

CHAOTIC POLYNOMIALS ON FRÉCHET SPACES

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ABSTRACT. Contrary to the case of polynomials on Banach spaces, in which it is known that no hypercyclic homogeneous polynomial of degree $m \geq 2$ exists on any Banach space, we construct for each $m \geq 2$ a chaotic m -homogeneous polynomial P on the Fréchet space $\mathcal{H}(\mathbb{C})$.

A map $T : X \rightarrow X$ on a metric space X is chaotic (see [3]) if (a) T is transitive (which for complete separable X is equivalent to the existence of $x \in X$ whose orbit $\text{Orb}(T, x) := \{x, Tx, T^2x, \dots\}$ is dense in X), (b) the periodic points of T are dense in X , and (c) T has sensitive dependence on initial conditions. Banks et al. [1] showed that (c) is redundant in the definition of Devaney.

We present an example of a chaotic m -homogeneous continuous polynomial $P : \mathcal{H}(\mathbb{C}) \rightarrow \mathcal{H}(\mathbb{C})$. This must be compared with a result of Bernardes [2], who showed that for $m > 1$ there are no continuous m -homogeneous polynomials admitting a vector with dense orbit (hypercyclic in the usual terminology) on any Banach space. The result of Bernardes is a consequence of the inequality $\|P^n x\| \leq \|P\|^{1+m+\dots+m^{n-1}} \|x\|^{m^n}$ for every $x \in X$ if P is a continuous m -homogeneous polynomial ($m > 1$) on a Banach space X , since no hypercyclic vector for P could lie on the ball centered at 0 of radius $r := 1/\|P\|$. This implies that P does not admit any hypercyclic vector.

On $\mathcal{H}(\mathbb{C})$ we consider the increasing sequence of norms $(\|\cdot\|_k)_k$ defined by

$$\|f\|_k := \sup_{j \geq 0} \frac{|f^{(j)}(0)|}{j!} k^j, \quad k \in \mathbb{N}, \quad f \in \mathcal{H}(\mathbb{C}),$$

which define the natural Fréchet topology on $\mathcal{H}(\mathbb{C})$.

Our notation is standard. We refer to the monograph [6] for Fréchet spaces and to [4] for polynomials on locally convex spaces.

Theorem 1. *For each m natural ($m \geq 2$) there exists a chaotic m -homogeneous polynomial $P : \mathcal{H}(\mathbb{C}) \rightarrow \mathcal{H}(\mathbb{C})$.*

Proof. Let us define $P : \mathcal{H}(\mathbb{C}) \rightarrow \mathcal{H}(\mathbb{C})$ by $(Pf)(z) := \sum_{j \geq 0} \frac{(f^{(j+1)}(0))^m}{j!} z^j$ for every $f \in \mathcal{H}(\mathbb{C})$ and for every $z \in \mathbb{C}$. P is obviously a well-defined m -homogeneous polynomial. We first prove that P is hypercyclic.

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Indeed, by the hypercyclicity of the derivative operator $D : \mathcal{H}(\mathbb{C}) \rightarrow \mathcal{H}(\mathbb{C})$ (see [5]), there exists $x \in \mathcal{H}(\mathbb{C})$, a hypercyclic vector for D . We denote by $x_j := x^{(j)}(0)$, $j \in \mathbb{N}$, and pick an increasing sequence $(n_k)_k$ of natural numbers such that $(D^{n_k} x)_k$ is dense in $\mathcal{H}(\mathbb{C})$ and

$$(i) \quad \left\| \sum_{j \geq n_{k+1} - n_k} \frac{x_{j+n_k}}{j!} z^j \right\|_k < \frac{1}{k},$$

$$(ii) \quad \sup \left\{ \frac{k^j}{j!} \mid j \geq n_{k+1} - n_k \right\} < \frac{1}{k},$$

for each $k \in \mathbb{N}$.

