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LACUNARY SETS BASED ON LORENTZ SPACES

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ABSTRACT. A new lacunary set for compact abelian groups is introduced; this is called a $\Lambda(p,q)$ set. This set is defined in terms of the Lorentz spaces and is shown to be a generalization of $\Lambda(p)$ sets and Sidon sets. A number of functional-analytic statements about $\Lambda(p,q)$ sets are established by making use of the structural similarities between L^p spaces and Lorentz spaces. These statements are analogous to several well-known properties of a set which are equivalent to the definition of a $\Lambda(p)$ set. Some general set-theoretic and arithmetic properties of $\Lambda(p,q)$ sets are also developed; these properties extend known results on the structure of $\Lambda(p)$ sets. Open problems and directions for further research are outlined.

1. INTRODUCTION

Throughout this paper G denotes an infinite compact abelian group and Γ its discrete dual group. If $X \subseteq L^1$ and $E \subseteq \Gamma$, let $X_E = \{f \in X : \hat{f}(\gamma) = 0$ for all $\gamma \notin E\}$. For $p \in (1,\infty)$ recall that a subset E of Γ is called a $\Lambda(p)$ set if $L^p_E = L^1_E$. The set E is called a Sidon set if $L^\infty_E \subseteq \{f \in L^1 : \hat{f} \in \ell^1(\Gamma)\}$. Sidon sets and $\Lambda(p)$ sets are the most widely studied types of lacunary sets for compact abelian groups; two standard references on the theory of these sets are [10] and [12]. In this paper we introduce and study a new type of lacunary set which is defined in terms of the Lorentz spaces, $L(p,q)$.

Definition 1.1. Let $p \in (1,\infty)$ and $q \in [1,\infty)$. A subset E of Γ is called a $\Lambda(p,q)$ set if $L(p,q)_E = L^1_E$.

What motivates the notation for $\Lambda(p,q)$ sets is the notation for Lorentz spaces. These spaces are a two-parameter family of function spaces which are closely related to the L^p spaces. In particular, they are intermediate to the L^p spaces in the sense that whenever $1 \le q < p < r \le \infty$,

$$L^\infty \subset \bigcup_{t>p} L^t \subseteq L(p,q) \subset L^p \subset L(p,r) \subseteq \bigcap_{s<p} L^s \subset L^1. \tag{1}$$

Furthermore, each L^p space is itself a Lorentz space as $L^p = L(p,p)$. It follows from this that every $\Lambda(p)$ set is also a $\Lambda(p,p)$ set. In [6, Section 3] two types of lacunary sets based on Lorentz spaces are introduced; these are called $\Lambda_1(p,q)$ and $\Lambda_2(p,q)$ sets. We shall see that these sets are also examples of $\Lambda(p,q)$ sets.

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There are a number of well-known functional-analytic properties of a set which are equivalent to the definition of a $\Lambda(p)$ set (see [10, 37.9] and [12, 5.3]). Since the Lorentz spaces are generalizations of the L^p spaces, it is not surprising that similar characterizations of $\Lambda(p,q)$ sets may be obtained by straightforward modifications of known results for $\Lambda(p)$ sets. An easy, yet important, theorem states that if $p > 2$, then the class of $\Lambda(p)$ sets is closed under the formation of finite unions. We will prove a similar result for $\Lambda(p,q)$ sets and discuss some related questions concerning unions. In [1], [3], [7], and [8] one finds a number of results on arithmetic and general set-theoretic properties of $\Lambda(p)$ sets. These properties deal with the structural nature of $\Lambda(p)$ sets. We will establish analogous properties for $\Lambda(p,q)$ sets.

2. Preliminaries

Let λ denote the normalized Haar measure on G and let $\|\cdot\|_p$ denote the usual p-norm where $p \in [1,\infty]$. Let T denote the set of trigonometric polynomials on G and M denote the set of complex bounded Borel measures on G. For the reader's convenience, we shall give the definition and state some basic properties of Lorentz spaces; further details on these spaces are found in [6], [11], and [13].

Let f be a complex-valued measurable function on G which is finite almost everywhere. The distribution function λ_f of f is defined by

$$\lambda_f(y) = \lambda\{x \in G : |f(x)| > y\} \quad \text{for } y \geq 0.$$

The non-increasing rearrangement of f is the function

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\} \quad \text{for } t \geq 0.$$

The Lorentz space $L(p,q)$ is defined as the set of equivalence classes of functions f such that $\|f\|^*_{p,q} < \infty$, where

$$\|f\|^*_{p,q} = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^1 \left[t^{1/p} f^*(t)\right]^q \frac{dt}{t}\right)^{1/q} & \text{if } 1 \leq p,q < \infty, \\ \displaystyle\sup_{t\in(0,\infty)} t^{1/p} f^*(t) & \text{if } 1 \leq p \leq \infty,\ q = \infty. \end{cases}$$

