

DENSELY HEREDITARILY HYPERCYCLIC SEQUENCES AND LARGE HYPERCYCLIC MANIFOLDS

LUIS BERNAL-GONZÁLEZ

(Communicated by David R. Larson)

ABSTRACT. We prove in this paper that if (T_n) is a hereditarily hypercyclic sequence of continuous linear mappings between two topological vector spaces X and Y , where Y is metrizable, then there is an infinite-dimensional linear submanifold M of X such that each non-zero vector of M is hypercyclic for (T_n) . If, in addition, X is metrizable and separable and (T_n) is densely hereditarily hypercyclic, then M can be chosen dense.

1. PRELIMINARIES

In this paper we make the following notations and abbreviations: \mathbb{R} = the real line, \mathbb{C} = the complex plane, \mathbb{N} = the set of positive integers, TVS = topological vector space, LCS = locally convex space, HC = hypercyclic, HHC = hereditarily hypercyclic, DHC = densely hypercyclic, DHHC = densely hereditarily hypercyclic.

Let X and Y be two TVSs over the same field \mathbb{K} ($= \mathbb{R}$ or \mathbb{C}) and $T_n : X \rightarrow Y$ ($n \in \mathbb{N}$) a sequence of continuous linear mappings. A vector $x \in X$ is said to be HC for (T_n) if the orbit $\{T_n x : n \in \mathbb{N}\}$ is dense in Y . Denote $HC((T_n)) = \{x \in X : x \text{ is HC for } (T_n)\}$. It is easy to prove that if Y is metrizable and separable, then $HC((T_n))$ is a G_δ subset. The sequence (T_n) is called HC whenever $HC((T_n))$ is not empty. Clearly, Y must be separable in this case. (T_n) is called HHC (see [An1, Section 3]) or strongly HC (see [BoS, Section 2]) whenever every subsequence (T_{n_k}) is HC. We say that (T_n) is DHC whenever $HC((T_n))$ is dense in X . Finally, we say that (T_n) is DHHC whenever every subsequence (T_{n_k}) is DHC. It is evident that if (T_n) is DHHC, then it is DHC and HHC, and if (T_n) is either DHC or HHC, then it is HC. If $X = Y$ and $T : X \rightarrow X$ is an operator ($=$ continuous linear selfmapping) on X , then a vector $x \in X$ is said to be HC for T if and only if it is HC for the sequence (T^n) of iterates of T . T is said to be HC (HHC, DHC, DHHC) whenever the sequence (T^n) is HC (HHC, DHC, DHHC, respectively). We denote $HC(T) = HC((T^n))$. It is well known [Kit, Theorem 4.2] (see also [Rol]) that no finite-dimensional TVS X supports a HC operator. The problem of the existence of a HC operator on X when X is infinite-dimensional –posed by Rolewicz [Rol] in

Received by the editors February 2, 1998.

1991 *Mathematics Subject Classification*. Primary 47B99; Secondary 46A99, 30E10, 32A07.

Key words and phrases. Hypercyclic vector, linear operator, densely hereditarily hypercyclic sequence, infinite-dimensional manifold, dense manifold, metrizable topological vector space, entire function of subexponential type, Runge domain, infinite order linear differential operator.

This research was supported in part by DGES grant #PB96–1348 and the Junta de Andalucía.

1969– has been recently solved in the affirmative for Banach spaces, Fréchet spaces and certain kinds of TVSs (see [An2], [Be1] and [BoP]).

