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IRREDUCIBLE REPRESENTATIONS OF THE CUNTZ ALGEBRA $\mathcal{O}_N$

EUI-CHAI JEONG

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Abstract. In this paper, we establish formulas for the configuration of a special class of irreducible representations of the Cuntz algebra $\mathcal{O}_N$, $N = 2, 3, \ldots, \infty$. These irreducible representations arise as subrepresentations of naturally occurring representations of $\mathcal{O}_N$ acting in $L^2(\mathbb{T})$ and arise from consideration of multiresolution wavelet filters.

1. Introduction

Recent papers (e.g., [BraJo97a], [BraJo97b], [Jor]) show that decomposition of representations of finitely generated C^*-algebras has applications to

1. filter functions for multiresolutions from wavelet theory;
2. limit problems in analytic number theory;
3. multiplicity problems from noncommutative harmonic analysis.

In this paper, we introduce a special class of representations related to 1–3 above. They can be described on the Hilbert space $\ell^2(S)$, where S is a discrete index set, or more specifically on $L^2(\mathbb{T}^n) \simeq \ell^2(\mathbb{Z}^n)$ where $S = \mathbb{Z}^n$, $n = 1, 2, \ldots$. Recall that the Cuntz algebra $\mathcal{O}_N$, $N = 2, 3, \ldots$, is the C^*-algebra generated by isometries $s_1, \ldots, s_N$ satisfying

$$s_i^* s_j = \delta_{ij}\mathbb{1} \quad \text{and} \quad \sum_{i=1}^{N} s_i s_i^* = \mathbb{1}$$

for $i, j \in \{0, 1, \ldots, N-1\}$ [Cun77]. The C^*-algebra $\mathcal{O}_\infty$ is the one generated by isometries s_i, $i \in \mathbb{Z}$, satisfying $s_i^* s_j = \delta_{ij}\mathbb{1}$. We will say that φ is a nondegenerate representation of $\mathcal{O}_\infty$, with a slight abuse of terminology, if φ is a representation with $\sum_{i\in\mathbb{Z}} \varphi(s_i s_i^*) = \mathbb{1}$, where the sum is in the strong operator topology.

2. Representations of $\mathcal{O}_N$ arising in multiresolution wavelets in scale N

Let $\mathcal{H}$ be the Hilbert space $L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the torus, and let $\{z^n\}_{n=-\infty}^{\infty}$ be the usual orthonormal basis of Fourier analysis: the convention is $z = e^{it}$, $t \in \mathbb{R}$.

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The Haar measure on $\mathbb{T}$ will be normalized. Let functions $\{m_i\}_{i=0}^{N-1} \subset L^2(\mathbb{T})$ be given such that the corresponding $N \times N$ matrix

$$\left(m_k\left(e^{i\frac{2\pi l}{N}}z\right)\right)_{k,l=0}^{N-1} \tag{2.1}$$

is unitary. The representation of $\mathcal{O}_N$ in question is defined from the function m_i as follows:

$$S_i\xi(z) = \sqrt{N}m_i(z)\,\xi\left(z^N\right),$$

where $S_i = \varphi(s_i)$. These representations were introduced in [Jor], and arise in the study of filter-banks in wavelet theory: see [Dau92], [CoRy], [JoPe94], and [JoPe96]. While not wavelets, the representations coming from (2.2) are closely related to so-called "tight frame" wavelet bases. For the wavelets, we must require the added so-called "low pass" condition $|m_0(1)| = 1$ [Dau92] which is of course not satisfied for $m_i(z) = N^{-\frac{1}{2}}z^{d_i}$. We review a simple case. Let $N \in \{2, 3, \dots\}$ and let $D = \{d_0, d_1, \dots, d_{N-1}\} \subset \mathbb{Z}$ be a set of integers such that any two distinct members are mutually incongruent modulo N. Then the operators

$$S_i\colon z^n \longmapsto z^{d_i+Nn} \tag{2.2}$$

satisfy the Cuntz relations

$$S_i^*S_j = \delta_{ij}\mathbb{1} \quad \text{and} \quad \sum_{i=0}^{N-1} S_iS_i^* = \mathbb{1}. \tag{2.3}$$

These representations induce naturally a function system on the index set $\mathbb{Z}$ of the orthonormal basis:

$$\begin{aligned} \sigma\colon \mathbb{Z} &\longrightarrow \mathbb{Z} \quad N \text{ to } 1 \text{ with} \\ \sigma_i\colon \mathbb{Z} &\longrightarrow \mathbb{Z} \quad \text{satisfying } \sigma \cdot \sigma_i = \mathrm{id}_{\mathbb{Z}} \end{aligned}$$

for $i = 0, 1, \dots, N-1$, or

$$S_i(z^n) = z^{\sigma_i(n)} \tag{2.4}$$

and

$$S_i^*(z^n) = z^{\sigma(n)}$$

for $z \in \mathbb{T}$. (See [BraJo97b], [JeongI].)