Since $\lambda_{f^*} = \lambda_f$, it follows that $\|f\|^*_{p,p} = \|f\|_p$ and hence $L(p,p) = L^p$ for all $p \in [1,\infty]$. The function $f \mapsto \|f\|^*_{p,q}$ is a quasi-norm for $L(p,q)$, but is not generally a norm. However, $L(p,q)$ does have a norm which is related to $\|\cdot\|^*_{p,q}$. To define this norm, consider a function f and its averaging function f^{**} where

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds \quad \text{for } t > 0.$$

Then $L(p,q)$ can be taken as the set of equivalence classes of functions f such that $\|f\|_{(p,q)} < \infty$, where

$$\|f\|_{(p,q)} = \begin{cases} \left(\displaystyle\int_0^\infty \left[t^{1/p} f^{**}(t)\right]^q \frac{dt}{t}\right)^{1/q} & \text{if } 1 \leq p,q < \infty, \\ \displaystyle\sup_{t\in(0,\infty)} t^{1/p} f^{**}(t) & \text{if } 1 \leq p \leq \infty,\ q = \infty. \end{cases}$$

If $p = q \in \{1, \infty\}$ or if $p \in (1, \infty)$ and $q \in [1, \infty)$, then $L(p, q)$ is a Banach space with norm $\|\cdot\|_{(p,q)}$. The quasi-norm and norm are related by the inequality

$$\left(\frac{p}{q}\right)^{1/q} \|f\|^*_{p,q} \leq \|f\|_{(p,q)} \leq p' \left(\frac{p}{q}\right)^{1/q} \|f\|^*_{p,q} \tag{2}$$

where $p \in (1, \infty)$ and $q \in [1, \infty]$. Note that p' is the index conjugate to p and $\left(\frac{p}{q}\right)^{1/q} = 1$ if $q = \infty$. The most useful inequalities for the quasi-norm and norm are as follows. For $p \in [1, \infty]$ and $1 \leq q_1 < q_2 \leq \infty$,

$$\|f\|^*_{p,q_2} \leq \|f\|^*_{p,q_1} \tag{3}$$

and, for $p \in (1, \infty)$,

$$\|f\|_{(p,q_2)} \leq \left(\frac{q_1}{p}\right)^{(q_1^{-1} - q_2^{-1})} \|f\|_{(p,q_1)}. \tag{4}$$

If $1 < p_1 < p_2 < \infty$ and $q \in [1, \infty)$, then

$$\|f\|^*_{p_1,q} \leq \left(\frac{p_2}{p_2 - p_1}\right)^{1/q} \|f\|^*_{p_2,\infty}. \tag{5}$$

A consequence of this last inequality is the proper inclusion

$$L(p_2, q_2) \subset L(p_1, q_1) \tag{6}$$

whenever $1 < p_1 < p_2 < \infty$ and $q_1, q_2 \in [1, \infty]$. As in [6, p. 368] define a total ordering on the set $J = (1, \infty) \times [1, \infty)$ by $(r, s) > (p, q)$ if and only if either $r > p$ or $r = p$ and $s < q$. Using this ordering and the inclusions in (1) and (6), it follows that

$$L(r, s) \subset L(p, q) \text{ if and only if } (r, s) > (p, q). \tag{7}$$

3. Basic properties of $\Lambda(p, q)$ sets

If E is a finite subset of Γ, then E is a $\Lambda(p, q)$ set for all $(p, q) \in J$ since $L(p, q)_E = L^1_E = T_E$. In contrast, the group Γ itself is not a $\Lambda(p, q)$ set for any $(p, q) \in J$ since (1) shows that each such space $L(p, q)$ is always a proper subset of L^1. Our first results give simple relationships between $\Lambda(p, q)$ sets, $\Lambda(p)$ sets, and Sidon sets.

Theorem 3.1. *Let $(p, q), (r, s) \in J$ and let $E \subset \Gamma$.*
(a) *If E is a $\Lambda(p, q)$ set, then E is a $\Lambda(r, s)$ set whenever $(p, q) > (r, s)$.*
(b) *If E is a $\Lambda(p)$ set, then E is a $\Lambda(r, s)$ set whenever $(p, p) > (r, s)$.*
(c) *If E is a $\Lambda(p, q)$ set, then E is a $\Lambda(r)$ set whenever $(p, q) > (r, r)$.*

Proof. This is evident from (7) and the definition of $\Lambda(p)$ sets and $\Lambda(p, q)$ sets. □

Theorem 3.2. *Each Sidon set is a $\Lambda(p, q)$ set for all $(p, q) \in J$.*

Proof. If E is a Sidon set, then [10, 37.10] shows that E is a $\Lambda(r)$ set for each $r \in (1, \infty)$. Given $(p, q) \in J$, if $r > p$, then $(r, r) > (p, q)$ and thus E is a $\Lambda(p, q)$ set by Theorem 3.1(b). □

Corollary 3.3. *Each infinite subset of Γ contains an infinite set which is a $\Lambda(p, q)$ set for every $(p, q) \in J$.*

Proof. By [10, 37.18] each infinite subset of Γ contains an infinite Sidon set; hence the result follows from Theorem 3.2. □

Theorem 3.2 and Corollary 3.3 are generalizations of classical results about Sidon sets and $\Lambda(p)$ sets. The following result is also such a generalization.

