{J_1}\) for \(t > 0\). Thus we have \(x_1^0 \geq \{ \phi^{I_1}_{I_1}(x_1^0, v) \}_{I_1} \geq p_{J_1}\) and \(v \leq \{ \phi^{I_1}_{I_1}(x_1^0, v) \}_{I_1} \leq p_{J_1}\). It follows from the type-\(K_1\) increase of \(\phi^{I_1}_{I_1}(x_1^0, v)\) that

\[ \lim_{t \to +\infty} \phi^{I_1}_{I_1}(x_1^0, v) = u_{I_1}. \]

Clearly, \(u_{I_1} > 0\), that is, \((S_{I_1})\) has a positive steady state \(u_{I_1}\).

We claim that \(f_{I_1}(u_{I_1}, 0) \leq 0\) is impossible. If not, then \((u_{I_1}, 0)\) is globally asymptotically stable by Theorem 2.1, contradicting the assumption. Thus, there exists \(J_2 = \{ j \in I_1 : f_j(u_{I_1}, 0) > 0 \} \neq \emptyset\). If \(J_2 = I_1\), then we choose \(L = I_1\). Otherwise, setting \(L = I_2 = I_1 \cup J_2\), we consider the subsystem

\[ (S_{I_2}) \]
\[ \dot{x}_{I_2} = \text{diag}(x_{I_2}) f_{I_2}(x_{I_2}, 0). \]

Since \(f_j(u_{I_1}, 0, 0) > 0\) for \(j \in J_2\) and \(f_j(u_{I_2}, 0, 0) = 0\) for \(j \in I_1\), we have \(f_{I_2}(u_{I_1}, 0, 0) \leq K_2, 0\) where \(K_2 = R^k_+ \times (-R^p_{\#(J_1 \cup J_2)})\). It is easy to see that \(f_j(p_{J_1}, p_{J_2}, 0) = f_j(p_{J_1}, p_{J_2}, p_{J_2}) = 0\) for \(j \in J_1 \cup J_2\) because of \(\partial f_j/\partial x_{J_2} \leq 0\) for \(j \in J_1 \cup J_2\) and \(f_j(p_{J_1}, p_{J_2}, 0) \geq f_j(p_{J_1}, p_{J_2}, p_{J_2}) = 0\) for \(j \in I\). Then \(f_{I_2}(p_{J_1}, p_{J_2}, 0) \geq K_2, 0\) and \(\phi^{I_2}_{I_2}(x_1^0, v) \geq p_{J_2}\) for \(t > 0\). Thus we have \(x_1^0 \geq \{ \phi^{I_2}_{I_2}(x_1^0, v) \}_{I_2} \geq p_{J_2}\) and \(v \leq \{ \phi^{I_2}_{I_2}(x_1^0, v) \}_{I_2} \leq p_{J_2}\). It follows from the type-\(K_2\) increase of \(\phi^{I_2}_{I_2}(x_1^0, v)\) that

\[ \lim_{t \to +\infty} \phi^{I_2}_{I_2}(x_1^0, v) = u_{I_2}. \]

Similarly, \(f_{I_2}(u_{I_2}, 0) \leq 0\) cannot hold. Then there exists \(J_3 = \{ j \in I_2 : f_j(u_{I_2}, 0) > 0 \} \neq \emptyset\). This process must stop at most the \((n - k - 1)\)th-step. We assume the process stops at the \(n\)th-step. Let \(L = I \cup J_1 \cup \cdots \cup J_n\). Then the subsystem \((S_L)\) has a positive steady state \(u^0\) such that \(f_L(u^0, 0) > 0\). Clearly, \(L \supset I\).

The necessity of (2) can be proved by using the same method.
(Sufficiency) In fact, the main idea to prove the sufficiency is taken from [5], [7]. First, we prove that if \((S^*)\) has a positive steady state \(\bar{x}\), then \(\bar{x}\) is globally asymptotically stable in \(\text{Int} R^n_+\).

Since \(\bar{x}\) is a positive steady state, for any \(x \in R^n_+\), we have

\[
f(x) - f(\bar{x}) = \left[\int_0^1 Df(sx + (1-s)\bar{x})ds\right](x - \bar{x}).
\]

The condition (C) shows that

\[
\int_0^1 Df(sx + (1-s)\bar{x})ds \leq K M,
\]

which implies that

\[
\text{diag}(x) f(x) \leq K \text{diag}(x) M(x - \bar{x}) \quad \text{for} \quad x \geq_K \bar{x}, \quad x \geq 0,
\]

and

\[
\text{diag}(x) f(x) \geq K \text{diag}(x) M(x - \bar{x}) \quad \text{for} \quad x \leq_K \bar{x}, \quad x \geq 0.
\]

Make a Lotka–Volterra system

\[(S) \quad \dot{x} = \text{diag}(x)(r + Mx), \quad x \in R^n_+ \quad \text{and} \quad r \in R^n,
\]

where \(r = -M\bar{x}\). Obviously, \(\bar{x}\) is a positive steady state for \((S)\) and attracts all points in \(\text{Int} R^n_+\) by the result of Takeuchi and Adachi [1] mentioned in the introduction. Hence it follows that

\[
\lim_{t \to +\infty} \phi_t(x) = \bar{x} \quad \text{for} \quad x \in \text{Int} R^n_+ \quad \text{with} \quad x \geq_K \bar{x} \quad \text{or} \quad x \leq_K \bar{x}
\]

from standard differential inequality arguments.