From (2.2), σ_i, $i = 0, 1, \dots, N-1$, and σ, can be described as follows:

$$\begin{aligned} \sigma_i(n) &= Nn + d_i, \quad i = 0, 1, \dots, N-1, \\ \sigma(n) &= m, \end{aligned}$$

where $m = \frac{n-d_{i_0}}{N}$ with unique $d_{i_0} \in \{d_0, d_1, \dots, d_{N-1}\}$ satisfying $n = d_{i_0}$ modulo N.

Following [BraJo97b], we define two equivalence relations $\sim$ and $\approx$ on $\mathbb{Z}$:

> $n \sim m$ if there exist k_1 and k_2 such that $n = \sigma_{i_{k_1}} \cdots \sigma_{i_2}\sigma_{i_1}\sigma^{k_2}m$ for some $n, m \in \mathbb{Z}$ (equivalently $z^n = S_{i_{k_1}} \cdots S_{i_2}S_{i_1}S_{j_{k_2}}^* \cdots S_{j_2}^*S_{j_1}^*z^m$, $z \in \widetilde{\mathbb{T}}$, where $\widetilde{\mathbb{T}} = \{t \in \mathbb{T} \mid t = 0 \text{ modulo } 2\pi\}$);$n \approx m$ if furthermore $k_1 = k_2$.

Consider the gauge group ("the automorphism group") of $\mathcal{O}_N$, denoted by $(r_z)_{z\in\mathbb{T}}$, which is determined by $r_z(S_i) = zS_i$, $z \in \mathbb{T}$. The subalgebra UHF_N of gauge-invariant elements is

$$\mathrm{UHF}_N := \{a \in \mathcal{O}_N \mid r_z(a) = a \;\forall z \in \mathbb{T}\}.$$

It is a UHF algebra of Glimm type M_{n^∞} (see, e.g., [Cun77], [BJP], and [BraJo]). For general representations $\pi \in \mathrm{Rep}(\mathcal{O}_N, \mathcal{H})$, the case when (UHF_N) is weakly*-dense in $\pi(\mathcal{O}_N)$ is studied in [BEEK] and [Pow88]. In fact, if we consider a separable Hilbert space $\mathcal{H}$ with an orthonormal basis $\{e_n \mid n \in \mathbb{Z}\}$, then the corresponding representation in (2.2) or (2.4) can be written:

$$\begin{aligned} S_i(e_n) &= e_{Nn+d_i} \\ &= e_{\sigma_i(n)} \end{aligned}$$

and

$$S_i^*(e_n) = e_{\sigma(n)}.$$

Theorem 2.1 ([BraJo, Theorem 2.7]). *The closure of any subspace of $\mathcal{H}$ spanned by vectors e_n, where n runs through a $\sim$-equivalence ($\approx$-equivalence) class, is an irreducible $\mathcal{O}_N$-module* (UHF_N*-module, respectively*).

Proof. [BraJo]. □

Therefore, φ splits any representations given by (2.2) into a direct sum of irreducible mutually inequivalent representations of $\mathcal{O}_N$,

$$\varphi = \sum_i^{\oplus} \varphi_i.$$

Theorem 2.2. *Suppose S is the index set of an orthonormal basis of a separable Hilbert space. If S has a radix-representation in base N with set $\{a_0, a_1, \ldots, a_{N-1}\}$ of coefficients, i.e., for every $s \in S$ there exists a unique polynomial $P_s(N) = a_0 + a_1N + \cdots + a_lN^l$ such that $P_s(N) = s$, then the representation coming from* (2.2) *splits into finitely many irreducible subrepresentations with $N < \infty$.*

