PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 12, Pages 3461–3466
S 0002-9939(99)04898-4
Article electronically published on May 13, 1999

EXISTENCE OF HOMOGENEOUS IDEALS FITTING INTO LONG BOURBAKI SEQUENCES

MUTSUMI AMASAKI

(Communicated by Wolmer V. Vasconcelos)

Abstract. For any finitely generated torsion-free graded module over a polynomial ring, there exists a homogeneous ideal fitting into an exact sequence similar to a Bourbaki sequence even though its height is not restricted to two.

Introduction

Given a homogeneous ideal I of height $p \geq 2$ in a polynomial ring $R := k[x_1, \dots, x_r]$, there is a finitely generated torsion-free graded R-module M with no free direct summand satisfying $\mathrm{Ext}^i_R(M, R) = 0$ for $i = 1, \dots, p-1$ that fits into an exact sequence of the form

$$(*) \qquad 0 \longrightarrow S_{p-1} \longrightarrow S_{p-2} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \oplus M \longrightarrow I(c) \longrightarrow 0,$$

where c is an integer and S_i $(0 \leq i \leq p-1)$ are finitely generated graded free R-modules (see e.g. [6, Section 1], [12, Section 1]). By this sequence one obtains

$$H^{i-1}_{\mathfrak{m}}(R/I)(c) \cong H^i_{\mathfrak{m}}(M) \quad \text{for} \quad i = 1, \dots, \dim(R/I) = r - p.$$

Since $H^i_{\mathfrak{m}}(M) = 0$ for $i = r-p+1, \dots, r-1$ and $i = 0$ by local duality, considering the local cohomologies of R/I is the same thing as considering those of M.

Keeping the above observation in mind we are interested in the following problem. First, fix an integer $p \geq 2$ and a finitely generated torsion-free graded R-module M with no free direct summand satisfying $\mathrm{Ext}^i_R(M, R) = 0$ for $i = 1, \dots, p-1$. Next, let $\mathfrak{I}(M, p)$ be the set of all homogeneous ideals I in R of height p fitting into exact sequences of the form $(*)$. With this notation, describe all the members of $\mathfrak{I}(M, p)$ in full generality.

Perhaps the most popular way to study the structure of $\mathfrak{I}(M, p)$ is to do so in the framework of even linkage theory (see [8, 11, 12, 14]). But we want to propose another approach based on the analysis of basic sequences. Roughly speaking, the basic sequence $B_R(I) = (\bar{n}^1; \bar{n}^2; \cdots; \bar{n}^{r+1})$ of a homogeneous ideal I is a sequence of integers obtained by arranging in a suitable order the degrees of the Gröbner basis of I with respect to generic coordinates, where $\bar{n}^i$ denotes a subsequence for each i (see [3, Section 1], [5, Example 4.1]). A similar sequence can also be defined for an arbitrary finitely generated graded R-module (see [5, Section 2]) and, if $I \in \mathfrak{I}(M, p)$

Received by the editors September 26, 1997 and, in revised form, February 10, 1998.
1991 *Mathematics Subject Classification.* Primary 13D02; Secondary 13D03.

with $B_R(I) = (\bar{n}^1; \bar{n}^2; \cdots; \bar{n}^{r+1})$, $B_R(M) = (\bar{\nu}^1; \bar{\nu}^2; \cdots; \bar{\nu}^{r+1})$, then we have

$$(**) \qquad \begin{cases} \bar{n}^p = (\bar{w}', \bar{\nu}^p + c) & \text{up to permutation and} \\ \bar{n}^i = \bar{\nu}^i + c & \text{for } i = p+1, \dots, r+1 \end{cases}$$

with a suitable sequence of integers $\bar{w}'$, where $\bar{a} + c = (a_1 + c, \dots, a_l + c)$ for a sequence $\bar{a} = (a_1, \dots, a_l)$ (see [6, Section 2]). This formula, however, is not enough for characterizing all possible basic sequences of the elements of $\mathfrak{I}(M, p)$.