Theorem 3.4. *There exists a subset of Γ which is a $\Lambda(p,q)$ set for all $(p,q) \in J$ but is not a Sidon set.*

Proof. By [12, 5.14] Γ contains an infinite subset E which is a $\Lambda(r)$ set for all $r \in (1,\infty)$ but is not a Sidon set. If $(p,q) \in J$, then $(r,r) > (p,q)$ whenever $r > p$, and thus E is a $\Lambda(p,q)$ set by Theorem 3.1(b). □

In [6, Section 3] the following pair of lacunary sets based on Lorentz spaces are introduced.

Definition 3.5. Let $E \subset \Gamma$ and let $(p,q) \in J$.

(a) The set E is a $\Lambda_1(p,q)$ set if there exists $s \in (1,p)$ such that $L(p,q)_E = L(s,q)_E$.

(b) The set E is a $\Lambda_2(p,q)$ set if there exists $r \in (q,\infty)$ such that $L(p,q)_E = L(p,r)_E$.

As shown for $\Lambda(p,q)$ sets, each finite subset of Γ is a $\Lambda_1(p,q)$ set and a $\Lambda_2(p,q)$ set for every $(p,q) \in J$. As well, Γ itself is not a $\Lambda_1(p,q)$ set or a $\Lambda_2(p,q)$ set for any $(p,q) \in J$. The $\Lambda_2(p,q)$ sets are used in [6] to establish a characterization theorem for Lorentz-improving measures [6, 3.4]. Some inclusions between $\Lambda_2(p,q)$ sets, $\Lambda_1(p,q)$ sets, and $\Lambda(p,q)$ sets are given in the following result.

Theorem 3.6. *Let $E \subset \Gamma$ and let $(p,q) \in J$.*

(a) *If E is a $\Lambda_1(p,q)$ set, then E is a $\Lambda(r,s)$ set whenever $(r,s) < (u,u)$ and $1 < u < p$.*

(b) *If E is a $\Lambda(p,q)$ set, then E is a $\Lambda_1(r,s)$ set whenever $(p,q) \geq (r,s)$.*

(c) *If E is a $\Lambda(p,q)$ set, then E is a $\Lambda_2(r,s)$ set whenever $(p,q) \geq (r,s)$.*

Proof. (a) If E is a $\Lambda_1(p,q)$ set, then [6, 3.1(b)] shows that E is a $\Lambda(u)$ set for every $u \in (1,p)$. By Theorem 3.1(b) E is also a $\Lambda(r,s)$ set whenever $(r,s) < (u,u)$.

(b) Assume E is a $\Lambda(p,q)$ set and let $(p,q) \geq (r,s)$. Then for each $w \in (1,r)$,

$$L^1_E \subseteq L(p,q)_E \subseteq L(r,s)_E \subseteq L(w,s)_E \subseteq L^1_E$$

and thus E is a $\Lambda_1(r,s)$ set.

(c) If E is a $\Lambda(p,q)$ set and $(p,q) \geq (r,s)$, then

$$L^1_E \subseteq L(p,q)_E \subseteq L(r,s)_E \subseteq L(r,w)_E \subseteq L^1_E$$

for each $w \in (s,\infty)$. This shows E is a $\Lambda_2(r,s)$ set. □

4. Equivalent statements for $\Lambda(p,q)$ sets

There are a number of functional-analytic properties of a subset E of Γ which are equivalent to the definition of a $\Lambda(p,q)$ set. These properties are analogous to some well-known and useful statements about a set which are equivalent to the definition of a $\Lambda(p)$ set (see [10, 37.7 and 37.9] and [12, 5.3]). The first of two results gives a characterization of $\Lambda(p,q)$ sets in terms of norms and quasi-norms, and is almost a full generalization of [10, 37.7].

Theorem 4.1. *Let $E \subset \Gamma$, let $(p,q),(r,s) \in J$, and assume $r > p$. The following assertions are equivalent:*

(i) *E is a $\Lambda(r,s)$ set;*

(ii) *$L(r,s)_E = L(p,q)_E$;*

(iii) *there exists a constant k such that $\|f\|^*_{r,s} \leq k\|f\|^*_{p,q}$ for every $f \in T_E$;*

(iv) *there exists a constant k such that $\|f\|^*_{r,s} \leq k\|f\|_1$ for every $f \in T_E$;*

(v) *$M_E = L(r,s)_E$.*

Proof. We will just outline the proof of each implication; the reader who is interested in the details of the original proof for $\Lambda(p)$ sets can refer to [10, 37.7].

