AN EMBEDDING THEOREM FOR LIE ALGEBRAS

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ABSTRACT. In this paper we give a sufficient condition for a restricted enveloping algebra to be quasi-elementary. We also prove that every finite dimensional p-nilpotent Lie algebra can be embedded in a finite dimensional p-nilpotent quasi-elementary Lie algebra.

1. INTRODUCTION

Elementary abelian p-groups play an important role in understanding the mod-p cohomology of finite groups. For notation suppose that G is a finite group and that k is an algebraically closed field of characteristic $p > 0$. The most fundamental result in this direction was proved by Quillen [7, 8] and says that the minimal primes of the mod-p cohomology ring $H^*(G, k)$ are precisely the inverse images under restriction of the radicals of the rings $H^*(E, k)$ for a set $\{E\}$ of representatives of the conjugacy classes of maximal elementary abelian p-subgroups of G. Among other things this establishes a one-to-one correspondence between the conjugacy classes of maximal elementary abelian p-subgroups and the components of the maximal ideal spectrum of $H^*(G, k)$. Also if M is a finite dimensional kG-module, then an element in $\mathrm{Ext}^*_{kG}(M, M)$ is nilpotent if and only if its restriction to every elementary abelian p-subgroup of G is nilpotent [7, 8, 1].

In [5] and [6] Palmieri and Nakano showed that analogous theorems hold for finite dimensional, graded, cocommutative Hopf algebras, once the correct generalization of an elementary abelian p-group has been identified. Palmieri calls such a generalization a *quasi-elementary* (QE) Hopf algebra. His definition, which is given below, is somewhat technical, and hence it would be helpful to be able to explicitly identify the QE Hopf algebras that arise in a particular context. This is done for finite p-groups in Quillen's work using Serre's theorem [9]. In this case quasi-elementary means elementary. Also, in [5] and [6], the QE subalgebras of the Steenrod algebra are explicitly identified. The answer here is somewhat complicated, but it is still the case that a sub-Hopf algebra of a quasi-elementary algebra is quasi-elementary.

This note grew out of an attempt to understand the structure of the QE Hopf subalgebras of the restricted enveloping algebras of certain p-nilpotent Lie algebras.

In our investigation we came across a rather curious result stated as Theorem 2.4 below. This result shows just how spectacularly the analogy between elementary abelian groups and quasi-elementary Hopf algebras can fail. The p-nilpotent Lie algebras are in some ways the p-restricted Lie algebra analogs of p-groups for group algebras, and we might expect similar behavior. Even though in this context it is clear that QE does not mean E the differences from the p-group case are even greater. Briefly stated, our main result is that any p-nilpotent Lie algebra can be embedded in one which is QE. This is not surprising (indeed it is well known) for arbitrary finite dimensional p-restricted Lie algebras because any one such can be embedded in $\mathfrak{sl}(n)$ for some n. The latter is QE because it is generated as an algebra by its null cone which is irreducible (see Theorem 2.1 below). However the statement is also comparable to the assertion that any finite group can be embedded in a simple group (whose group algebra is necessarily QE by Palmieri's definition, given below).

Throughout this paper $\mathfrak{g}$ is a finite dimensional restricted p-nilpotent Lie algebra over an algebraically closed field k of characteristic $p > 0$. Let $V(\mathfrak{g})$ denote the restricted enveloping algebra of $\mathfrak{g}$. We say that $\mathfrak{g}$ is QE if $V(\mathfrak{g})$ is QE. The cohomology ring $\mathrm{Ext}_{V(\mathfrak{g})}(k,k)$ is denoted by $H^*(\mathfrak{g})$. Let $H^{\mathrm{ev}}(\mathfrak{g})$ be the subring of $H^*(\mathfrak{g})$ generated by elements of even degree. We define

$$H^\bullet(\mathfrak{g}) = \begin{cases} H^*(\mathfrak{g}) & \text{if } p = 2, \\ H^{\mathrm{ev}}(\mathfrak{g}) & \text{if } p \text{ is odd,} \end{cases}$$

and we denote the variety corresponding to $H^\bullet(\mathfrak{g})$ by $|\mathfrak{g}|$. An important tool here is the theorem of Jantzen's [2] which allows us to work with $\mathcal{N}(\mathfrak{g})$, the null cone of $\mathfrak{g}$, rather than directly with $|\mathfrak{g}|$.