There are at this time a number of known results about the size and the structure of the sets $HC(T)$ and $HC((T_n))$ if hypercyclicity happens. For instance, if X is a separable TVS, then $HC(T)$ is a dense G_δ (so residual if X is, in addition, Baire) subset of X [GeS, Section 2]. Indeed, if for some vector $x \in X$ we have that $A = \{T^n x : n \in \mathbb{N}\}$ is dense in X , then the set $T^m(A)$ is also dense for each $m \in \mathbb{N}$, because T (and so T^2, T^3, \dots) has dense range; therefore each member $T^m x$ of A is HC, so $HC(T)$ is dense. Consequently, every HC operator on a separable TVS is DHC, and a similar argument shows that every HHC operator is DHHC. The structure of $HC((T_n))$ is far from linear; in fact, if X is Baire, Y is metrizable separable and (T_n) is DHC, then $X = HC((T_n)) + HC((T_n))$ (see [Gro, Satz 1.4.3] and [BoS, Section 1]). Nevertheless, if X is a LCS and T is a HC operator on X , then there is a dense T -invariant linear manifold $M \subset X$ such that $M \setminus \{0\} \subset HC(T)$ (see [Bes], [Bou] and [Her, Section 4]). Obviously, M must be infinite-dimensional. From now on, we shall say that a linear submanifold M of X is HC for a sequence $T_n : X \rightarrow Y$ ($n \in \mathbb{N}$) whenever $M \setminus \{0\} \subset HC((T_n))$. None of the two latter results can be extended to a sequence (T_n) of operators. Indeed, let X be a two dimensional Hilbert space, with orthonormal basis $\{e, f\}$, (x_n) a countable dense subset of X and, for each n , let y_n be a vector of norm n that is orthogonal to x_n . Then the sequence (T_n) of operators on X defined by $T_n(\alpha e + \beta f) = \alpha x_n + \beta y_n$ (α, β scalars) (see [GoS, Section 1]) is HC but $HC((T_n)) = \{\text{non-zero scalar multiples of } e\}$, which is not dense. That is, (T_n) is not DHC. Note that it is not HHC either. This example can be easily generalized to an infinite-dimensional separable Hilbert space, giving also a set $HC((T_n))$ with essentially one HC vector.

We state in our terminology the next nice result by Grosse-Erdmann [Gro, Satz 1.4.2]), which is easy to prove: Let X be an F -space (=complete metrizable TVS), Y a separable metrizable TVS and $T_n : X \rightarrow Y$ ($n \in \mathbb{N}$) a HC sequence of continuous linear mappings. If there is a dense subset $D \subset X$ such that $\lim_{n \rightarrow \infty} T_n x$ exists in Y for all $x \in D$, then (T_n) is DHC. Clearly, under the same hypotheses, (T_n) is DHHC if it is HHC. By the way, Ansari provides in [An1] sufficient conditions for a scalar multiple of an operator T on a Banach space to be HHC. Another sufficient condition for a sequence (T_n) of operators on a separable F -space X to be DHHC (see [GeS, Theorem 2.2] or [GoS, Corollary 1.4]) is the following:

- (A) There are dense subsets $D_1, D_2 \subset X$ and a sequence of maps $S_n : D_2 \rightarrow X$ ($n \in \mathbb{N}$) (possibly nonlinear, possibly discontinuous) such that $\lim_{n \rightarrow \infty} T_n x = 0$ for each $x \in D_1$, $\lim_{n \rightarrow \infty} S_n y = 0$ for each $y \in D_2$ and $T_n S_n = \text{identity on } D_2$.

A great deal of operators appearing in the literature, such as those of derivation and translation on the Fréchet space of complex entire functions, satisfy the hypercyclicity criterion (A) and, consequently, they are DHHC. In 1996, Montes [Mon] isolated sufficient conditions for an operator (or a sequence of operators) on a Banach space X to have an infinite-dimensional Banach subspace of HC vectors. His main result (see [Mon, Theorem 2.2 and Remarks 1 and 2]) states that if X is a separable Banach space, (T_n) is a sequence of operators on X satisfying (A) and there is an infinite-dimensional Banach subspace X_0 such that $\lim_{n \rightarrow \infty} T_n x = 0$ for every $x \in X_0$, then there is a HC infinite-dimensional Banach subspace X_1 for (T_n) . The last conclusion remains obviously true if the hypotheses are satisfied just

for a fixed subsequence (n_k) of positive integers (see [LeM]). Some applications of Montes' result can be found in [Mon, Section 3] and [LeM, Sections 2-4], in relation to composition operators on H^p spaces, compact perturbations of an operator of norm ≤ 1 and weighted backward shifts on sequence spaces. An example of an operator on a Banach space for which all HC Banach subspaces are finite-dimensional is provided in [Mon, Theorem 3.4] (see also the remark following Proposition 3.1 in [LeM]).

The positive and negative results quoted in this section suggest the following natural question: Are there large (i.e., infinite-dimensional or even dense) HC submanifolds for every HHC (or DHHC) sequence $T_n : X \rightarrow Y$ ($n \in \mathbb{N}$)? In this paper, we give an affirmative answer and prove that, under smooth hypotheses on X and Y , such manifolds always exist (Section 2). An example taken from Function Theory is supplied in Section 3.