(i $\Rightarrow$ ii) If E is a $\Lambda(r,s)$ set where $r > p$, then $(r,s) > (p,q)$ and hence $L(r,s)_E = L^1_E$ and $L(r,s)_E \subseteq L(p,q)_E$ by (7). Since $L(p,q)_E \subseteq L^1_E$, assertion (ii) follows trivially.

(ii $\Rightarrow$ iii) If $L(r,s)_E = L(p,q)_E$ where $r > p$, then $(r,s) > (p,q)$ and thus the mapping $f \mapsto f$ of $L(r,s)_E$ into $L(p,q)_E$ is surjective and continuous. Since $L(r,s)_E$ and $L(p,q)_E$ are Banach spaces [5, 14.2], it follows from the open mapping theorem that there exists a constant k such that

$$\|f\|_{(r,s)} \leq k\ \|f\|_{(p,q)} \ \text{ for all } \ f \in L(p,q)_E.$$

Assertion (iii) is now clear as a result of this inequality, (2), and the fact that $T_E \subseteq L(p,q)_E$.

(iii $\Rightarrow$ iv) Since $L(r,s)_E \subseteq L(p,q)_E$ and T_E is dense in both of these spaces, it follows from (iii) that $L(r,s)_E = L(p,q)_E$. Let p_1 and p_2 satisfy $p < p_1 < p_2 < r$. Then $L(r,s)_E \subseteq L^{p_2}_E \subseteq L^{p_1}_E \subseteq L(p,q)_E$ and hence $L^{p_2}_E = L^{p_1}_E$. By [10, 37.7] $L^{p_2}_E = L^1_E$ and thus $L(r,s)_E = L(p,q)_E = L^1_E$. By the open mapping theorem there exists a constant k such that $\|f\|_{(r,s)} \leq k\ \|f\|_1$ for all $f \in L^1_E$. Combining this inequality with those in (2) and assertion (iii) yields (iv).

(iv $\Rightarrow$ v) From (1) and the Radon-Nikodym theorem, $L(r,s)_E$ may be regarded as a subset of M_E. We will show that $M_E \subseteq L(r,s)_E$. Let $\mu \in M_E$ and let $h \in T$. Then $\mu * h \in T_E$ and from (iv),

$$\|\mu * h\|^*_{r,s} \leq k\ \|\mu * h\|_1 \leq k\ \|\mu\|\ \|h\|_1.$$

It follows from this inequality and (2) that

$$\sup\{\|\mu * h\|_{(r,s)} : h \in T, \|h\|_1 \leq 1\} < \infty.$$

A straightforward modification of [10, 35.11] shows there exists a function $g \in L(r,s)$ such that $d\mu = g\ d\lambda$. Since $\hat{\mu}$ vanishes off of E, $\hat{g}$ does also and thus $g \in L(r,s)_E$. This proves $\mu \in L(r,s)_E$ which establishes (v).

(v $\Rightarrow$ i) If $L(r,s)_E = M_E$, then $L^1_E \subseteq M_E = L(r,s)_E \subseteq L^1_E$ and hence $L(r,s)_E = L^1_E$ which gives (i). □

The proof of Theorem 4.1 shows that assertions (i), (iv), and (v) there are equivalent under the weaker hypothesis $(r,s) > (p,q)$. However, the full equivalence of assertions (i)–(v) has not been established for pairs $(r,s),(p,q)$ where $p = r$ and $q < s$. It is to this extent that Theorem 4.1 is not a complete generalization of [10, 37.7].

The next result gives additional properties of a set which are equivalent to the definition of a $\Lambda(p,q)$ set. These properties complement those in Theorem 4.1 in that they characterize $\Lambda(p,q)$ sets in terms of the dual of a Lorentz space. This result is almost a full generalization of the equivalences for $\Lambda(p)$ sets as found in [10, 37.9].

Theorem 4.2. *Let $E \subset \Gamma$ and let $(p,q) \in J$. The following assertions are equivalent:*

(i) *E is a $\Lambda(p,q)$ set;*

(ii) *for each $g \in L(p',q')$ there exists an $h \in L^\infty$ such that $\hat{g}(\gamma) = \hat{h}(\gamma)$ for all $\gamma \in E$;*

(iii) *for each $g \in L(p',q')$ there exists a continuous function h such that $\hat{g}(\gamma) = \hat{h}(\gamma)$ for all $\gamma \in E$.*

If $p > 2$, then statements (i)–(iii) *above are equivalent to the assertion:*

(iv) *for each $g \in L(p',q')$, $\sum_{\gamma \in E} |\hat{g}(\gamma)|^2 < \infty$.*

Proof. Again we will just outline the proof of each implication and refer the interested reader to [10, 37.9] for the original details concerning $\Lambda(p)$ sets.