Now we select $(y_j)_{j \geq 0}$ such that

$$y_j^{m^{n_k}} = x_j, \quad n_k \leq j < n_{k+1} \quad (n_0 := 0) \quad \forall k \in \mathbb{N}.$$

Then

$$y := y(z) := \sum_{j \geq 0} \frac{y_j}{j!} z^j \in \mathcal{H}(\mathbb{C}),$$

since $|y_j| \leq 1 + |x_j|$, $j \geq 0$. Moreover, the selection of $(y_j)_{j \geq 0}$ and the definition of P imply

$$(iii) \quad (P^{n_k} y)^{(j)}(0) = (D^{n_k} x)^{(j)}(0), \quad 0 \leq j < n_{k+1} - n_k, \quad \forall k \in \mathbb{N}.$$

Given $n \in \mathbb{N}$, $f \in \mathcal{H}(\mathbb{C})$ and $\varepsilon > 0$, we fix $k > n$ such that $1/k < \varepsilon/4$ and $\|D^{n_k} x - f\|_n < \varepsilon/4$. Then, from (i), (ii) and (iii), we get

$$\begin{aligned} & \|P^{n_k} y - f\|_n \\ \leq & \|P^{n_k} y - \sum_{n_{k+1} - n_k > j \geq 0} \frac{x_{j+n_k}}{j!} z^j\|_n + \left\| \sum_{j \geq n_{k+1} - n_k} \frac{x_{j+n_k}}{j!} z^j \right\|_n + \|D^{n_k} x - f\|_n \\ < & \|P^{n_k} y - \sum_{n_{k+1} - n_k > j \geq 0} \frac{(D^{n_k} x)^{(j)}(0)}{j!} z^j\|_k + \frac{1}{k} + \frac{\varepsilon}{4} \\ < & \sup \left\{ \frac{|y_{j+n_k}|^{m^{n_k}}}{j!} k^j \mid j \geq n_{k+1} - n_k \right\} + \frac{\varepsilon}{2} \\ \leq & \sup \left\{ \frac{1 + |x_{j+n_k}|}{j!} k^j \mid j \geq n_{k+1} - n_k \right\} + \frac{\varepsilon}{2} \\ < & \sup \left\{ \frac{|x_{j+n_k}|}{j!} k^j \mid j \geq n_{k+1} - n_k \right\} + \frac{1}{k} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This implies that P is hypercyclic. To conclude the proof we show that the periodic vectors of P are dense in $\mathcal{H}(\mathbb{C})$. We are done if, given $m \in \mathbb{N}$ and an arbitrary

polynomial $p(z) = \sum_{j=1}^n \frac{\alpha_j}{j!} z^j$, we find $u \in \mathcal{H}(\mathbb{C})$ periodic for P such that $\|u - p\|_m < 1/m$. To do this, take $k > n$ satisfying

$$(iv) \quad \sup \left\{ \frac{M}{j!} m^j \mid j \geq k \right\} < \frac{1}{m},$$

where $M := \sup\{1 + |\alpha_j| \mid 1 \leq j \leq n\}$. Choose $(u_j)_{j \geq 0}$ such that

$$u_j := \alpha_j \quad (\alpha_j := 0 \quad \text{if } j > n), \quad u_{j+r k}^{m^k} = u_{j+(r-1)k} \quad \text{if } 0 \leq j < k \quad \text{and } r \geq 1.$$

We have that $u(z) := \sum_{j \geq 0} \frac{u_j}{j!} z^j \in \mathcal{H}(\mathbb{C})$, since $|u_j| \leq 1 + |\alpha_j|$, $j \geq 0$. By the selection of $(u_j)_{j \geq 0}$ and (iv) we also get that u is k -periodic for P and

$$\|u - p\|_m = \left\| \sum_{j \geq k} \frac{u_j}{j!} z^j \right\|_m \leq \sup \left\{ \frac{M}{j!} m^j \mid j \geq k \right\} < \frac{1}{m}.$$

We conclude that P is chaotic. \square

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