2. EXISTENCE OF LARGE HYPERCYCLIC LINEAR MANIFOLDS

If (T_n) is HHC, the conditions to be imposed for the existence of infinite-dimensional HC submanifolds are very smooth and expected: X is infinite-dimensional and Y is metrizable. In fact, the hypothesis that (T_n) is HHC forces X to be infinite-dimensional. If, in addition, the space X is metrizable and separable and (T_n) is DHHC, then one can get a dense HC submanifold. This is the content of the two next results.

Theorem 1. *Let X and Y be two TVSs such that Y metrizable. If $T_n : X \rightarrow Y$ ($n \in \mathbb{N}$) is a HHC sequence of continuous linear mappings, then there is an infinite-dimensional HC linear submanifold of X for (T_n) . In particular, X must be infinite-dimensional.*

Proof. First, we pick a HC vector x_1 for (T_n) . Then we can find a subsequence $\{p(1, j) : j \in \mathbb{N}\}$ of positive integers such that

$$(1) \quad T_{p(1, j)} x_1 \rightarrow 0 \quad (j \rightarrow \infty).$$

Second, since (T_n) is HHC, the sequence $(T_{p(1, j)})$ is HC, so we can choose a vector $x_2 \in HC((T_{p(1, j)}))$. By (1) it is clear that x_2 is linear independent of x_1 , because $(T_{p(1, j)} x_2)$ cannot tend to zero, by density. Now choose a subsequence $\{p(2, j) : j \in \mathbb{N}\}$ of $(p(1, j))$ with

$$(2) \quad T_{p(2, j)} x_2 \rightarrow 0 \quad (j \rightarrow \infty).$$

Note that $T_{p(2, j)} x_1 \rightarrow 0$ ($j \rightarrow \infty$) too. The new sequence $(T_{p(2, j)})$ is HC. Choose a vector $x_3 \in HC((T_{p(2, j)}))$. By (1) and (2) it is clear that x_3 does not belong to the linear span of $\{x_1, x_2\}$.

It is evident that this process can be continued by induction, getting a sequence $\{x_N : N \in \mathbb{N}\} \subset X$ and a family $\{\{p(n, j) : j \in \mathbb{N}\} : n \in \mathbb{N}\}$ of sequences of positive integers satisfying

$$(3) \quad x_N \in G_{N-1} \quad \text{for all } N \in \mathbb{N},$$

$$(4) \quad x_N \in HC((T_{p(N-1, j)})) \quad \text{for all } N \in \mathbb{N}$$

and

$$(5) \quad T_{p(k,j)}x_N \rightarrow 0 \quad (j \rightarrow \infty) \quad \text{for all } k \geq N,$$

where $(p(0, j))$ is the whole sequence of positive integers, $G_0 = X$ and $G_N = X \setminus \text{span}(\{x_1, \dots, x_N\})$ for $N \in \mathbb{N}$. Define

$$M = \text{span}(\{x_N : N \in \mathbb{N}\}).$$

It is clear from (3) that M is an infinite-dimensional linear submanifold of X .

It remains to prove that each nonzero vector of M is HC for (T_n) . Fix $x \in M \setminus \{0\}$. Then there are finitely many scalars a_1, \dots, a_N with $a_N \neq 0$ such that $x = \sum_{k=1}^N a_k x_k$. We may assume that $a_N = 1$ because if λ is a nonzero scalar, then x is HC if and only if λx is HC. Let y be in Y . Let us show a subsequence $\{T_{r(j)} : j \in \mathbb{N}\}$ of (T_n) such that

$$T_{r(j)}x \rightarrow y \quad (j \rightarrow \infty).$$

By (4), there is a subsequence $(r(j))$ of $(p(N-1, j))$ such that

$$(6) \quad T_{r(j)}x_N \rightarrow y \quad (j \rightarrow \infty).$$

But, since $(r(j))$ is a subsequence of $(p(N-1, j))$, we see from (5) that $T_{r(j)}x_k \rightarrow 0$ ($j \rightarrow \infty$) for all $k \in \{1, \dots, N-1\}$, so $\sum_{k=1}^{N-1} a_k T_{r(j)}x_k \rightarrow 0$ ($j \rightarrow \infty$). Finally, by (6) and linearity,

$$T_{r(j)}x = T_{r(j)}x_N + \sum_{k=1}^{N-1} a_k T_{r(j)}x_k \rightarrow y + 0 = y \quad (j \rightarrow \infty),$$

as required. \square

Theorem 2. *Let X and Y be two separable metrizable TVSs. If $T_n : X \rightarrow Y$ ($n \in \mathbb{N}$) is a DHHC sequence of continuous linear mappings, then there is a dense HC linear submanifold of X for (T_n) .*

