

CONVERGENCE OF THE STEEPEST DESCENT METHOD FOR ACCRETIVE OPERATORS

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ABSTRACT. Let X be a uniformly smooth Banach space and let $A: X \rightarrow X$ be a bounded demicontinuous mapping, which is also α -strongly accretive on X . Let $z \in X$ and let x_0 be an arbitrary initial value in X . Then the approximating scheme

$$x_{n+1} = x_n - c_n(Ax_n - z), \quad n = 0, 1, 2, \dots,$$

converges strongly to the unique solution of the equation $Ax = z$, provided that the sequence $\{c_n\}$ fulfills suitable conditions.

1. INTRODUCTION

The main objective of this paper is to present some further findings concerning some recent works (see [15], [17]) in the area of the so-called steepest descent approximation method for accretive operators. First of all, we describe the general setting of our results.

Let X be a (real) Banach space and let D be a subset of X . An operator $A: D \rightarrow X$ is said to be k -accretive ($k \in \mathbb{R}$) if for each pair $u, v \in D$ there exists $j \in J(u - v)$ such that

$$(1) \quad \langle A(u) - A(v), j \rangle \geq k\|u - v\|^2,$$

where $J: X \rightarrow 2^{X^*}$ is the normalized duality mapping which is defined by

$$J(u) = \{j \in X^* : \langle u, j \rangle = \|u\|^2, \|j\| = \|u\|\}.$$

For an alternative equivalent definition see Morales [11]. Here $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach Theorem that $J(u)$ is nonempty for each $u \in X$. Moreover, it is known that $J(u)$ is single-valued if X is smooth (in particular when the dual space X^* is strictly convex), while if X^* is uniformly convex, then the mapping J is uniformly continuous on bounded sets. For $k > 0$ in inequality (1), we say that A is *strongly accretive*, while for $k = 0$, A is called *accretive*. There is a class of mappings intimately related to the k -accretive ones. These are the k -pseudo-contractions (for more details see [11]). This latter family is formed by mappings written as $I - A$ where I is the identity and A is k -accretive. Nevertheless, in the present paper,

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we are mainly interested in a class of operators somehow more general than the strongly accretive ones, which are defined as follows.

Let $\alpha: [0, \infty) \rightarrow [0, \infty)$ be a function for which $\alpha(0) = 0$ and the $\liminf_{r \rightarrow r_0} \alpha(r) > 0$ for every $r_0 > 0$. An operator $A: D(A) \subset X \rightarrow X$ is called α -strongly accretive if for each $u, v \in D(A)$ there exists $j \in J(u - v)$ so that

$$\langle Au - Av, j \rangle \geq \alpha(\|u - v\|)\|u - v\|.$$

In contrast with the terminology of [12] we do not assume that α is continuous at zero, nor do we assume that α is strictly increasing, another common assumption. Nevertheless, a mild additional assumption will be imposed on the function α . We also say that T is α -strongly pseudo-contractive if $I - T$ is α -strongly accretive.

Along with existence theory for zeros of monotone and accretive operators, we find through the years significant efforts to identify approximation schemes for such zeros. Particularly the so-called steepest descent method was introduced for monotone operators by Vainberg [16] and Zarantonello [18] independently, in the following terms.

Let A be a monotone and Lipschitz operator defined on a Hilbert space into itself. Then the approximating process

$$(2) \quad x_{n+1} = (1 - \lambda)x_n - \lambda Ax_n, \quad \lambda \in (0, 1),$$

converges strongly to a zero of A .

In the last 30 years the process described by (2) has evolved in various directions. Among them, we can find Petryshyn [13], Bruck [1], Crandall and Pazy [5], Dotson [7] and many others. Nowadays, this process is described for accretive operators as follows:

$$(3) \quad x_{n+1} = x_n - c_n Ax_n$$

for a suitable sequence $\{c_n\}$.

Recently Xu and Roach [17], and Chidume [4] have studied necessary and sufficient conditions for the convergence of the process (3) for these operators. The purpose of this paper is to continue this study. Among other things, we repair what seems to be an oversight in the proof of Theorem 1 of [17], which is also repeated in [4]. Specifically, in trying to show the boundedness of the sequence $\{x_n\}$ on page 347 of [17], the authors establish formula (2.11) for $n = n_j$, but not necessarily for its successor. The step seems crucial and is referred to later in the argument. In what follows, we give an alternative proof of boundedness of this sequence which appears shorter and simpler, although our proof is somewhat inspired by the original ideas. At the same time we also extend some results of [3] and [6].

We will need a few preliminary facts, in particular, the following lemma.