So far we have been successful only in two special cases. In the case $p = 2$, the formula $(**)$ can further be developed to give a complete description of the basic sequences of the elements of $\mathfrak{I}(M, 2)$ (see [7, Section 2]). Our results describe also in a different way the main theorem in the above-mentioned two codimensional even linkage theory. In particular, one can understand the numerical function θ_X defined by Nollet in [13] from a structural point of view (see [7, Theorem 3.3]). When M is Buchsbaum, or equivalently, when the ring R/I is Buchsbaum, the relation $(**)$, together with the Borel fixedness of generic initial ideals, determines almost completely the basic sequences of the elements of $\mathfrak{I}(M, p)$ (see [1, Sections 2 and 3], [2], [3, Sections 5 and 6]).

Before proceeding further, we have to know first whether the set $\mathfrak{I}(M, p)$ is empty or not. The aim of this paper is to settle this question. In fact we will prove that $\mathfrak{I}(M, p) \neq \emptyset$ for an arbitrary finitely generated torsion-free graded R-module M with no free direct summand satisfying $\operatorname{Ext}^i_R(M, R) = 0$ for $i = 1, \dots, p-1$. To the best of our knowledge, there seems no published proof of this fact except for the case $p = 2$ (see e.g. [9, Chapitre VII, Section 4, Théorème 6]).

Modules and ideals via complexes

Given integers $p \geq 2$, $s \leq -1$ and a homogeneous ideal $\mathfrak{a} \subset R$ of height larger than or equal to p, let $\mathcal{C}(p, s, \mathfrak{a})$ denote the set of complexes $F_\bullet$ of finitely generated graded free R-modules bounded on both sides satisfying the following conditions:

(i) $H_i(F_\bullet) = 0$ for $i \geq 0$,
(ii) $F_s \neq 0$ and $F_i = 0$ for $i < s$,
(iii) $\mathfrak{a}^l H_i(F_\bullet) = 0$ for all $i < 0$ and $l \gg 0$.

Lemma 1. *Let $p \geq 2$, $s \leq -1$ be integers, $\mathfrak{a}$ a homogeneous ideal in R of height larger than or equal to p, and $F_\bullet$ an element of $\mathcal{C}(p, s, \mathfrak{a})$.*

(1) *If $s < -1$, then there exist a homogeneous ideal $\mathfrak{a}' \subset \mathfrak{a}$ of height larger than or equal to p and a complex $F'_\bullet \in \mathcal{C}(p, s+1, \mathfrak{a}')$ such that $F'_i = F_i$ for $i > p-1$.*

(2) *If $s = -1$ and $\operatorname{rank}_R(F_{-1}) > 1$, then there exist a homogeneous ideal $\mathfrak{a}' \subset \mathfrak{a}$ of height p and a complex $F'_\bullet \in \mathcal{C}(p, -1, \mathfrak{a}')$ such that $\operatorname{rank}_R(F'_{-1}) = \operatorname{rank}_R(F_{-1}) - 1$ and $F'_i = F_i$ for $i > p-1$.*

(3) *If $s = -1$ and $\operatorname{rank}_R(F_{-1}) = 1$, then there exist a homogeneous ideal $\mathfrak{a}' \subset \mathfrak{a}$ of height p and a complex $F'_\bullet \in \mathcal{C}(p, -1, \mathfrak{a}')$ such that $\operatorname{rank}_R(F'_{-1}) = 1$, $\operatorname{codim}(H_{-1}(F'_\bullet)) = p$, and $F'_i = F_i$ for $i > p-1$.*

Proof. We denote by $\varphi_i : F_i \longrightarrow F_{i-1}$ ($i \in \mathbf{Z}$) the differentials of $F_\bullet$. By the Artin-Rees lemma, there exists for each $i < 0$ an integer l_i such that $\mathfrak{a}^l F_i \cap \operatorname{Ker}(\varphi_i) = \mathfrak{a}^{l-l_i}(\mathfrak{a}^{l_i} F_i \cap \operatorname{Ker}(\varphi_i))$ for all $l \geq l_i$. Since $\mathfrak{a}^l H_i(F_\bullet) = 0$ for all $i < 0$ and $l \gg 0$ and $F_i = 0$ for $i < s$, there is therefore an integer $\bar{l}$ such that

$$(1.1) \qquad \mathfrak{a}^{\bar{l}} F_i \cap \operatorname{Ker}(\varphi_i) \subset \mathfrak{a}^{\bar{l}-l_i} \operatorname{Ker}(\varphi_i) \subset \operatorname{Im}(\varphi_{i+1})$$