Proof. The technique is very similar to that of the proof of the latter theorem, but there are several small changes. We omit the details of some steps. Choose a dense sequence (z_n) in X and denote by D a distance on X compatible with its topology. As in the proof of Theorem 1, we can get a sequence $\{x_N : N \in \mathbb{N}\} \subset X$ and corresponding sequences $(p(N, j))$ of positive integers satisfying (3), (4) and (5), where now G_N is the open ball

$$G_N = \{x \in X : D(x, z_N) < 1/N\}$$

for $N \in \mathbb{N}$. Observe that this time (3) is possible because (T_n) is DHHC. Define again

$$M = \text{span}(\{x_N : N \in \mathbb{N}\}).$$

Then M is a linear submanifold of X and from (4) and (5) we obtain that each non-zero element of M is HC for (T_n) . Finally, the set $\{x_n : n \in \mathbb{N}\}$ is dense in X because $\{z_n : n \in \mathbb{N}\}$ is dense and $D(x_n, z_n) < 1/n \rightarrow 0$ ($n \rightarrow \infty$) by (3). Thus, M is also dense and the proof is complete. \square

It is evident that the conclusions of Theorems 1, 2 hold if we just assume that some subsequence (T_{n_k}) of (T_n) is HHC or DHHC, respectively. We also point out here that the conclusion of Theorem 2 is false if it is just assumed that (T_n) is HHC: for instance, Herzog [Hez, Section 4] has shown that there is a sequence $(a_n) \subset \mathbb{D}$ with $|a_n| \rightarrow 1$ ($n \rightarrow \infty$) such that the sequence $T_n : f \in H^\infty \mapsto T_n f \in H(\mathbb{D})$ defined as

$$(T_n f)(z) = f' \left(\frac{z - a_n}{1 - \overline{a_n} z} \right) \quad (z \in \mathbb{D}; n \in \mathbb{N})$$

is not DHC (so there can be no dense HC manifold). Nevertheless, it is HHC by Theorem 3 of [Hez] applied on $X = A(\mathbb{D}) \subset H^\infty$. Here \mathbb{D} is, as usual, the open unit disk, $H(\mathbb{D})$ is the space of analytic functions in \mathbb{D} with the compact-open topology, H^∞ is the space of bounded analytic functions with the maximum norm and $A(\mathbb{D})$ is the space of continuous functions on the closed unit disk which are analytic in \mathbb{D} , with the maximum norm.

3. AN EXAMPLE

In this section we show that certain sequences of infinite-order linear differential operators with constant coefficients on the space of holomorphic functions on a Runge domain of \mathbb{C}^N ($N \in \mathbb{N}$) have dense HC manifolds. We refer the reader to [Be2] for notations and methods. The proofs of the theorems of this section are very similar to some of [Be2], so they will be omitted.