Lemma 1 (Reich [14]). *Let X be a uniformly smooth Banach space. Then there exists a nondecreasing continuous function $b: [0, \infty) \rightarrow [0, \infty)$ which satisfies the following:*

- (i) $b(ct) \leq cb(t)$ for all $c \geq 1$;
- (ii) $\lim_{t \rightarrow 0^+} b(t) = 0$; and
- (iii) $\|x + y\|^2 \leq \|x\|^2 + 2 \operatorname{Re}\langle y, J(x) \rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|)$ for all $x, y \in X$.

Throughout the paper we assume that X is a real Banach space whose dual space X^* is uniformly convex. This latter statement is equivalent to saying that X is uniformly smooth. In addition, we say that an operator A is *demicontinuous* if it is continuous from the strong topology of X into the weak topology of X . We

also say that A is *bounded* if it maps bounded sets into bounded sets. We denote the distance between the sets A and B by $\text{dist}(A, B)$, where

$$\text{dist}(A, B) = \inf\{\|a - b\| : a \in A, b \in B\}.$$

2. MAIN RESULTS

Theorem 2. *Let X be a uniformly smooth Banach space, let b be the function of Lemma 1, and let $A: X \rightarrow X$ be a bounded demicontinuous mapping, which is also α -strongly accretive on X . Let $z \in X$ and let x_0 be an arbitrary initial value in X for which the $\liminf_{r \rightarrow \infty} \alpha(r) > \|A(x_0)\|$. Then the approximating scheme*

$$(4) \quad x_{n+1} = x_n - c_n(Ax_n - z), \quad n = 0, 1, 2, \dots,$$

converges strongly to the unique solution of the equation $Ax = z$, provided that the sequence $\{c_n\}$ of positive real numbers satisfies the following:

- (i) $\{c_n\}$ is bounded above by the constant r_0 defined in (6);
- (ii) $\sum_{n=1}^{\infty} c_n = \infty$; and
- (iii) $\sum_{n=0}^{\infty} c_n b(c_n) < \infty$.

We begin with some preliminary notions and notations that will be used in the proof of Theorem 2. We first observe that due to assumption (iii) on $\{c_n\}$, we easily derive that $\lim_{n \rightarrow \infty} c_n = 0$.

For the initial guess x_0 of the iterative process defined by (4), we define the positive constants m_0 and r_0 as follows. We may assume without loss of generality that $z = 0$ and consequently that $\|Ax_0\| > 0$. Then we set

$$a_0 = \sup\{r : \alpha(r) \leq \|Ax_0\|\},$$

which allows us to define

$$(5) \quad m_0 = \sup\{\|Au\| : \|u - x_0\| \leq 3a_0\}$$

and

$$(6) \quad r_0 = \frac{1}{m_0} \min \left\{ \sup b^{-1} \left[\frac{a_0 \alpha(a_0)}{\max\{a_0, 1\} m_0} \right], a_0 \right\}.$$

Proposition 3. *Let X be a Banach space, whose dual X^* is uniformly convex. Let $A: D(A) \subset X \rightarrow X$ be a bounded and α -strongly accretive operator. Suppose there exists a zero of A and a sequence defined by*

$$x_{n+1} = x_n - c_n Ax_n, \quad n = 0, 1, 2, \dots,$$

with $c_n \leq r_0$ for $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded.

Proof. We denote by x^* the unique zero of A . Then

$$\alpha(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \langle Ax_0, J(x_0 - x^*) \rangle \leq \|Ax_0\| \|x_0 - x^*\|.$$

This implies that $\alpha(\|x_0 - x^*\|) \leq \|Ax_0\|$, and thus $\|x_0 - x^*\| \leq a_0$. Suppose now that the sequence $\{x_n\}$ is not bounded. So, let n_0 be the first natural number for which

$$(7) \quad \|x_{n_0} - x^*\| \geq a_0.$$

Then $\|x_{n_0-1} - x^*\| < a_0$, and thus $\|x_{n_0-1} - x_0\| \leq 2a_0$. This implies that $\|Ax_{n_0-1}\| \leq m_0$. Therefore,

$$(8) \quad \begin{aligned} \|x_{n_0+1} - x^*\|^2 &= \|x_{n_0} - c_{n_0}Ax_{n_0} - x^*\|^2 \\ &\leq \|x_{n_0} - x^*\|^2 - 2c_{n_0}\langle Ax_{n_0}J(x_{n_0} - x^*) \rangle \\ &\quad + \max\{\|x_{n_0} - x^*\|, 1\}c_{n_0}\|Ax_{n_0}\|b(c_{n_0}\|Ax_{n_0}\|). \end{aligned}$$