for all $i < 0$. Since $\mathrm{ht}(\mathfrak{a}^{\bar{l}}) = \mathrm{ht}(\mathfrak{a}) \geq p$, we can pick p homogeneous elements $f_1, \ldots, f_p \in \mathfrak{a}^{\bar{l}}$ which form an R-regular sequence. Using these elements, let $K_\bullet := K(f_1, \ldots, f_p; R)_\bullet$ be the Koszul complex with differential $\partial_\bullet$. Note that

$$H_i(K_\bullet) = 0 \quad \text{for} \quad i \neq 0 \quad \text{and} \quad (f_1, \ldots, f_p)H_0(K_\bullet) = 0.$$

From now on, given a complex $C_\bullet$ of graded R-modules and an integer n, we denote by $C_\bullet[n]$ (resp. $C_\bullet(n)$) the complex $C'_\bullet$ (resp. $C''_\bullet$) such that $C'_i = C_{i+n}$ (resp. $C''_i = C_i(n)$).

(1) For proving the first assertion, we construct a chain map $\mu_\bullet : K_\bullet[-s] \otimes_R F_s \longrightarrow F_\bullet$ such that μ_s is an isomorphism and then consider its mapping cone. To begin with, let $\mu_s : K_0 \otimes F_s = F_s \longrightarrow F_s$ be the identity mapping and $\mu_i := 0$ for $i < s$. Let $j \geq s$ be an integer. If we have already defined homomorphisms $\mu_i : K_{i-s} \otimes F_s \longrightarrow F_i$ for all $i \leq j$ satisfying $\mu_{i-1} \circ (\partial_{i-s} \otimes \mathrm{id}_{F_s}) = \varphi_i \circ \mu_i$, then, since

$$\begin{aligned}\mathrm{Im}(\mu_j \circ (\partial_{j+1-s} \otimes \mathrm{id}_{F_s})) &\subset (f_1, \ldots, f_p)F_j \cap \mathrm{Ker}(\varphi_j)\\ &\subset \mathfrak{a}^{\bar{l}} F_j \cap \mathrm{Ker}(\varphi_j) \subset \mathrm{Im}(\varphi_{j+1})\end{aligned}$$

by (1.1) and condition (i), there exists a homomorphism $\mu_{j+1} : K_{j+1-s} \otimes F_s \longrightarrow F_{j+1}$ satisfying $\mu_j \circ (\partial_{j+1-s} \otimes \mathrm{id}_{F_s}) = \varphi_{j+1} \circ \mu_{j+1}$. Thus, we obtain a desired chain map $\mu_\bullet$ inductively. Let $\mathrm{con}\mu_\bullet$ be its mapping cone with differential $\lambda_\bullet$. Since $K_{i-s} = 0$ for $i < s,\ i \geq p-1$, we have $\mathrm{con}(\mu_\bullet)_i = F_i$ for $i > p-1$ and $\mathrm{con}(\mu_\bullet)_i = 0$ for $i < s$. Moreover, it follows from the long exact sequence arising from

$$0 \longrightarrow F_\bullet \longrightarrow \mathrm{con}\mu_\bullet \longrightarrow K_\bullet[-s-1] \otimes_R F_s \longrightarrow 0$$

that $H_i(\mathrm{con}\mu_\bullet) = 0$ for $i \geq 0$ and that $(f_1, \ldots, f_p)^l H_i(\mathrm{con}\mu_\bullet) = 0$ for all $i < 0$ and $l \gg 0$. Since $\lambda_{s+1}|_{K_0 \otimes F_s} = \mu_s$ is an isomorphism, we can cancel out free direct summands $K_0 \otimes F_s$ and $\mathrm{con}(\mu_\bullet)_s = F_s = \lambda_{s+1}(K_0 \otimes F_s)$ from $\mathrm{con}(\mu_\bullet)_{s+1}$ and $\mathrm{con}(\mu_\bullet)_s$ respectively, to obtain a new complex $F'_\bullet$ such that $F'_i = \mathrm{con}(\mu_\bullet)_i$ for $i > s+1$, $F'_{s+1} = \mathrm{con}(\mu_\bullet)_{s+1}/K_0 \otimes F_s$, $F'_i = 0$ for $i < s+1$, and $H_i(\mathrm{con}\mu_\bullet) \cong H_i(F'_\bullet)$ for all $i \in \mathbf{Z}$. Let $\mathfrak{a}' := (f_1, \ldots, f_p)$. Since $\mathrm{rank}_R(F'_{s+1}) = \mathrm{rank}_R(F_{s+1}) > 0$ by conditions (ii) and (iii), we obtain $F'_\bullet \in \mathcal{C}(p, s+1, \mathfrak{a}')$ as desired.