The definition of QE is the following:

Definition 1.1 (Palmieri [6]). A Hopf algebra C over k is *elementary* if it is bicommutative and has $x^p = 0$ for all $x \in IC$, the augmentation ideal of C. Let $C(x)$ denote a monogenic elementary Hopf algebra, generated by x; i.e., $C(x)$ is isomorphic as an algebra to $k[x]/(x^n)$, where n is p or 2. (Note here that the case $n = 2$ ($p \neq 2$) occurs only for genuinely graded-cocommutative Hopf algebras, not just cocommutative ones, and will not be of concern here.) For a Hopf algebra E, a nonzero element $v \in \mathrm{Ext}^2_E(k,k)$ is called a *Serre element* if there is a Hopf algebra extension

$$E' \hookrightarrow E \twoheadrightarrow C(x)$$

so that under the induced map in Ext, v is the image of a nonzero element of $\mathrm{Ext}^2_{C(x)}(k,k)$. A Hopf algebra E is *quasi-elementary* if no product of Serre elements is nilpotent.

2. The theorem

As before, let $\mathcal{N}(\mathfrak{g}) = \{x \in \mathfrak{g} : x^{[p]} = 0\}$ be the null cone of $\mathfrak{g}$. By a theorem of Friedlander and Parshall [4], $H^\bullet(\mathfrak{g})$ is a finitely generated module over the symmetric algebra $S^*(\mathfrak{g})$ on the vector space $\mathfrak{g}$. Hence the map of varieties $|\mathfrak{g}| \longrightarrow \mathfrak{g}$ is finite-to-one. Jantzen's theorem says that the image of $|\mathfrak{g}|$ in $\mathfrak{g}$ is $\mathcal{N}(\mathfrak{g})$.

We need the following.

Theorem 2.1. *Let $C \subseteq \mathcal{N}(\mathfrak{g})$ be an irreducible closed connected set. Then $\mathfrak{h}_C$, the restricted Lie algebra generated by C, is a QE Lie algebra.*

Proof. If B is a sub-Hopf algebra of A, then we have a restriction map $\mathrm{res}_{A,B}$: $H^*(A) \to H^*(B)$, and hence a corresponding map of varieties $\mathrm{res}^*_{A,B} : |B| \longrightarrow |A|$. Suppose that $\mathfrak{h}_C$ is not a QE Lie algebra. Then by Definition 1.1, some product $v_1, \ldots, v_t$ of Serre elements is zero. Each v_i corresponds to a subalgebra $E_i \subseteq V(\mathfrak{h}_C)$. Then

$$|\mathfrak{h}_C| = \bigcup_{i=1}^{t} \mathrm{res}^*_{V(\mathfrak{h}_C), E_i} |E_i| .$$

This result, though implicit in the proof of Theorem 1.2 of [6], is not actually stated there. The point is that every element of $H^\bullet(\mathfrak{g})$ whose restriction to every E_i is nilpotent, must be nilpotent, and hence must be contained in every maximal ideal. So any maximal ideal I must contain the kernel of the restriction to some E_i. Otherwise for each i there would exist an element $x_i \notin I$ which is in the kernel of the restriction to E_i. But then we have a contradiction because the product $x_1 \cdots x_t$ is not in I.

By Milnor and Moore [3, Theorem 6.11] $E_i = V(\mathfrak{l}_i)$ for some Lie algebra $\mathfrak{l}_i < \mathfrak{h}_C$. It follows that

$$C \subseteq \bigcup_{i=1}^{t} \mathcal{N}(\mathfrak{l}_i),$$

and, of course, $\mathcal{N}(\mathfrak{l}_i) \subseteq \mathfrak{l}_i$. Since C is irreducible $C \subseteq \mathfrak{l}_i$ for some i. This last is not possible because C generates $\mathfrak{h}_C$. Therefore $\mathfrak{h}_C$ is a QE algebra. □

Corollary 2.2. *Let G be a connected algebraic group which acts on a restricted p-Lie algebra $\mathfrak{g}$. Let $C \subseteq \mathcal{N}(\mathfrak{g})$ be an irreducible closed set. Then the orbit $\mathcal{O}_C$ of C under G generates a QE subalgebra.*

Proof. The point is that $\mathcal{O}_C$ is the image of the map $G \times C \longrightarrow \mathcal{N}(\mathfrak{g})$ which is irreducible because $G \times C$ is irreducible. So the corollary follows from the theorem. □

Remark 2.3. The results in Palmieri's paper have the additional hypothesis that the Hopf algebra be graded. However there is no problem for us. For a finite dimensional, genuinely cocommutative Hopf algebra such as $V(\mathfrak{g})$ we can assume that everything is in degree zero. In addition the results which we need from [6] do not depend on the Hopf algebra being connected.

