PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 12, Pages 3707–3709
S 0002-9939(99)04914-X
Article electronically published on May 13, 1999

NEW Σ^1_3 FACTS

SY D. FRIEDMAN

(Communicated by Carl G. Jockusch, Jr.)

ABSTRACT. We use "iterated square sequences" to show that there is an L-definable partition $n : L\text{-}Singulars \to \omega$ such that if M is an inner model not containing $0^{\#}$:
(a) For some $k, M \models \{\alpha | n(\alpha) \leq k\}$ is stationary.
(b) For each k there is a generic extension of M in which $0^{\#}$ does not exist and $\{\alpha | n(\alpha) \leq k\}$ is non-stationary.
This result is then applied to show that if M is an inner model without $0^{\#}$, then some Σ^1_3 sentence not true in M can be forced over M.

Assume that $0^{\#}$ exists and that M is an inner model of ZFC, $0^{\#} \notin M$. Then of course M is not lightface Σ^1_3-correct: the true Σ^1_3 sentence "$0^{\#}$ exists" is false in M. In this article we use a result about L-definable partitions (which may be of independent interest) to show that in fact M fails to satisfy some Σ^1_3 sentence true in a forcing extension of M. We work in Morse-Kelley class theory (though Gödel-Bernays with a satisfaction predicate for V will suffice).

Theorem 1. *Assume that $0^{\#}$ exists. There exists an ω-sequence of true Σ^1_3 sentences $\langle \varphi_n \mid n \in \omega \rangle$ such that if M is an inner model, $0^{\#} \notin M$:*

(a) *φ_n is false in M for some n.*
(b) *For each n, some generic extension of M satisfies φ_n.*

Moreover if $M = L[R]$, R a real, then these generic extensions can be taken to be inner models of $L[R, 0^{\#}]$.

The above result is based on the next result, concerning L-definable partitions.

Theorem 2. *There exists an L-definable function $n : L\text{-}Singulars \to \omega$ such that if M is an inner model, $0^{\#} \notin M$:*

(a) *For some k, $M \models \{\alpha \mid n(\alpha) \leq k\}$ is stationary.*
(b) *For each k there is a generic extension of M in which $0^{\#}$ does not exist and $\{\alpha \mid n(\alpha) \leq k\}$ is non-stationary.*

Remark. "Stationary in M" means: intersects every M-definable (with parameters) closed unbounded class of ordinals.

Proof. We define $n(\alpha)$. Let $\langle C_\alpha \mid \alpha\ L\text{-singular}\rangle$ be an L-definable $\square$-sequence: C_α is closed unbounded in α, ordertype $C_\alpha < \alpha$ and $\bar{\alpha} \in \lim C_\alpha \to C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$. Let

Received by the editors November 25, 1997 and, in revised form, February 13, 1998.
1991 *Mathematics Subject Classification.* Primary 03E45, 03E55, 03E15, 03D60.
Key words and phrases. Class forcing, absoluteness, partitions.
The author's research was supported by NSF Contract #9625997-DMS.

$\operatorname{ot} C_\alpha$ denote the ordertype of C_α. If $\operatorname{ot} C_\alpha$ is L-regular, then $n(\alpha) = 0$. Otherwise $n(\alpha) = n(\operatorname{ot} C_\alpha) + 1$.

(a) is clear, as otherwise (using the fact that we are working in Morse- Kelley class theory) there is a closed unbounded $C \subseteq L$-regulars amenable to M, contradicting the Covering Theorem and the hypothesis that $0^\#$ does not belong to M.

Now we prove (b). Fix $n \in \omega$. In M let P consist of closed, bounded $p \subseteq \mathrm{ORD}$ such that $\alpha \in p \to \alpha$ L-regular or $n(\alpha) \geq n+1$, ordered by $p \leq q$ iff p end extends q.

We claim that P is ∞-distributive in M. Suppose that $p \in P$ and $\langle D_\alpha \mid \alpha < \kappa \rangle$ is a definable sequence of open dense subclasses of P, κ regular. We wish to find $q \leq p$, $q \in D_\alpha$ for all $\alpha < \kappa$. Let C be the class of all strong limit cardinals β such that $D_\alpha \cap V_\beta$ is dense in $P \cap V_\beta$ for all $\alpha < \kappa$, a closed unbounded class of ordinals. It suffices to show that $C \cap \{\beta \mid n(\beta) \geq n+1\}$ has a closed subset of ordertype $\kappa + 1$, for then p can be successively extended κ times meeting the D_α's, to conditions with maximum in $\{\beta \mid n(\beta) \geq n+1\}$; the final condition (at stage κ) extends p and meets each D_α.