(i $\Rightarrow$ ii) This follows very closely the proof that (i) implies (ii) in [10, 37.9] except that Hölder's inequality for Lorentz spaces [13, 3.5] is used in place of the standard Hölder inequality.

(ii $\Rightarrow$ iii) Assume (ii) and let $g \in L(p',q')$. A factorization theorem for Lorentz spaces ([5, 14.3] and [10, 32.33(d)]) shows there exist an $f \in L^1$ and a $g_1 \in L(p',q')$ such that $g = f * g_1$. Now (ii) implies there is an $h_1 \in L^\infty$ such that $\hat{h_1}(\gamma) = \hat{g_1}(\gamma)$ for all $\gamma \in E$. Let $h = f * h_1$ and note from [9, 20.16] that h is continuous. One easily checks that $\hat{g}(\gamma) = \hat{h}(\gamma)$ for all $\gamma \in E$ and this establishes (iii).

(iii $\Rightarrow$ i) The proof of this implication follows very closely the proof that (iii $\Rightarrow$ i) in [10, 37.9]. For convenience and completeness we outline the argument here for Lorentz spaces. Using [10, 35.7(c)] it is straightforward to verify that $L_{E^c}(p',q')$ is a closed two-sided ideal in $L(p',q')$ where E^c denotes the complement of E in Γ. As well, one checks that for a function $g \in L(p',q')$, $g + L_{E^c}(p',q') = h + L_{E^c}(p',q')$ precisely when $\hat{g}(\gamma) = \hat{h}(\gamma)$ for all $\gamma \in E$. There exists a constant k such that, for each $g \in L(p',q')$, there is some continuous function h which satisfies $h + L_{E^c}(p',q') = g + L_{E^c}(p',q')$ and $\|h\|_u \le k\ \|g\|_{(p',q')}$. Here $\|\cdot\|_u$ denotes the uniform norm. Now let $f \in T_E$, let $g \in L(p',q')$, and consider the function h above. Since $\hat{h}(\gamma) = \hat{g}(\gamma)$ for all $\gamma \in E$, it follows from [10, 34.33(ii)] that

$$\int_G \bar{f}(s)(g(s) - h(s))d\lambda(s) = 0.$$

We see that

$$\left| \int_G \bar{f}(s)g(s)d\lambda(s) \right| \le \|h\|_u\ \|f\|_1 \le k\ \|g\|_{(p',q')}\ \|f\|_1.$$

From this inequality and the duality result in [2, (2.5), p. 9], it follows that $\|f\|_{(p,q)} \le k\|f\|_1$. Assertion (i) is now evident from (2) and Theorem 4.1.

For the final implication assume that $p > 2$.

(iii $\Rightarrow$ iv) If $g \in L(p',q')$, consider a continuous function h such that $\hat{g}(\gamma) = \hat{h}(\gamma)$ for all $\gamma \in E$. Then $\sum_{\gamma \in E} |\hat{g}(\gamma)|^2 = \sum_{\gamma \in E} |\hat{h}(\gamma)|^2 \le \|h\|_2^2 < \infty$ which yields (iv).

(iv $\Rightarrow$ i) Following [10, 37.9] suppose that $\sum_{\gamma \in E} |\hat{g}(\gamma)|^2 < \infty$ for each $g \in L(p',q')$. Since $p > 2$, (1) shows that $L^2 \subset L(p',q')$ and thus the mapping $f \mapsto f$ from L^2 into $L(p',q')$ is continuous. By [10, 28.43] there exists an $h \in L^2$ such that $\hat{h}(\gamma) = \hat{g}(\gamma)$ for all $\gamma \in E$ and $\hat{h}$ vanishes off of E. It follows from the proof of (iii $\Rightarrow$ i) and (2) that there exists a constant k such that $\|f\|_{p,q}^* \le k\|f\|_2$ for all $f \in T_E$. Theorem 4.1 now shows that E is a $\Lambda(p,q)$ set which establishes (i). ∎

Theorem 4.2 is not a complete generalization of [10, 37.9] as the equivalence of assertions (i)–(iv) has not been established for (p, q) where $p = 2$ and $q < 2$.

5. Set-theoretic and arithmetic properties

An important problem in the theory of lacunary sets is the so-called "union problem". The problem is to determine whether a particular type of lacunary set is closed under finite unions. The following instances of this problem are well-known. If $p > 2$ and if E_1 and E_2 are $\Lambda(p)$ sets, then $E_1 \cup E_2$ is also a $\Lambda(p)$ set [10, 37.21]. If $p \in (1, \infty)$, then $E_1 \cup E_2$ is a $\Lambda(p)$ set whenever E_1 is a $\Lambda(p)$ set consisting of non-negative integers and E_2 is a $\Lambda(p)$ set consisting of negative integers [14, 4.4]. Lastly, if E_1 and E_2 are Sidon sets, then so is $E_1 \cup E_2$ [12, 3.5]. These three examples motivate the following results for $\Lambda(p, q)$ sets.