Proof. Let the set $D = \{d_0, d_1, \ldots, d_{N-1}\}$ be a residue set modulo N in S and representations be defined by

$$S_i(e_n) = e_{nN+d_i},$$

$i = 0, 1, \ldots, N-1$. For $s \in S$, define $\deg(s) :=$ degree of s in the form of the polynomials in N. Note that $\sigma(s) = \frac{s-d_{i_0}}{N}$ for some $i_0 \in \{0, 1, \ldots, N-1\}$ such that $s = d_{i_0}$ modulo N. Thus $\deg(\sigma(s)) < \deg(s)$ if $\deg(d_i) \le \deg(s)$, and $\deg(s) < \deg(\sigma(s))$ if $\deg(s) < \deg(d_i) - 1$ for every $i = 0, 1, \ldots, N-1$. Thus, for every $s \in S$ there exists a positive integer n_s such that

$$m - 1 \text{ (or 0 if } m = 0) \le \deg\left(\sigma^k(s)\right) \le M - 1,$$

for every integer $k \ge n_s$, where $m = \min\{\deg(d_i) \mid i = 0, 1, \ldots, N-1\}$ and $M = \max\{\deg(d_i) \mid i = 0, 1, \ldots, N-1\}$. Therefore, for every $s \in S$ we have an element $x \in B(p) := \left\{a_0 + a_1N + \cdots + a_lN^l \mid a_l \ne 0,\; m-1 \le l \le M-1\right\}$, which implies the representation coming from (2.2) splits into at most $\#(B(p))$ irreducible subrepresentations. Since the cardinality of the set $B(p)$ is finite, this completes the proof. □

Corollary 2.3. *The representation of $\mathcal{O}_N$, $N < \infty$, on $\ell^2(\mathbb{Z})$ coming from* (2.2) *splits into finitely many irreducible subrepresentations.*

Proof. From Theorem 2.2 above, we need to build a radix-representation of $\mathbb{Z}$ in base N. If $2 < N < \infty$, then take a digit set $\{a_0, a_1, \ldots, a_{N-1}\} := \mathbb{Z} \cap \left(-\frac{N}{2}, \frac{N}{2}\right]$. With this digit set, any $s \in \mathbb{Z}$ has a unique radix-representation in N. (For details see [JeongII].) Now suppose $N = 2$. If we take a digit set $\{0, 1\}$, the argument in the proof of Theorem 2.2 holds for the index set $\mathbb{Z}_+ \cup \{0\}$. Thus the number of irreducible subrepresentations is at most $2 \times \#(B(p))$. □

Remark 2.4. Not every representation of $\mathcal{O}_N$, either $N < \infty$ or $N = \infty$, splits into finitely many irreducible subrepresentations. For $N < \infty$, representations of $\mathcal{O}_N$ on a separable Hilbert space $\ell^2(\mathbb{Z}^n)$ for $n > 1$ can have infinitely many irreducible subrepresentations. (See Chapter 3, Example 3.)

3. Irreducible representations of $\mathcal{O}_N$, $N < \infty$

We first consider representations of $\mathcal{O}_N$, $N < \infty$, on the Hilbert space $\ell^2(\mathbb{Z})$. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the largest (smallest) integer such that $\lfloor x \rfloor \leq x$ ($\lceil x \rceil \geq x$, respectively). Extending a theorem in [BraJo], we have

Theorem 3.1. *For $N < \infty$, and the digit set $\{d_0, d_1, \ldots, d_{N-1}\}$, the number of irreducible subrepresentations of the representation of $\mathcal{O}_N$ resulting from* (2.2) *is at most*

$$1 + \left\lfloor -\frac{d_m}{N-1} \right\rfloor - \left\lceil -\frac{d_M}{N-1} \right\rceil,$$

where $d_m = \min\{d_i \mid i = 0, \ldots, N-1\}$ and $d_M = \max\{d_i \mid i = 0, \ldots, N-1\}$.