However, for further estimates of inequality (8), we need to show that $\|Ax_{n_0}\| \leq m_0$. To see this, let

$$\begin{aligned} \|x_{n_0} - x_0\| &\leq \|x_{n_0} - x_{n_0-1}\| + \|x_{n_0-1} - x^*\| + \|x^* - x_0\| \\ &\leq c_{n_0-1}\|Ax_{n_0-1}\| + 2a_0 \leq r_0m_0 + 2a_0 \leq 3a_0, \end{aligned}$$

which means $\|Ax_{n_0}\| \leq m_0$. Now if we denote $k_0 = \max\{\|x_{n_0} - x^*\|, 1\}$, then we may re-write the inequality (8) as

$$(9) \quad \begin{aligned} \|x_{n_0+1} - x^*\|^2 &\leq \|x_{n_0} - x^*\|^2 - 2c_{n_0}\alpha(\|x_{n_0} - x^*\|)\|x_{n_0} - x^*\| \\ &\quad + k_0c_{n_0}m_0b(c_{n_0}m_0) \\ &\leq \|x_{n_0} - x^*\|^2 - c_{n_0}[2a_0\alpha(a_0) - k_0m_0b(c_{n_0}m_0)]. \end{aligned}$$

However, due to the choice of r_0 in (6), it is easy to see that

$$2a_0\alpha(a_0) - k_0m_0b(c_{n_0}m_0) \geq 0.$$

Therefore

$$\|x_{n_0+1} - x^*\| \leq \|x_{n_0} - x^*\|.$$

To complete the proof, we first defined ρ_n by $\|x_n - x^*\|$. Then we have that $\rho_{n_0+1} \leq \rho_{n_0}$. In addition, it can be easily derived that

$$\|Ax_{n_0+1}\| \leq m_0.$$

This means, if we assume that $\rho_{n_0+1} \geq a_0$, then the previous argument holds and thus $\rho_{n_0+2} \leq \rho_{n_0+1}$. On the other hand, if $\rho_{n_0+1} < a_0$, then either $\rho_n < a_0$ for all $n \geq n_0 + 1$, in which case the proof is complete, or there exists $j \in \mathbb{N}$ such that $\rho_j \geq a_0$ while $\rho_{j-1} < a_0$. In the latter case, if $\|Ax_{j-1}\| \leq m_0$ we can return to the previous argument given in (7)–(9). To this end, we note that

$$\begin{aligned} \|x_j - x^*\| &\leq \|x_j - x_{j-1}\| + \|x_{j-1} - x^*\| \\ &\leq c_{j-1}\|Ax_{j-1}\| + a_0, \end{aligned}$$

while,

$$\|x_{j-1} - x_0\| \leq \|x_{j-1} - x^*\| + \|x^* - x_0\| \leq 2a_0.$$

This implies that $\|Ax_{j-1}\| \leq m_0$, and therefore $\|x_n - x^*\| \leq 2a_0$ for all $n \in \mathbb{N}$, proving that the sequence $\{x_n\}$ is bounded.

Proof of Theorem 2. Due to Theorem 1 of Kartsatos [8], we conclude that $A(X)$ is open in X . Since $A(X)$ is also clearly closed, A must be surjective. This means, the equation $Ax = z$ has a unique solution for an arbitrary $z \in X$ and without loss of generality we may assume that $z = 0$.

Due to Proposition 3 we know that $\{x_n\}$ is bounded and thus so is $\{Ax_n\}$. Therefore, this and (9) imply

$$(10) \quad \|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - 2c_n\alpha(\|x_n - x^*\|)\|x_n - x^*\| + Mc_nb(c_n)$$

for a suitable positive constant M . It follows recursively that for $n > 1$

$$\|x_{n+1} - x^*\|^2 \leq \|x_1 - x^*\|^2 - 2 \sum_{j=1}^n c_j \alpha(\|x_j - x^*\|) \|x_j - x^*\| + M \sum_{j=1}^n c_j b(c_j).$$

Now, using condition (ii) of $\{c_n\}$ we obtain

$$\sum_{j=1}^{\infty} c_j \alpha(\|x_j - x^*\|) \|x_j - x^*\| < \infty,$$

and since $\sum_{j=1}^{\infty} c_j = +\infty$, we conclude that

$$\liminf_{n \rightarrow \infty} \alpha(\|x_n - x^*\|) \|x_n - x^*\| = 0.$$

Consequently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ for which

$$\lim_{k \rightarrow \infty} \|x_{n_k} - x^*\| = 0.$$

To complete the proof, let $\varepsilon > 0$ be arbitrary. Then there exists a $k_0 \in \mathbb{N}$ such that

$$\|x_{k_0} - x^*\| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and} \quad \sum_{j=k_0}^{\infty} c_j b(c_j) < \frac{\varepsilon^2}{2M}.$$

Now we let $n \geq k_0$ and apply a recursive process to (10). Then

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_{k_0} - x^*\|^2 - 2 \sum_{j=k_0}^n \alpha(\|x_j - x^*\|) \|x_j - x^*\| + M \sum_{j=k_0}^n c_j b(c_j) \\ &\leq \|x_{k_0} - x^*\|^2 + M \sum_{j=k_0}^{\infty} c_j b(c_j) < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2. \end{aligned}$$

Therefore the sequence $\{x_n\}$ converges strongly to the unique zero of the operator $A - z$.