Theorem 5.1. *Let $(p, q) \in J$ and assume $p > 2$. If E_1 and E_2 are $\Lambda(p, q)$ sets, then $E_1 \cup E_2$ is a $\Lambda(p, q)$ set.*

Proof. The argument is similar to that for $\Lambda(p)$ sets found in [10, 37.21]. Assume E_1 and E_2 are $\Lambda(p, q)$ sets and let $E = E_1 \cup E_2$; we may assume the union is disjoint. By Theorem 4.1 and (2) there exist constants k_j, $j = 1, 2$, such that $\|f\|_{(p,q)} \leq k_j \; \|f\|_2$ for each $f \in T_{E_j}$. Let $k = \max\{k_1, k_2\}$ and let $f \in T_E$. Since f can be written $f = f_1 + f_2$ where $f_j \in T_{E_j}$, it follows from the Peter-Weyl theorem that $\|f\|_2^2 = \|f_1\|_2^2 + \|f_2\|_2^2$ and thus $\|f_j\|_2 \leq \|f\|_2$. Now $\|f\|_{(p,q)} \leq \|f_1\|_{(p,q)} + \|f_2\|_{(p,q)} \leq 2k \; \|f\|_2$, hence Theorem 4.1 and (2) imply that E is a $\Lambda(p, q)$ set. □

An obvious consequence of Theorem 5.1 is that a finite union of $\Lambda(p, q)$ sets is itself a $\Lambda(p, q)$ set. However, this is a best possible result in the sense that one cannot generally replace the word "finite" with "infinite" as Γ is not a $\Lambda(p, q)$ set for any $(p, q) \in J$. Our next result is a generalization of [14, 4.4] where the analysis takes place on the circle group and its dual, the integers.

Theorem 5.2. *Let E_1 be a set of non-negative integers and let E_2 be a set of negative integers. If E_1 and E_2 are $\Lambda(p, q)$ sets for some $(p, q) \in J$, then the set $E = E_1 \cup E_2$ is also a $\Lambda(p, q)$ set.*

Proof. Choose $(r, s) \in J$ with $p > r$ and hence $(p, q) > (r, s)$. By Theorem 3.1(a) both E_1 and E_2 are $\Lambda(r, s)$ sets. It follows from Theorem 4.1 and (2) that there are constants k_j, $j = 1, 2$, with $\|f_j\|_{(p,q)} \leq k_j \|f_j\|_{(r,s)}$ for all $f_j \in T_{E_j}$. Let $f \in T_E$ and write $f = f_1 + f_2$ where $f_j \in T_{E_j}$. If $1 < r_1 < r < r_2 < \infty$, then $L^{r_2} \subseteq L(r, s) \subseteq L^{r_1}$, and thus the M. Riesz theorem [4, 12.9.1] shows there are constants c_1 and c_2 such that $\|f_j\|_{r_m} \leq c_m \|f\|_{r_m}$ where $j, m = 1, 2$. Interpolate via [11, p. 264] to conclude there exists a constant k such that $\|f_j\|_{(r,s)} \leq k \|f\|_{(r,s)}$. We see that

$$\|f\|_{(p,q)} \leq (k_1 + k_2) \|f_j\|_{(r,s)} \leq (k_1 + k_2) \|f\|_{(r,s)}.$$

Since $p > r$, it follows from Theorem 4.1 that E is a $\Lambda(p, q)$ set. □

If $E \subseteq \Gamma$ and $\tau \in \Gamma$, the translate of E by τ is the set $\tau E = \{\tau\gamma : \gamma \in E\}$. Our next result shows that $\Lambda(p, q)$ sets are translation-invariant.

Theorem 5.3. *If E is a $\Lambda(p, q)$ set for some $(p, q) \in J$, then τE is a $\Lambda(p, q)$ set for each $\tau \in \Gamma$.*

Proof. By (1) we need only verify the containment $L^1_{\tau E} \subseteq L(p,q)_{\tau E}$. If $f \in L^1_{\tau E}$ and $\gamma \in \Gamma$, then

$$\begin{aligned}(\tau^{-1}f)^\wedge(\gamma) &= \int_G \tau^{-1}(s)f(s)\overline{\gamma}(s)d\lambda(s)\\ &= \int_G f(s)\overline{(\tau\gamma)}(s)d\lambda(s) = \hat{f}(\tau\gamma).\end{aligned}$$

If $\gamma \notin E$, then $\tau\gamma \notin \tau E$ and thus $(\tau^{-1}f)^\wedge$ vanishes off of E. Since f and $\tau^{-1}f$ have equal distribution functions, we see that $\tau^{-1}f \in L^1_E$ and, as E is a $\Lambda(p,q)$ set, $f \in L(p,q)_{\tau E}$. □

If E is a $\Lambda(p,q)$ set and F is a finite subset of Γ, then $FE = \{\tau\gamma : \tau \in F, \gamma \in E\}$ is also a $\Lambda(p,q)$ set. As mentioned above, one cannot replace F with an arbitrary infinite subset of Γ as Γ is not a $\Lambda(p,q)$ set.

