

CALCULATING AND INTERPRETING THE MISLIN GENUS OF A SPECIAL CLASS OF NILPOTENT SPACES

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ABSTRACT. We prove that there is a bijection between the Mislin genus of a circle bundle over a certain nilpotent base space M , which is constructed from a nilpotent group N of a certain specified type, and the Mislin genus of N itself.

1. INTRODUCTION

In [9] Mislin introduced the concept of the genus of a finitely generated nilpotent group N . This is the set of isomorphism classes of finitely generated nilpotent groups M such that the localizations M_p and N_p are isomorphic at every prime p . Analogously the Mislin genus of a connected nilpotent space X (of the homotopy type of a CW-complex) of finite type (see [4]) is defined as the set of homotopy types of nilpotent spaces Y of finite type such that the localizations Y_p and X_p are homotopy equivalent at every prime p .

Now it is easy to see that, given a finitely generated nilpotent group N , one may construct the Eilenberg–Mac Lane space $K(N, 1)$, and there is then a bijection of Mislin genera $\mathcal{G}(N) \cong \mathcal{G}(K(N, 1))$. Our object in this paper is to establish a bijection of genera $\mathcal{G}(N) \cong \mathcal{G}(X)$ for nilpotent groups N of a special class and nilpotent spaces X constructed in a more subtle way from the group N . Moreover, we will be dealing with finitely generated nilpotent groups whose Mislin genera have already been calculated. Indeed, in a series of papers ([1], [2], [5], [7]), the authors have calculated the Mislin genus of any group in a certain class \mathcal{N}_1 of finitely generated nilpotent groups. The class \mathcal{N}_1 consists of those nilpotent groups N , given in terms of the associated short exact sequence

$$(1.1) \quad TN \twoheadrightarrow N \twoheadrightarrow FN,$$

where TN is the torsion subgroup and FN the torsionfree quotient, by the conditions

- (a) TN and FN are commutative;
- (b) (1.1) splits on the right; and
- (c) for the associated action $\omega: FN \rightarrow \text{Aut } TN$, the image ωFN is contained in the centre of $\text{Aut } TN$.

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In fact, in the presence of (a), condition (c) is equivalent to condition

(c') for all ξ in FN , there exists a positive integer u such that $\xi \cdot a = \omega(\xi)(a) = ua$ for all $a \in TN$.

It is shown in [6] that the genus $\mathcal{G}(N)$ of a group N in \mathcal{N}_1 is trivial unless FN is cyclic. Now we know (see [3], [9]) that, if $N \in \mathcal{N}_1$, then $\mathcal{G}(N)$ admits the structure of a finite abelian group (note that, since FN is commutative, the commutator subgroup of N is finite). Suppose then that FN is cyclic, generated by ξ , and let t be the order of u (see (c')) modulo m , where m is the exponent of TN . Then the calculation of the genus yields

$$(1.2) \quad \mathcal{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\},$$

where $(\mathbb{Z}/t)^*$ is the multiplicative group of units in the ring \mathbb{Z}/t . Moreover, we know how to modify the presentation of N , given by

$$TN \twoheadrightarrow N \twoheadrightarrow C = \langle \xi \rangle$$

with

$$\xi \cdot a = ua, \text{ for all } a \in TN,$$

to yield the other (isomorphism classes of) groups in the genus of N ; namely, if ℓ is prime to t , then the group N_ℓ corresponding under the isomorphism (1.2) is simply obtained from the short exact sequence

$$TN \twoheadrightarrow N_\ell \twoheadrightarrow C = \langle \xi \rangle$$

by imposing the action

$$\xi \cdot a = u^\ell a, \text{ for all } a \in TN.$$

In [2] it was shown, in the special case where TN is cyclic of prime power order, that we may construct a circle bundle X over a fixed base M to correspond to our group N . It is clear that this construction generalizes to any group N in \mathcal{N}_1 with FN cyclic. More precisely, we prescribe the 2-type of the connected nilpotent polyhedron M by requiring that