(2) In the second case, we construct a chain map $\mu_\bullet : K_\bullet[2](c) \longrightarrow F_\bullet$ with a suitable $c \in \mathbf{Z}$ such that the image of $\mu_{-1} : K_1(c) \longrightarrow F_{-1}$ is a free direct summand of F_{-1} of rank two, and then consider its mapping cone. To this end, let $R(-a) \oplus R(-b)$ be a free direct summand of F_{-1} with $a \geq b$, so that $F_{-1} = (R(-a) \oplus R(-b)) \oplus P$ for some graded free R-module P. By multiplying either f_1 or f_2 by a suitable homogeneous polynomial, if necessary, we may assume with no loss of generality that $\deg(f_1) - \deg(f_2) = a - b$. Let $c := \deg(f_1) - a$. Then, since $K_1 = \bigoplus_{i=1}^p R(-\deg(f_i))$, we have $K_1(c) = (R(-a) \oplus R(-b)) \oplus Q$ with $Q := \bigoplus_{i=3}^p R(c - \deg(f_i))$. Let $\mu_i := 0$ for $i < -1$, and let

$$\mu_{-1} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{pmatrix}.$$

Then, $\mathrm{Im}(\mu_{-1} \circ \partial_2) \subset \mathfrak{a}^{\bar{l}} F_{-1} \subset \mathrm{Im}(\varphi_0)$. There is therefore a homomorphism $\mu_0 : K_2(c) \longrightarrow F_0$ such that $\mu_{-1} \circ \partial_2 = \varphi_0 \circ \mu_0$. Since $H_i(F_\bullet) = 0$ for $i \geq 0$, we can construct μ_i $(i \geq 0)$ successively so that $\mu_\bullet$ becomes a chain map. Let $\mathrm{con}\mu_\bullet$ be its mapping cone with differential $\lambda_\bullet$. Then $\mathrm{con}(\mu_\bullet)_i = F_i$ for $i > p-1$ and

$\mathrm{con}(\mu_\bullet)_i = 0$ for $i < -1$ since $K_{i+2} = 0$ for $i \geq p-1,\ i < -2$. Moreover it follows from the long exact sequence arising from

$$0 \longrightarrow F_\bullet \longrightarrow \mathrm{con}\mu_\bullet \longrightarrow K_\bullet[1](c) \longrightarrow 0$$

that $H_i(\mathrm{con}\mu_\bullet) = 0$ for $i \geq 0$ and that $(f_1, \ldots, f_p)^l H_{-1}(\mathrm{con}\mu_\bullet) = 0$ for $l \gg 0$. Since μ_{-1} maps $R(-a) \oplus R(-b) \subset K_1(c)$ isomorphically onto $R(-a) \oplus R(-b) \subset F_{-1}$, we can cancel out $R(-a) \oplus R(-b)$ and $\lambda_0(R(-a) \oplus R(-b))$ from $\mathrm{con}(\mu_\bullet)_0$ and $\mathrm{con}(\mu_\bullet)_{-1}$ respectively, to obtain a complex $F'_\bullet$ such that $F'_i = \mathrm{con}(\mu_\bullet)_i$ for $i > 0$, $F'_0 = \mathrm{con}(\mu_\bullet)_0/(R(-a) \oplus R(-b))$, $F'_{-1} = \mathrm{con}(\mu_\bullet)_{-1}/\lambda_0(R(-a) \oplus R(-b)) \cong P \oplus K_0(c)$, and $H_i(\mathrm{con}\mu_\bullet) \cong H_i(F'_\bullet)$ for all $i \in \mathbf{Z}$. Let $\mathfrak{a}' := (f_1, \ldots, f_p)$. Since $\mathrm{rank}_R(F'_{-1}) = \mathrm{rank}_R(F_{-1}) + 1 - 2 = \mathrm{rank}_R(F_{-1}) - 1$, we get $F'_\bullet \in \mathcal{C}(p, -1, \mathfrak{a}')$ as desired.