We now consider arithmetic and geometric properties of $\Lambda(p,q)$ sets. It is known that Sidon sets and $\Lambda(p)$ sets for $p > 2$ do not contain large parts of certain sets that are themselves generalized arithmetic progressions [12, Chapter 6]. This result for $\Lambda(p)$ sets was generalized and extended for all $p > 1$ in [8].

Definition 5.4 ([8, 1.1 and 1.2])**.** A subset P of Γ is called a pseudo-parallelepiped of dimension N if $P = \prod_{i=1}^N \{\gamma_i, \tau_i\}$ where $\gamma_i, \tau_i \in \Gamma$, $1 \le i \le N$. A parallelepiped P of dimension N is a pseudo-parallelepiped of dimension N such that P consists of 2^N elements.

As remarked in [8, p. 144], pseudo-parallelepipeds are generalizations of arithmetic progressions of integers. Our first result is an easy generalization of [8, 1.2].

Theorem 5.5. *If E is a $\Lambda(p,q)$ set for some $(p,q) \in J$, then there exists an integer N such that E does not contain any parallelepipeds of dimension N.*

Proof. This follows immediately from [8, 1.2] by noting from Theorem 3.1(c) that E is a $\Lambda(r)$ set for each $r \in (1,p)$. □

The next theorem extends the result of [8, 2.4] which gives an upper bound on the cardinality of the intersection of a $\Lambda(p)$ set and a generalized arithmetic progression. If F is a finite subset of Γ, let $|F|$ denote the cardinality of F.

Theorem 5.6. *Let E be a $\Lambda(p,q)$ set for some $(p,q) \in J$. There exist constants $c > 0$ and $\epsilon \in (0,1)$ such that if S is an arithmetic progression in Γ of length N, then $|E \cap S| \le cN^\epsilon$.*

Proof. This follows easily from [8, 2.3 and 2.4] since E is a $\Lambda(r)$ set for each $r \in (1,p)$. □

A consequence of Theorem 5.6 is that, like $\Lambda(p)$ sets, a $\Lambda(p,q)$ set cannot contain a generalized arithmetic progression of arbitrary length. For $p \in (1,2]$ it is not known whether the union of two $\Lambda(p)$ sets is itself a $\Lambda(p)$ set. However [8, 2.11] shows that $E_1 \cup E_2$ does not contain parallelepipeds of arbitrarily large dimension if E_1 and E_2 are $\Lambda(p)$ sets for $p > 1$. It is not difficult to see that this result is also true for $\Lambda(p,q)$ sets for all $(p,q) \in J$. Further connections between parellelepipeds, $\Lambda(p)$ sets, and $\Lambda(p,q)$ sets are also seen in [8, 4.1].

There is an interesting dichotomy in regards to the question of whether one class of $\Lambda(p)$ sets is contained in another. If $p \in (1,2)$, then each $\Lambda(p)$ set is also a

$\Lambda(p+\epsilon)$ set for some $\epsilon > 0$ (see [1, Main Theorem] and [7, Main Theorem]). On the other hand, it is shown in [3, Theorem 2] that, for each $p > 2$, there exists a $\Lambda(p)$ subset of the integers which is not a $\Lambda(r)$ set for any $r > p$. Both of these results have consequences in our study of $\Lambda(p, q)$ sets.

Theorem 5.7. *If E is a $\Lambda(p_1, q_1)$ set for some $(p_1, q_1) \in J$ and $p_1 \in (1, 2)$, then E is a $\Lambda(p, q)$ set for some $(p, q) > (p_1, q_1)$.*

Proof. Consider first the case where $q_1 \in [1, p_1)$. By (1), $L(p_1, q_1) \subset L^{p_1}$ and thus E is a $\Lambda(p_1)$ set. From [7, Main Theorem] E is $\Lambda(p_1 + \epsilon)$ set for some $\epsilon > 0$ and hence the result follows easily by just letting $p = q = p_1 + \epsilon$. Suppose now that $q_1 > p_1$. By (7), $L(p_1, q_1) \subset L^r$ for all $r \in (1, p_1)$ and thus E is a $\Lambda(r)$ set for each such r. We will show that E is a $\Lambda(s)$ set for some $s > p_1$. The conclusion will then follow from the case above.