$$(1.3) \quad \pi_1 M = C_t = \langle \eta \rangle; \pi_2 M = TN; \eta \cdot a = ua \text{ for all } a \in TN,$$

where C_t denotes the cyclic group of order t . We then let g generate the summand $\text{Ext}(\mathbb{Z}/t, \mathbb{Z})$ of $H^2(M; \mathbb{Z})$, and think of g as the classifying map $g: M \rightarrow K(\mathbb{Z}, 2)$ for a circle bundle X over M . It is easy to see that the bundle map $q: X \rightarrow M$ induces isomorphisms of all homotopy groups in dimensions $n \geq 2$, while $\pi_1 X = C = \langle \xi \rangle$ and q maps ξ onto η .

If we pick ℓ prime to t , and choose ℓ' so that $\ell\ell' \equiv 1 \pmod{t}$, then we may take $\ell'g$ to be the classifying map for a circle bundle X_ℓ over M , and we obtain a covering map $h: X \rightarrow X_\ell$. In this way we set up a bijective correspondence between the elements of $\mathcal{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\}$ and certain elements of the genus of X ; for one may show that X_ℓ is in the genus of X and has the homotopy type of $X_{\bar{\ell}}$ if and only if $\ell \equiv \pm \bar{\ell} \pmod{t}$.

In this generality it would not be reasonable to expect that the entire genus of X would be obtained in this way. However, we have conjectured that this would be so if M contained no non-trivial homotopy beyond that specified by (1.3), that is, if M (and thus also X) has only the two non-trivial homotopy groups π_1 and π_2 . Our object in this note is to prove this conjecture:

Theorem. *If M is a connected nilpotent polyhedron with 2-type as specified in (1.3) and no other non-trivial homotopy groups, then the construction given above*

$$\mathcal{G}(N) \rightarrow \mathcal{G}(X): N_\ell \mapsto X_\ell$$

is a bijection of genera.

Our basic tool is a result of McGibbon ([8]), which extends a theorem of Zabrodsky ([10], [11]). Essentially, McGibbon’s result shows that, for spaces such as our connected nilpotent space X , whose rationalization is an H-space with only one non-vanishing homotopy group, the genus of X may (like the genus of N) be given the structure of a finite abelian group and that there is a right exact sequence, again very like that first noticed and adopted by Mislin ([9]), for specifying the genus group $\mathcal{G}(X)$. More precisely, there is a right exact sequence

$$(1.4) \quad s\text{-Equ}(X) \xrightarrow{d} (\mathbb{Z}/s)^*/\{\pm 1\} \twoheadrightarrow \mathcal{G}(X),$$

where s is a positive integer depending on X and $s\text{-Equ}(X)$ refers to the set of homotopy classes of self-maps of X which are p -equivalences for the prime divisors p of s . Here we have not quoted McGibbon’s result in its full generality in [8], being content to specialize to our situation. Thus our task is simply to use (1.4) to show that

$$\mathcal{G}(X) \cong (\mathbb{Z}/t)^*/\{\pm 1\}.$$

For we have already found a set of homotopy types of spaces X_ℓ , in the genus of X , in bijective correspondence with the elements of $(\mathbb{Z}/t)^*/\{\pm 1\}$.

In Section 2 we interpret the group N as a function of the space X ; we will use this interpretation to analyse the first homomorphism of the sequence (1.4). In Section 3 we complete the analysis of the terms in (1.4) and thus achieve the proof of our main result.

2. A LEMMA ON FREE HOMOTOPY GROUPS

The following result deserves to be well-known – and perhaps is. Let A, X be pointed, path-connected topological spaces. Denote by $[A, \Omega X]$ the set of based homotopy classes of maps from A to ΩX , and let $[A, \Omega X]_{\text{fr}}$ be the set of free homotopy classes of maps from A to ΩX . Then both these sets receive a group structure from the group structure of ΩX in the based homotopy category; and $\pi_1 X$ acts on $[A, \Omega X]$ by means of the rule $[g] \mapsto [\ell g \ell^{-1}]$, where ℓ is a loop, ℓ^{-1} is its reverse and $(\ell g \ell^{-1})(a) = \ell g(a) \ell^{-1}$, for all $a \in A$.