(3) In the last case, we construct a chain map $\mu_\bullet : K_\bullet[2](c) \longrightarrow F_\bullet$ such that μ_{-1} is a surjection and then consider its mapping cone. Let a be an integer with $F_{-1} = R(-a)$ and let $c := \deg(f_1) - a$. Let further $\mu_i := 0$ for $i < -1$, $P := \bigoplus_{i=2}^p R(c - \deg(f_i))$, and $\mu_{-1} := (1, 0)$ a homomorphism from $K_1(c) = R(-a) \oplus P$ to $F_{-1} = R(-a)$. Then, the $\mathrm{Im}(\mu_{-1} \circ \partial_2)$ is contained in $\mathrm{Im}(\varphi_0)$ as in the preceding case. Hence we can construct μ_i $(i \geq 0)$ successively so that $\mu_\bullet$ becomes a chain map. Let $\mathrm{con}\mu_\bullet$ be its mapping cone with differential $\lambda_\bullet$. Then, again as in the preceding case, we have $\mathrm{con}(\mu_\bullet)_i = F_i$ for $i > p-1$, $\mathrm{con}(\mu_\bullet)_i = 0$ for $i < -1$, $H_i(\mathrm{con}\mu_\bullet) = 0$ for $i \geq 0$, and moreover, it follows from the exact sequence

$$0 \longrightarrow H_{-1}(F_\bullet) \longrightarrow H_{-1}(\mathrm{con}\mu_\bullet) \longrightarrow H_0(K_\bullet(c)) \longrightarrow 0$$

that $(f_1, \ldots, f_p)^l H_{-1}(\mathrm{con}\mu_\bullet) = 0$ for all $l \gg 0$ and that $\mathrm{codim}(H_{-1}(\mathrm{con}\mu_\bullet)) = p$ in $\mathrm{Spec}(R)$. Since $\mu_{-1}|_{R(-a)}$ is an isomorphism, we can cancel out $R(-a)$ and $\lambda_0(R(-a))$ from $\mathrm{con}(\mu_\bullet)_0$ and $\mathrm{con}(\mu_\bullet)_{-1}$ respectively, to obtain a complex $F'_\bullet$ such that $F'_i = \mathrm{con}(\mu_\bullet)_i$ for $i > 0$, $F'_0 = \mathrm{con}(\mu_\bullet)_0/R(-a)$, $F'_{-1} = \mathrm{con}(\mu_\bullet)_{-1}/\lambda_0(R(-a)) \cong K_0(c)$, $\mathrm{rank}_R(F'_{-1}) = -1$, $H_i(\mathrm{con}\mu_\bullet) \cong H_i(F'_\bullet)$ for all $i \in \mathbf{Z}$. Hence $F'_\bullet \in \mathcal{C}(p, -1, \mathfrak{a}')$ with $\mathfrak{a}' := (f_1, \ldots, f_p)$ as desired.

Lemma 2. *Let $p \geq 2$, $s \leq -1$, $\mathfrak{a}$, and $F_\bullet \in \mathcal{C}(p, s, \mathfrak{a})$ be as in the previous lemma. In each of the three cases there, we may assume that the differential $\varphi'_\bullet$ of $F'_\bullet$ satisfies $\mathrm{Im}({\varphi'_p}^\vee) = \mathrm{Im}({\varphi_p}^\vee)$ and $\varphi'_i = \varphi_i$ for $i > p$.*

Proof. In each case, the differential φ'_p, obtained by the method described in the proof above, is of the form

$$\begin{pmatrix} \varphi_p \\ 0 \end{pmatrix} : F_p \longrightarrow F_{p-1} \oplus L$$

with a graded free R-module L. Hence $\mathrm{Im}({\varphi'_p}^\vee) = \mathrm{Im}({\varphi_p}^\vee)$. Likewise $\varphi'_i = \varphi_i$ for all $i > p$. □

Theorem 3. *Let $p \geq 2$ be an integer and let M be a finitely generated torsion-free graded R-module with no free direct summand satisfying $\mathrm{Ext}^i_R(M, R) = 0$ for $i = 1, \ldots, p-1$. Then there exists a homogeneous ideal I in R of height p which fits into an exact sequence of the form*

$$0 \longrightarrow S_{p-1} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \oplus M \longrightarrow I(c) \longrightarrow 0,$$

where c is an integer and S_i $(0 \leq i \leq p-1)$ are finitely generated graded free R-modules.