For any \(x \in \text{Int} R^n_+\), there exist \(y \geq_K \bar{x}\) and \(y > 0\), \(z \leq_K \bar{x}\) and \(z > 0\) such that \(z \leq_K x \leq_K y\). Applying Theorem 3.1, we have

\[
\phi_t(z) \leq_K \phi_t(x) \leq_K \phi_t(y) \quad \text{for} \quad t \geq 0.
\]

Because

\[
\lim_{t \to +\infty} \phi_t(z) = \lim_{t \to +\infty} \phi_t(y) = \bar{x},
\]

we have established that

\[
\lim_{t \to +\infty} \phi_t(x) = \bar{x} \quad \text{for} \quad x \in \text{Int} R^n_+.
\]

Since \(Df(\bar{x}) \leq_K M\), we have \(s(Df(\bar{x})) \leq s(M) < 0\) from Perron-Frobenius theory (see [2, Thm.2.2]) which implies that \(\bar{x}\) is asymptotically stable. Thus \(\bar{x}\) is globally asymptotically stable.

Next, we prove that \((0, w^0) \leq_K (u^0, 0)\).

Since \(u^0\) and \(w^0\) are positive steady states of \((S_L)\) and \((S_P)\) respectively, we deduce that

\[
\lim_{t \to +\infty} \phi_t^L(u) = u^0 \quad \text{for} \quad u \in \text{Int} R^n_+^L,
\]

and

\[
\lim_{t \to +\infty} \phi_t^P(w) = w^0 \quad \text{for} \quad w \in \text{Int} R^n_+^P.
\]
from the above result. It is easy to choose \( u \in \text{Int } R^+_L \) and \( w \in \text{Int } R^+_P \) such that \((0, w) \leq_K (u, 0)\). Then
\[
(0, \phi^P_t (w)) \leq_K (\phi^L_t (u), 0) \quad \text{for } t \geq 0
\]
by Theorem 3.1. Therefore, we conclude that
\[
(0, w^0) \leq_K (u^0, 0).
\]

Finally, we prove that \((S^+)\) has a unique positive steady state which is globally asymptotically stable in \( R^+_0 \).

Since \((0, w^0) \leq_K (u^0, 0)\), we have \((0, w^0) \leq_K (z, w^0) \leq_K (u^0, 0)\) for sufficiently small \( z > 0 \). From condition (2), for above \( z \), it is easy to obtain that \( f(z, w^0) \geq_K 0 \) from the continuity and type-\( K \) monotonicity of \( f \). Then \( \phi_t(z, w^0) \) is type-\( K \) nondecreasing for \( t \geq 0 \) by Theorem 3.2. We deduce that \((0, w^0) \leq_K \phi_t(z, w^0) \leq_K (u^0, 0)\) for all \( t \geq 0 \) from Theorem 3.1. Hence, \( \phi_t(z, w^0) \) is bounded. Consequently, \( \phi_t(z, w^0) \) converges to some point \( \tilde{x} \) as \( t \to +\infty \) and \((0, w^0) \leq_K \tilde{x} \leq_K (u^0, 0)\). It is clear that \( \tilde{x}_I \geq \{(z, w^0)^I \} \geq 0 \). Similarly, from the condition (1), for sufficiently small \( v \), we have \( \phi_t(u^0, v) \to \tilde{x} \) as \( t \to +\infty \), \((0, w^0) \leq_K \tilde{x} \leq_K (u^0, 0) \) and \( \tilde{x}_J \geq \{(u^0, v)^J \} \geq 0 \).

Choose \( z > 0, v > 0 \) to be small enough such that \((z, w^0) \leq_K (u^0, v)\). Then \( \phi_t(z, w^0) \leq_K \phi_t(u^0, v) \) for \( t > 0 \). So \( \tilde{x} \leq_K \tilde{x} \), namely, \((\tilde{x}_I, \tilde{x}_J) \leq_K (\tilde{x}_I, \tilde{x}_J) \). This means

\[
\tilde{x}_I \geq \tilde{x}_I > 0, \quad \tilde{x}_J \geq \tilde{x}_J > 0.
\]

Hence, we conclude that \( \tilde{x} > 0 \) and \( \tilde{x} > 0 \). From the first paragraph of the proof of sufficiency, it follows that \( \tilde{x} = \tilde{x} \) which is globally asymptotically stable in \( R^+_0 \).

The proof is completed. \( \square \)

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, PEOPLE’S REPUBLIC OF CHINA

E-mail address: jiangjf@math.ustc.edu.cn