Theorem 2.1. *There is a natural isomorphism*

$$(2.1) \quad \theta: [A, \Omega X]_{\text{fr}} \cong [A, \Omega X] \rtimes \pi_1 X,$$

where the group on the right of (2.1) is the semidirect product for the action of $\pi_1 X$ on $[A, \Omega X]$, and θ is given by

$$\theta[f] = ([f \ell^{-1}], [\ell]),$$

where $\ell = f(a_0)$, with a_0 the base point of A .

Proof. Let $\theta[f_i] = ([f_i \ell_i^{-1}], [\ell_i])$, for $i = 1, 2$. Then

$$\theta[f_1 f_2] = ([f_1 f_2 \ell_2^{-1} \ell_1^{-1}], [\ell_1 \ell_2]).$$

On the other hand,

$$\begin{aligned} ([f_1\ell_1^{-1}], [\ell_1])([f_2\ell_2^{-1}], [\ell_2]) &= ([f_1\ell_1^{-1}\ell_1f_2\ell_2^{-1}\ell_1^{-1}], [\ell_1\ell_2]) \\ &= ([f_1f_2\ell_2^{-1}\ell_1^{-1}], [\ell_1\ell_2]). \end{aligned}$$

Thus θ is a homomorphism. Now define

$$\bar{\theta}: [A, \Omega X] \rtimes \pi_1 X \rightarrow [A, \Omega X]_{\text{fr}}$$

by

$$\bar{\theta}([g], [\ell]) = [g\ell].$$

It is easily verified that $\bar{\theta}$ is a two-sided inverse to θ , so that θ is an isomorphism. \square

We may apply this theorem to the space X constructed in the Introduction out of the group N ; we will specifically suppose that M (and hence X too) has vanishing homotopy groups in dimensions $n \geq 3$. We take $A = S^1$, so that θ is an isomorphism between $[S^1, \Omega X]_{\text{fr}}$ and $\pi_2 X \rtimes \pi_1 X$. But

$$\pi_2 X \rtimes \pi_1 X = TN \rtimes C = N.$$

Thus we have

Corollary 2.2. $[S^1, \Omega X]_{\text{fr}} \cong N$ and the split short exact sequence

$$(2.2) \quad \pi_2 X \hookrightarrow [S^1, \Omega X]_{\text{fr}} \twoheadrightarrow \pi_1 X$$

coincides with the split short exact sequence (1.1)

$$TN \hookrightarrow N \twoheadrightarrow FN. \quad \square$$

3. PROOF OF THE MAIN RESULT

We first use the procedure given on p. 297 of [8]¹ to calculate s in (1.4) for our particular space X . We must calculate certain integers $s_1(X)$, $s_2(X)$ and then s is the least common multiple of the product $s_1(X)s_2(X)$ and those primes which divide the exponent of the torsion in $QH^1(X; \mathbb{Z})$ and $QH^2(X; \mathbb{Z})$. (Any notational obscurities are cleared up in [8].)