The study of convergence of the Mann process for operators defined on closed and convex subsets of a Banach space into itself has been of interest in recent years (see for instance [2], [9], [15]). Our next result deals precisely with the case that the operator A is not necessarily defined in the whole Banach space X .

Before we state our next theorem, we need the following existence result for zeros of an operator, which also appears to be a new result.

Proposition 4. *Let K be a closed convex subset of a Banach space X , let $A: K \rightarrow X$ be a continuous and α -strongly accretive mapping, where the $\liminf_{r \rightarrow \infty} \alpha(r) > \|A(x_0)\|$ for some $x_0 \in K$. Suppose in addition that*

$$(11) \quad \lim_{h \rightarrow 0} \frac{1}{h} \text{dist}(x - hAx; K) = 0$$

for every $x \in K$. Then A has a unique zero in K .

Proof. Without loss of generality we may assume $x_0 = 0$. Since the operator $-A$ fulfills the assumptions of Theorem 6 of Martin [10], we conclude that $K \subset (I + A)(K)$. Also the fact that $I + A$ is invertible implies that $g = (I + A)^{-1}$ is a non-expansive mapping from K into K . Since the fixed points of g are the zeros of A , it is sufficient to show that g has a fixed point in K . To this end, we first show that the set

$$E = \{x \in K : Ax = tx \text{ for some } t < 0\}$$

is bounded. To see this, let $x \in E$. Then $Ax = tx$ for some $t < 0$ and,

$$\langle Ax - A(0), j \rangle \geq \alpha(\|x\|)\|x\|,$$

for some $j \in J(x)$. This implies,

$$\alpha(\|x\|)\|x\| \leq t\|x\|^2 + \|A(0)\|\|x\|.$$

Since $t < 0$, $\alpha(\|x\|) \leq \|A(0)\|$, and this implies that E is bounded. As a consequence of this we can easily show that the set

$$F = \{y \in K : g(y) = \lambda y \text{ for some } \lambda > 1\}$$

is also bounded. Next, we show that $(I - g)(K)$ is a closed set of X . Suppose $\{y_n\}$ is a sequence in K so that $y_n - g(y_n) \rightarrow u$, for some $u \in X$. Let $g(y_n) = x_n$. Then $y_n - g(y_n) = Ax_n \rightarrow u$. But

$$\langle Ax_n - Ax_m, j \rangle \geq \alpha(\|x_n - x_m\|)\|x_n - x_m\|,$$

which implies that $\alpha(\|x_n - x_m\|) \leq \|Ax_n - Ax_m\|$, and thus $\{x_n\}$ is a Cauchy sequence. This means $x_n \rightarrow x$ for some $x \in K$, and since $I + A$ is continuous, $y_n \rightarrow y$ for some $y \in K$. This implies $u = (I - g)(y)$.

Now, let $t_n \in (0, 1)$ so that $t_n \rightarrow 1^-$ as $n \rightarrow \infty$. Then $t_n g(y_n) = y_n$ for some $y_n \in K$, which implies that

$$y_n - g(y_n) = (1 - t_n^{-1})y_n.$$

Since $\{y_n\} \subset F$, $y_n - g(y_n) \rightarrow 0 \in (I - g)(K)$.

Proposition 4 is an extension of Theorem 2 of Deimling [6], who assumes in addition that either there exists a number $R > 0$ and $j \in J(x)$ so that $\langle Ax, j \rangle \geq 0$ for $\|x\| \geq R$ or $\|Ax\| \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

Corollary 5. *Let X and K be as in Proposition 4. Let $T: K \rightarrow K$ be a continuous and α -strongly pseudo-contractive mapping. Then T has a unique fixed point in K .*

We conclude with a result that improves Theorem 1 of [3].

Theorem 6. *Let X be a uniformly smooth Banach space and let K be a closed convex subset of X . Suppose $A: K \rightarrow X$ is a bounded, continuous and α -strongly accretive operator for which $I - A$ maps K into K . Then the iterative process (3) converges strongly to the unique zero of A provided that $\{c_n\}$ satisfies the same assumptions as in Theorem 2.*

Proof. Due to Corollary 5, the operator A has a unique zero in K , and since continuity implies demicontinuity, the conclusion follows from Theorem 2.

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