Now we can state our main theorem.

Theorem 2.4. *Any finite dimensional p-nilpotent Lie algebra $\mathfrak{g}$ can be embedded in a finite dimensional p-nilpotent QE Lie algebra $\mathfrak{q}$.*

Note that by the Poincare-Birkhoff-Witt Theorem, this is equivalent to saying that $V(\mathfrak{g})$ can be embedded in $V(\mathfrak{q})$.

Proof. Recall that k is an algebraically closed field of characteristic p. We can assume that $\mathfrak{g}$ is the restricted Lie algebra of strictly upper triangular $n \times n$ matrices over k, since all p-nilpotent finite dimensional restricted Lie algebras over k can be embedded in such a $\mathfrak{g}$. Denote $\mathfrak{g}$ by $\mathfrak{g}_n$. For all n and p we will construct a QE Lie algebra $\mathfrak{q}_{(n,p)}$ and display an embedding $\theta : \mathfrak{g}_n \hookrightarrow \mathfrak{q}_{(n+(n-1)(p-1),p)}$.

We begin by constructing $\mathfrak{q}_{(n,p)}$.

Definition 2.5. Write $n = pq + r$ where q, r are nonnegative integers with $r < p$. Let x be the following $n \times n$ block diagonal matrix:

$$x = \begin{pmatrix} J_1 & 0 & \ldots & 0 & 0 \\ 0 & J_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & J_q & 0 \\ 0 & 0 & \ldots & 0 & J_{q+1} \end{pmatrix}$$

where each J_i for $1 \le i \le q$ is a $p \times p$ matrix, J_{q+1} is an $r \times r$ matrix, and each J_i, for $1 \le i \le q+1$ has ones down the superdiagonal and zeros elsewhere.

We note that $x^p = 0$ and so

$$x \in \mathcal{N}(\mathfrak{g}_n) = \{x \in \mathfrak{g}_n \,:\, x^{[p]} = 0\}.$$

Let G denote the group of upper triangular matrices with coefficients in k. Then G is a connected algebraic group and G acts on $\mathcal{N}(\mathfrak{g}_n)$ by conjugation. Let C denote the line through the origin and x, and let $\mathcal{O}_C$ denote the orbit of the line C under the given action. Define $\mathfrak{q}_{(n,p)}$ to be the restricted Lie algebra generated by $\overline{\mathcal{O}_C}$. By Corollary 2.2 $\mathfrak{q}_{(n,p)}$ is a QE Lie algebra.

It is an exercise in linear algebra to check that $\mathfrak{q}_{(n,p)}$ consists of all $n \times n$ block upper triangular matrices of the form

$$\begin{pmatrix} R_{1,1} & R_{1,2} & \ldots & R_{1,q} & R_{1,q+1} \\ 0 & R_{2,2} & \ldots & R_{2,q} & R_{2,q+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & R_{q,q} & R_{q,q+1} \\ 0 & 0 & \ldots & 0 & R_{q+1,q+1} \end{pmatrix}$$

where each $R_{i,i}$ for $1 \le i \le q$ is a $p \times p$ strictly upper triangular matrix with arbitrary elements above the diagonal, $R_{q+1,q+1}$ is an $r \times r$ strictly upper triangular matrix, the superdiagonal blocks $R_{i,i+1}$ for $1 \le i \le q$ have arbitrary entries except for a zero in the lower left hand corner, and the remaining blocks are arbitrary.

Now we define the embedding. Let $C \in \mathfrak{g}_n$ and write $C = (c_{i,j})$. Then $\theta(C) = B = (b_{s,t}) \in \mathfrak{q}_{(n+(n-1)(p-1),p)}$ where

$$b_{s,t} = \begin{cases} c_{i,j} & \text{if } s = i + (i-1)(p-1) \text{ and } t = j + (j-1)(p-1), \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

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