Proof. If $M = 0$, then any Cohen-Macaulay homogeneous ideal I of height p will do. Suppose $M \neq 0$. Let

$$\cdots \xrightarrow{\varphi_{p+1}} F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0$$

be a minimal free resolution of M over R. Let further

$$\cdots \xrightarrow{\varphi_{-2}^\vee} F_{-2}^\vee \xrightarrow{\varphi_{-1}^\vee} F_{-1}^\vee \xrightarrow{\varphi_0^\vee} F_0^\vee \xrightarrow{\varphi_1^\vee} \mathrm{Im}(\varphi_1^\vee) \longrightarrow 0$$

be a minimal free resolution of $\mathrm{Im}(\varphi_1^\vee)$ over R. Connecting the former to the dual of the latter, we obtain a complex

$$F_\bullet : \cdots \xrightarrow{\varphi_{p+1}} F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F_{-1} \xrightarrow{\varphi_{-1}} F_{-2} \xrightarrow{\varphi_{-2}} \cdots$$

bounded on both sides (cf. [4]). Since $H_i(F_\bullet) = \mathrm{Ext}_R^{p-i}(\mathrm{Coker}(\varphi_p^\vee), R)$ for $i < p$, the codimension of $H_0(F_\bullet)$ in $\mathrm{Spec}(R)$ is larger than or equal to p. Besides,

$$0 \longrightarrow H_0(F_\bullet) \longrightarrow M = F_0/\,\mathrm{Im}(\varphi_1) \longrightarrow F_{-1}$$

is exact and M is torsion-free. Hence $H_0(F_\bullet) = 0$ and $\mathrm{rank}_R(F_{-1}) > 0$. On the other hand, $\mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) = 0$ $(1 \leq i \leq p-1)$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) \leq p-1$ by hypothesis, so that there is a homogeneous ideal $\mathfrak{a}$ of height larger than or equal to p such that $M_\mathfrak{p}$ is free if $\mathfrak{p} \in \mathrm{Spec}(R)$ and $\mathfrak{p} \not\supset \mathfrak{a}$. By our construction of the complex $F_\bullet$, this means that the localization $(F_\bullet)_\mathfrak{p}$ is split exact for all prime ideals $\mathfrak{p}$ not containing $\mathfrak{a}$. The support of $H_i(F_\bullet)$ is therefore contained in $\mathrm{Spec}(R/\mathfrak{a})$; in other words, $\mathfrak{a}^l H_i(F_\bullet) = 0$ for all $i < 0$ and $l \gg 0$. Thus $F_\bullet \in \mathcal{C}(p, s, \mathfrak{a})$ for some $s \leq -1$. Applying Lemmas 1 and 2 to $F_\bullet$, we can obtain a complex $G_\bullet \in \mathcal{C}(p, -1, \mathfrak{b})$ such that $\mathrm{rank}_R(G_{-1}) = 1$, $\mathrm{codim}(H_{-1}(G_{-1})) = p$, $\mathrm{Im}(\psi_p^\vee) = \mathrm{Im}(\varphi_p^\vee)$, and $G_i = F_i$, $\psi_{i+1} = \varphi_{i+1}$ for $i > p-1$, where $\mathfrak{b}$ is a homogeneous ideal in R of height larger than or equal to p and $\psi_\bullet$ is the differential of $G_\bullet$. In fact, if $s < -1$, then apply (1) repeatedly to reduce s to -1. If $s = -1$ but $\mathrm{rank}_R(F_{-1}) > 1$, then apply (2) repeatedly to reduce $\mathrm{rank}_R(F_{-1})$ to one. If $s = -1$ and $\mathrm{rank}_R(F_{-1}) = 1$ but $\mathrm{codim}(H_{-1}(F_\bullet)) > p$, then apply (3) to reduce $\mathrm{codim}(H_{-1}(F_\bullet))$ to p. For such a $G_\bullet$, let c be the integer such that $G_{-1} = R(c)$ and $I \subset R$ the ideal of height p such that $\mathrm{Im}(\psi_0) = I(c)$. Now, reverse the procedure. The complex