A domain (= nonempty connected open subset) $G \subset \mathbb{C}^N$ is said to be a Runge domain if each analytic function on G can be approximated uniformly by polynomials on every compact subset of G (see [Hor, pp. 52–59]). When $N = 1$, the Runge domains are precisely the simply connected domains. Denote by $H(G)$, as usual, the Fréchet space of analytic functions on G endowed with the compact-open topology. For $1 \leq j \leq N$ let D_j denote complex partial differentiation with respect to the j th coordinate. A multi-index is an N -tuple $p = (p_1, \dots, p_N)$ of nonnegative integers. Denote $|p| = p_1 + \dots + p_N$, $D^p = D_1^{p_1} \circ \dots \circ D_N^{p_N}$ ($D^0 = I =$ the identity operator), $z^p = z_1^{p_1} \cdot \dots \cdot z_N^{p_N}$ and $|z| = (|z_1|^2 + \dots + |z_N|^2)^{1/2}$ if $z = (z_1, \dots, z_N)$. An entire function $\Phi(z) = \sum_{|p| \geq 0} a_p z^p$ on \mathbb{C}^N is said to be of exponential type whenever there exist positive constants A and B such that $|\Phi(z)| \leq A e^{B|z|}$ ($z \in \mathbb{C}^N$). We say that Φ is of subexponential type whenever, given $\varepsilon > 0$, there exists a positive constant $K = K(\varepsilon)$ such that $|\Phi(z)| \leq K e^{\varepsilon|z|}$ ($z \in \mathbb{C}^N$). Each entire function of subexponential type is obviously of exponential type. In [Be2, Section 2], it is shown that if $G \subset \mathbb{C}^N$ is a nonempty open subset and $\Phi(z) = \sum_{|p| \geq 0} a_p z^p$ is an entire function of subexponential type, then the series $\Phi(D) = \sum_{|p| \geq 0} a_p D^p$ defines an operator on $H(G)$. If Φ is of exponential type, then the latter series defines an operator on $H(\mathbb{C}^N)$. Note that for $N = 1$ we have the following special cases: $\Phi(D)$ is the differentiation operator $f \mapsto f'$ if $\Phi(z) = z$ and, for fixed $a \in \mathbb{C} \setminus \{0\}$, $\Phi(D)$ is the translation operator $f(z) \mapsto f(z + a)$ if $\Phi(z) = e^{az}$.

Next, we furnish an eigenvalue criterion in order that a sequence (T_n) can be DHHC. Its statement and proof are similar to Theorem 7 of [Be2]. Recall that, in a TVS, a subset is said to be total whenever its linear span is dense. If T is an operator and e is an eigenvector, then we denote by $\lambda(T, e)$ its corresponding eigenvalue.

Theorem 3. Let X be a separable Fréchet space and (T_n) a sequence of operators on X . Assume that there are two subsets \mathcal{A}, \mathcal{B} of X satisfying:

- a) Each vector of $\mathcal{A} \cup \mathcal{B}$ is an eigenvector for every T_n in such a way that $\lim_{n \rightarrow \infty} \lambda(T_n, a) = 0$ for all $a \in \mathcal{A}$ and $\lim_{n \rightarrow \infty} \lambda(T_n, b) = \infty$ for all $b \in \mathcal{B}$.
 b) \mathcal{A} and \mathcal{B} are total in X .

Then (T_n) is DHHC.

Before continuing, let us state four conditions that may or may not be satisfied by a sequence $\{\Phi_n\}_1^\infty$ of entire functions on \mathbb{C}^N . Recall that if $\Phi(z) = \sum_{|p| \geq 0} a_p z^p \in H(\mathbb{C}^N)$ and Φ is not identically zero, then its multiplicity for the zero at the origin is $m(\Phi) = \min\{|p| : a_p \neq 0\}$:

- (P) There are two nonempty open subset A, B of \mathbb{C}^N such that $\lim_{n \rightarrow \infty} \Phi_n(a) = 0$ for all $a \in A$ and $\lim_{n \rightarrow \infty} \Phi_n(b) = \infty$ for all $b \in B$.
 (P') There are two subsets A, B of \mathbb{C}^N each of them with at least one finite accumulation point such that $\lim_{n \rightarrow \infty} \Phi_n(a) = 0$ for all $a \in A$ and $\lim_{n \rightarrow \infty} \Phi_n(b) = \infty$ for all $b \in B$.
 (Q) $m(\Phi_n) \rightarrow \infty$ ($n \rightarrow \infty$) and there is a nonempty open subset $B \subset \mathbb{C}^N$ such that $\lim_{n \rightarrow \infty} \Phi_n(b) = \infty$ for all $b \in B$.
 (Q') $m(\Phi_n) \rightarrow \infty$ ($n \rightarrow \infty$) and there is a subset $B \subset \mathbb{C}^N$ with at least one finite accumulation point such that $\lim_{n \rightarrow \infty} \Phi_n(b) = \infty$ for all $b \in B$.