Lemma 3. *Suppose $m \geq k$, α is regular and C is a closed set of ordertype $\alpha^{+m}+1$, consisting of ordinals greater than α^{+m} (where $\alpha^{+0} = \alpha$, $\alpha^{+(p+1)} = (\alpha^{+p})^+$). Then $C \cap \{\beta \mid n(\beta) \geq k\}$ has a closed subset of ordertype $\alpha^{+(m-k)} + 1$.*

Proof of Lemma 3. By induction on k. Suppose $k = 0$. Let $\beta = \max C$. Then β is singular and hence singular in L. So C_β is defined and $\lim(C_\beta \cap C)$ is a closed set of ordertype $\alpha^{+m} + 1$ consisting of L-singulars. So $\lim(C_\beta \cap C) \subseteq C \cap \{\gamma \mid n(\gamma) \geq 0\}$ satisfies the lemma.

Suppose the lemma holds for k and let $m+1 \geq k+1$, C a closed set of ordertype $\alpha^{+(m+1)} + 1$ consisting of ordinals greater than $\alpha^{+(m+1)}$. Let $\beta = \max C$. Then C_β is defined and $D = \lim(C_\beta \cap C)$ is a closed set of ordertype $\alpha^{+(m+1)} + 1$. Let $\bar\beta = (\alpha^{+m} + \alpha^{+m} + 1)$st element of D. Then

$$\bar D = \{\operatorname{ot} C_\gamma \mid \gamma \in D, (\alpha^{+m} + 1) \text{ st element of } D \leq \gamma \leq \bar\beta\}$$

is a closed set of ordertype $\alpha^{+m} + 1$ consisting of ordinals greater than α^{+m}. By induction there is a closed $\bar D_0 \subseteq \bar D \cap \{\gamma \mid n(\gamma) \geq k\}$ of ordertype $\alpha^{+(m-k)} + 1$. But then $D_0 = \{\gamma \in D \mid \operatorname{ot} C_\gamma \in \bar D_0\}$ is a closed subset of $C \cap \{\gamma \mid n(\gamma) \geq k+1\}$ of ordertype $\alpha^{+(m-k)} + 1$. As $\alpha^{+(m-k)} = \alpha^{+((m+1)-(k+1))}$ we are done. □

By the lemma, $C \cap \{\beta \mid n(\beta) \geq n\}$ has arbitrary long closed subsets for any n, for any closed unbounded $C \subseteq \mathrm{ORD}$. It follows that P is ∞-distributive. Now to prove (b), we apply the forcing P to M, producing C witnessing the nonstationarity of $\{\alpha \mid n(\alpha) \leq n\}$, and then follow this with the forcing to code $\langle M, C\rangle$ by a real, making C definable. Of course this will not produce $0^\#$ as every successor to a strong limit cardinal is preserved in the coding. □

We also note that in Theorem 2 the generic extension can be formed in $L[R, 0^\#]$ in the case $M = L[R]$, R a real, using the fact that in $L[R, 0^\#]$, generics can be constructed for P (an "Amenable" forcing) and for Jensen coding (see [99, Friedman]).

Proof of Theorem 1. We use David's trick (see [98, Friedman]). Let φ_n be the sentence: $\exists R \forall \alpha$(If $L_\alpha[R] \models ZF^-$, then $L_\alpha[R] \models \beta$ a limit cardinal $\to \beta$ L-regular or $n(\beta) \geq n$). (This is equivalent to a Σ^1_3 sentence as it is of the form $\exists R \psi(R)$ where $\psi(R)$ is Π_1 in the sense of Lévy and hence equivalent to a Π^1_2 sentence.)

By Theorem 2(b) and cardinal collapsing (to guarantee that limit cardinals β are either L-regular or satisfy $n(\beta) \geq n$), M has a generic extension $L[R] \models \beta$ a limit cardinal $\rightarrow \beta\, L$-regular or $n(\beta) \geq n$ (inside $L[S, 0^\#]$ if $M = L[S]$, S a real). By David's trick we can in fact obtain φ_n in $L[R]$. $\square$

Question. Can the generic extensions in Theorem 1(b) be taken to have the same cofinalities as M, in case M satisfies GCH?

References

[82] René David, A Very Absolute Π^1_2-Singleton, Annals of Pure and Applied Logic **23**, pp. 101–120. MR **84m:**03057

[98] Sy D. Friedman, David's Trick, to appear, Proceedings of the European Summer Meeting of the ASL, Leeds, England, 1998.

[99] Sy D. Friedman, *Fine Structure and Class Forcing*, book, rough draft.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

E-mail address: sdf@math.mit.edu