$$\cdots \xrightarrow{\psi_{p+1}} G_p \xrightarrow{\psi_p} G_{p-1} \xrightarrow{\psi_{p-1}} \cdots \xrightarrow{\psi_1} G_0 \xrightarrow{\psi_0} G_{-1} \longrightarrow \mathrm{Coker}(\psi_0) \longrightarrow 0$$

is a free resolution of $\mathrm{Coker}(\psi_0) = R/I(c)$, and

$$\cdots \xrightarrow{\varphi_{-1}^\vee} F_{-1}^\vee \xrightarrow{\varphi_0^\vee} F_0^\vee \xrightarrow{\varphi_1^\vee} \cdots \xrightarrow{\varphi_{p-1}^\vee} F_{p-1}^\vee \xrightarrow{\varphi_p^\vee} \mathrm{Im}(\varphi_p^\vee) = \mathrm{Im}(\psi_p^\vee) \longrightarrow 0$$

is a free resolution of $\mathrm{Im}(\psi_p^\vee)$. Since $F_i = G_i$ for all $i > p-1$, our assertion follows from [6, Lemma 1.3]. □

Remark 4. One can give another proof of [10, Theorem 1.3] by an argument similar to that in the proof of (2) of Lemma 1.

References

[1] M. Amasaki, *On the structure of arithmetically Buchsbaum curves in* $\mathbf{P}^3_k$, Publ. RIMS, Kyoto Univ. **20** (1984), 793 – 837. MR **86a:**14027

[2] M. Amasaki, *Integral arithmetically Buchsbaum curves in* $\mathbf{P}^3$, J. Math. Soc. Japan **41**, No. 1 (1989), 1 – 8. MR **90c:**14016

[3] M. Amasaki, *Application of the generalized Weierstrass preparation theorem to the study of homogeneous ideals*, Trans. AMS **317** (1990), 1 – 43. MR **90d:**13002

[4] M. Amasaki, *Free complexes defining maximal quasi-Buchsbaum graded modules over polynomial rings*, J. Math. Kyoto Univ. **33**, No. 1 (1993), 143 – 170. MR **94a:**13020
[5] M. Amasaki, *Generators of graded modules associated with linear filter-regular sequences*, J. Pure Appl. Algebra **114** (1996), 1 – 23. CMP 97:05
[6] M. Amasaki, *Basic sequences of homogeneous ideals in polynomial rings*, J. Algebra **190** (1997), 329 – 360. MR **98c:**13029
[7] M. Amasaki, *Basic sequence and Nollet's* θ_X *of a homogeneous ideal of height two*, preprint (August, 1996).
[8] E. Ballico, G. Bolondi and J. C. Migliore, *The Lazarsfeld-Rao problem for liaison classes of two-codimensional subschemes of* $\mathbf{P}^n$, Amer. J. Math. **113** (1991), 117–128. MR **92c:**14047
[9] N. Bourbaki, "Algèbre Commutative", Masson, Paris, 1985.
[10] A. V. Geramita and J. C. Migliore, *A generalized liaison addition*, J. Algebra **163** (1994), 139 – 164. MR **94m:**14066
[11] M. Martin-Deschamps et D. Perrin, *Sur la classification des courbes gauches*, Astérisque 184 – 185, Société Mathématique de France, 1990. MR **91h:**14039
[12] S. Nollet, *Even linkage classes*, Trans. AMS **348** (1996), 1137 – 1162. MR **96h:**14069
[13] S. Nollet, *Integral subschemes of codimension two*, preprint.
[14] A. P. Rao, *Liaison equivalence classes*, Math. Ann. **258** (1981), 169 – 173. MR **83j:**14045

Faculty of School Education, Hiroshima University, 1-1-1 Kagamiyama, Higashi-Hiroshima 739-8524, Japan

E-mail address: amasaki@ipc.hiroshima-u.ac.jp