Trivially (P) implies (P') and (Q) implies (Q'). Easy examples can be found showing that (P) and (Q) ((P') and (Q')), respectively) are not comparable. We only sketch the proof of our final result (Theorem 4): By Theorem 2, it suffices to show that $(\Phi_n(D))$ is DHHC. But this can be done by applying Theorem 3 on suitable families \mathcal{A} and \mathcal{B} made with exponentials related to A and B or with polynomials, in the same way as Theorems 8, 9 of [Be2] are derived from Theorem 7 of the same reference.

Theorem 4. Assume that G is a Runge domain of \mathbb{C}^N and Φ_n ($n \in \mathbb{N}$) are entire functions on \mathbb{C}^N . a) Suppose that every Φ_n is of subexponential (exponential, resp.) type and the sequence $\{\Phi_n\}$ satisfies either (P) or (Q). Then there is a dense HC submanifold of $H(G)$ ($H(\mathbb{C}^N)$, resp.) for $(\Phi_n(D))$. b) For $N = 1$ the statement of a) still holds if (P) and (Q) are respectively changed to (P') and (Q').

ACKNOWLEDGEMENT

I am grateful to Dr. Grosse-Erdmann for providing me with a strong simplification of original statements and proofs of Theorems 1 and 2.

REFERENCES

- [An1] S. I. ANSARI, *Hypercyclic and cyclic vectors*, J. Funct. Anal. **128** (1995), 374–383. MR **96h**:47002
 [An2] S. I. ANSARI, *Existence of hypercyclic operators on topological vectors spaces*, J. Funct. Anal. **148** (1997), 384–390. MR **98h**:47028
 [Be1] L. BERNAL-GONZÁLEZ, *On hypercyclic operators on Banach spaces*, Proc. Amer. Math. Soc., to appear. CMP 98:03
 [Be2] L. BERNAL-GONZÁLEZ, *Hypercyclic sequences of differential and antidifferential operators*, J. Approx. Theory, to appear.
 [Bes] J. BES, *Invariant manifolds of hypercyclic vectors for the real scalar case*, Proc. Amer. Math. Soc., to appear. CMP 98:05

- [BoP] J. BONET and A. PERIS, *Hypercyclic operators on non-normable Fréchet spaces*, J. Funct. Anal., to appear.
- [Bou] P. S. BOURDON, *Invariant manifolds of hypercyclic vectors*, Proc. Amer. Math. Soc. **118** (1993), 845–847. MR **93i**:47002
- [BoS] P. S. BOURDON and J. H. SHAPIRO, *Cyclic phenomena for composition operators*, Memoirs of the Amer. Math. Soc., No. 596, Providence, RI, 1997. MR **97h**:47023
- [GeS] R. M. GETHNER and J. H. SHAPIRO, *Universal vectors for operators on spaces of holomorphic functions*, Proc. Amer. Math. Soc. **100** (1987), 281–288. MR **88g**:47060
- [GoS] G. GODEFROY and J. H. SHAPIRO, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. **98** (1991), 229–269. MR **92d**:47029
- [Gro] K. G. GROSSE-ERDMANN, *Holomorphe Monster und universelle Funktionen*, Mitt. Math. Sem. Giessen 176, 1987. MR **88i**:30060
- [Her] D. A. HERRERO, *Limits of hypercyclic and supercyclic operators*, J. Funct. Anal. **99** (1991), 179–190. MR **92g**:47026
- [Hez] G. HERZOG, *On a theorem of Seidel and Walsh*, Period. Math. Hungar. **30** (1995), 205–210. MR **96e**:30116
- [Hor] L. HORMANDER, *An Introduction to Complex Analysis in Several Variables*, Van Nostrand, Princeton, NJ, 1966. MR **34**:293
- [Kit] C. KITAI, *Invariant closed sets for linear operators*, Dissertation, University of Toronto, 1982.
- [LeM] F. LEÓN-SAAVEDRA and A. MONTES-RODRÍGUEZ, *Linear structure of hypercyclic vectors*, J. Funct. Anal. **148** (1997), 524–545. MR **98h**:47028b
- [Mon] A. MONTES-RODRÍGUEZ, *Banach spaces of hypercyclic vectors*, Michigan Math. J. **43** (1996), 419–436. MR **98g**:47027
- [Rol] S. ROLEWICZ, *On orbits of elements*, Studia Math. **32** (1969), 17–22. MR **39**:3292

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE SEVILLA, APDO. 1160, SEVILLA 41080, SPAIN

E-mail address: `lbernal@cica.es`