As motivated by [7], for $r \in (1, p_1]$ and $n \geq 2$, let s be defined by the equation

$$\frac{1}{s} = \frac{1}{r} + \left(\frac{2-r}{4r}\right) c(n) \text{ where } c(n) = \frac{\log(1 - \frac{1}{n^2})}{\log(2n^2)}. \tag{8}$$

Note that $s > r$ since $c(n) < 0$. Rearrange (8) so that $s = s(r, n) = \dfrac{4r}{4 + (2-r)c(n)}$. By letting $k(r, r/2; E) = \inf\{\|f\|_r : f \in T_E, \|f\|_{r/2} \leq 1\}$ we see from [7, Remark, p. 7] that for each $r \in (1, p_1)$, if $n \geq 4(k(r, r/2; E))^r$, then E is a $\Lambda(s)$ set. For $n \geq 2$, let $r_n = p_1 - \frac{1}{2}[s(p_1, n) - p_1]$. It is clear that $r_n < p_1$. Since $c(n) \to 0^-$ as $n \to \infty$, it follows that $r_n > 1$ for all sufficiently large n. Consequently for these such n, E is a $\Lambda(s)$ set where $s = s(r_n, n)$. A straightforward calculation shows that $s(r_n, n) > p_1$ if n is sufficiently large. This completes the proof. □

Theorem 5.8. *For each $(p_1, q_1) \in J$ where $q_1 \geq p_1 > 2$, there exists a $\Lambda(p_1, q_1)$ set in Γ which is not a $\Lambda(p, q)$ set for any $(p, q) \in J$ such that $p > p_1$.*

Proof. Let $(p_1, q_1) \in J$ where $q_1 \geq p_1 > 2$. By [3, Theorem 1] there exists a subset E of Γ which is a $\Lambda(p_1)$ set but is not a $\Lambda(r)$ set for any $r > p_1$. It follows from (1) that E is a $\Lambda(p_1, q_1)$ set and from (7) that E is not a $\Lambda(p, q)$ set if $p > p_1$. □

6. Remarks

A pervasive theme in this paper has been to generalize results for $\Lambda(p)$ sets by expressing them as theorems about $\Lambda(p, q)$ sets. This theme has been developed by capitalizing on the structural similarities existing between L^p spaces and $L(p, q)$ spaces. A similar idea is exploited in [5] and [6] where the theory of L^p-improving measures is extensively generalized and subsequently leads to the theory of Lorentz-improving measures.

Theorem 5.7 and the inclusions found in (1) suggest the following question. Suppose E is a $\Lambda(2, q_1)$ set where $q_1 > 2$. Is it true that E is also a $\Lambda(2, q)$ set for some $q \in [1, q_1)$? As mentioned just before Theorem 5.7, the analogous question for $\Lambda(p)$ sets has been answered affirmatively: if $p \in (1, 2)$, then each $\Lambda(p)$ set is also a $\Lambda(p+\epsilon)$ set for some $\epsilon > 0$ (see [1, Main Theorem] and [7, Main Theorem]). The techniques used in these two papers for the analysis of $\Lambda(p)$ sets do not yet appear to be modifiable so as to yield a definite answer to the above question for $\Lambda(p, q)$ sets. Theorem 5.7 is, however, a natural result in this direction. Note that if E is a $\Lambda(2, q_1)$ set for some $q_1 > 2$, then E is also a $\Lambda(p)$ set for all $p \in (1, 2)$ since $L(2, q_1) \subset L^p$. Consequently, if there does exist such a set E which is not a

$\Lambda(2,q)$ set for any $q \in [1,q_1)$, then E also has the interesting property that it is a $\Lambda(p)$ set for all $p \in (1,2)$ yet is not a $\Lambda(2)$ set. The reader is referred to [8] for some further issues regarding open questions on the relationship between $\Lambda(p)$ sets for $p \in (1,2)$, and $\Lambda(2)$ sets.

A major contribution to the theory of $\Lambda(p)$ sets is [3] where a solution to the celebrated $\Lambda(p)$ set problem is presented. The solution asserts that, for each $p > 2$, there is a $\Lambda(p)$ set of integers which is not a $\Lambda(r)$ set for any $r > p$. The analogous problem for $\Lambda(p,q)$ sets is as follows. Does there exist a $\Lambda(p_1,q_1)$ set for some $(p_1,q_1) > (2,2)$ which is not a $\Lambda(p,q)$ set for any $(p,q) > (p_1,q_1)$? Theorem 5.8 is a minor result concerning this question. It is seen in [3] that the solution to the $\Lambda(p)$ set problem is obtained via intricate probabilistic methods. It is quite unclear as to whether these kinds of techniques can be modified so as to provide a solution to the problem for $\Lambda(p,q)$ sets.

A popular and successful direction for lacunary research is to generalize results for the integers and rephrase them in terms of the dual of a compact abelian group. A further degree of extension is attained when results for the dual of a compact abelian group can be abstracted in terms of the dual object of a compact group or hypergroup. There are a number of possibilities for further research suggested by these remarks and the results for $\Lambda(p,q)$ sets we have developed in this paper.

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