We consider the homomorphisms $\sigma_1: \pi_1 X \rightarrow PH_1(X; \mathbb{Z})/\text{torsion}$ and $\sigma_2: \pi_2 X \rightarrow PH_2(X; \mathbb{Z})/\text{torsion}$, which are obvious quotients of the Hurewicz homomorphism. Plainly $\ker \sigma_1 = \{1\}$, $\text{coker } \sigma_2 = \{1\}$ and $s_1(X) = 1$, since there is no torsion in $\pi_1 X$. As to $s_2(X)$, we must look at the lower central series, in the sense of [4, p. 34], for the action of $\pi_1 X$ on $\pi_2 X$. Plainly the product of the exponents of the quotients of the terms of the lower central series is m itself, since, in order that N be nilpotent, it is necessary and sufficient that m divides some power of $(u-1)$. This establishes that $s_2(X) = m$. Now we know that the set of prime divisors of m is, of course, just the set of primes p such that N has p -torsion, and no other primes can enter into the torsion in the (co)homology of X . Thus $s = m$ and the semigroup $s\text{-Equ}(X)$ of (1.4) is just the semigroup under composition of homotopy classes of self-maps of X , which are p -equivalences for all primes $p \in T(N)$, where $T(N)$ denotes the set of all primes p for which N has p -torsion.

¹We write s, s_i in the place of t, t_i in [8] to avoid confusion with our use of t in this part.

Finally, it remains to analyse the homomorphism d of (1.4). We consider an m -equivalence (that is, a $T(N)$ -equivalence) $f: X \rightarrow X$ and must calculate its effect on $\pi_1 X \otimes \mathbb{Q}$. However such an f acts on the sequence (2.2), producing

$$\begin{array}{ccccc} TN & \twoheadrightarrow & N & \twoheadrightarrow & FN = \pi_1 X \\ \downarrow \cong & & \downarrow f_* & & \downarrow f_* \\ TN & \twoheadrightarrow & N & \twoheadrightarrow & FN = \pi_1 X \end{array}$$

and $f_*: N \rightarrow N$ is an element of $T(N)$ -Aut N . Moreover, it was proved in [7] that, to compute the image of f_* in the sequence

$$T(N)\text{-Aut } N \rightarrow (\mathbb{Z}/e)^*/\{\pm 1\} \rightarrow \mathcal{G}(N),$$

which is used to yield $\mathcal{G}(N)$ an abelian group structure, it suffices to look at the determinant of $f_*: FN \rightarrow FN$. It follows that $f_*: \pi_1 X \rightarrow \pi_1 X$ is just multiplication by some integer that is congruent to 1 modulo t . Moreover, any such integer is realizable by some f (and any such f must be a $T(N)$ -equivalence). For if $\ell \equiv 1 \pmod t$, then we have a commutative diagram (recall that g is of order t)

$$\begin{array}{ccccccc} S^1 & \longrightarrow & X & \xrightarrow{q} & M & \xrightarrow{g} & K(\mathbb{Z}, 2) \\ \downarrow \ell & & \downarrow f & & \parallel & & \downarrow \ell \\ S^1 & \longrightarrow & X & \xrightarrow{q} & M & \xrightarrow{g} & K(\mathbb{Z}, 2). \end{array}$$

Finally, we have the short exact sequence

$$K \twoheadrightarrow (\mathbb{Z}/m)^*/\{\pm 1\} \twoheadrightarrow \mathcal{G}(X),$$

in which K consists of the residues, modulo ± 1 , which are congruent to 1 mod t . We claim that $\mathcal{G}(X)$ is just $(\mathbb{Z}/t)^*/\{\pm 1\}$. For consider the homomorphism

$$\rho: (\mathbb{Z}/m)^*/\{\pm 1\} \rightarrow (\mathbb{Z}/t)^*/\{\pm 1\}$$

which simply takes a residue class mod m and regards it as a residue class mod t . (Recall that $t \mid m$.) The kernel of ρ is of course K . On the other hand, it follows easily from Dirichlet's Theorem that ρ is surjective. Thus the quotient of $(\mathbb{Z}/m)^*/\{\pm 1\}$ by K is $(\mathbb{Z}/t)^*/\{\pm 1\}$, so that

$$\mathcal{G}(X) \cong (\mathbb{Z}/t)^*/\{\pm 1\},$$

as claimed. It therefore follows that the method of construction of spaces in the genus of X described in the Introduction produces, in fact, the entire genus and realizes the bijection $\mathcal{G}(X) \cong \mathcal{G}(N)$, as claimed.

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