Corollary 3.2. *Using the digit set $\{d_0, d_1, \ldots, d_{N-1}\}$, the representation of $\mathcal{O}_N$, $N < \infty$, from* (2.2) *is irreducible if one of the following conditions is fulfilled:*

1. $\left\lfloor -\frac{d_m}{N-1} \right\rfloor = \left\lceil -\frac{d_M}{N-1} \right\rceil$.
2. *There exists an integer n_0 such that*
$$\max_i |d_i - n_0| < N - 1.$$
3. $d_0 \neq 0$ *modulo N and $d_i = d_0 + i$, $i = 0, 1, \ldots, N-1$.*

Proof. It is easy to see that if any one of the three conditions 1–3 holds, then $1 + \left\lfloor -\frac{d_m}{N-1} \right\rfloor - \left\lceil -\frac{d_M}{N-1} \right\rceil$ is equal to 1. Thus the representation coming from (2.2) is itself irreducible. □

Proof of Theorem 3.1. If $x > -\frac{d_m}{N-1}$, $x \in \mathbb{R}$, then

$$Nx - x + d_k \geq (N-1)\left(x - \frac{-d_m}{N-1}\right) > 0, \quad k = 0, 1, \ldots, N-1.$$

If $n > \left\lfloor -\frac{d_m}{N-1} \right\rfloor$, $n \in \mathbb{Z}$, then $n > -\frac{d_m}{N-1}$ and

$$\sigma_k(n) - n = Nn - n + d_k > 0, \quad k = 0, 1, \ldots, N-1.$$

Thus $\sigma_k(n) > n$, $k = 0, 1, \ldots, N-1$, for every integer $n > \left\lfloor -\frac{d_m}{N-1} \right\rfloor$. If we take a joint left inverse σ, we have

$$n = \sigma(\sigma_k(n)) > \sigma(n).$$

Similarly, if $n < \left\lceil -\frac{d_M}{N-1} \right\rceil$, then $n < \sigma(n)$. We have shown that for each $n \in \mathbb{Z}$ there is a nonnegative integer m_0 such that

$$\left\lceil -\frac{d_M}{N-1} \right\rceil \leq \sigma^k(n) \leq \left\lfloor -\frac{d_m}{N-1} \right\rfloor$$

for all $k \geq m_0$. For any $n \in \mathbb{Z}$ the sequence $\left\{\sigma^k(n)\right\}_{k=0}^{\infty}$ is eventually in the set $F := \left\{\left\lceil -\frac{d_M}{N-1} \right\rceil, \dots, \left\lfloor -\frac{d_m}{N-1} \right\rfloor\right\}$. In other words, for $n \in \mathbb{Z}$, there is an element $f_0 \in F$ such that $n \sim f_0$. Hence the number of $\sim$-equivalence classes in $\mathbb{Z}$, or equivalently the number of irreducible subrepresentations of the representation of $\mathcal{O}_N$, $N < \infty$, coming from (2.2), is at most the cardinality of the set F, which is $1 + \left\lfloor -\frac{d_m}{N-1} \right\rfloor - \left\lceil -\frac{d_M}{N-1} \right\rceil$. □

Thus every representation of $\mathcal{O}_N$, $N < \infty$, on $\ell^2(\mathbb{Z})$ defined by (2.2) decomposes into finitely many irreducible subrepresentations. In contrast, the representations of $\mathcal{O}_N$, $N < \infty$, on $\ell^2(\mathbb{Z}^n)$ have examples which have infinitely many irreducible subrepresentations. Let $\mathbf{M}$ be an $n \times n$ integer matrix with $|\det \mathbf{M}| = N > 1$. Consider a digit set $\{d_0, d_1 \dots, d_{N-1}\}$, a complete residue set modulo $\mathbf{M}\mathbb{Z}^n$ in $\mathbb{Z}^n$. The representation of $\mathcal{O}_N$, $N < \infty$, on $\ell^2(\mathbb{Z}^n)$ that is defined by

$$S_i(e_x) = e_{\mathbf{M}x + d_i}, \quad i = 0, 1, \dots, N-1, \tag{3.1}$$

for $x \in \mathbb{Z}^n$ has a corresponding function system $\sigma_i(x) = \mathbf{M}x + d_i$ and $\sigma(x) = \mathbf{M}^{-1}(x - d_{i_0})$ for unique $d_{i_0} \in \{d_0, d_1, \dots, d_{N-1}\}$ such that $x = d_{i_0}$ modulo $\mathbf{M}\mathbb{Z}^n$ in $\mathbb{Z}^n$. Let U be the open closed hypercube $\prod_{i=1}^{n}\left(-\frac{1}{2}, \frac{1}{2}\right]$ in $\mathbb{R}^n$ and $C = \{0\} \cup \{(0, \dots, 0, x_i, 0, \dots, 0) \mid x_i = -1 \text{ or } 1 \text{ for } i = 1, \dots, n\}$.

Theorem 3.3. *If an $n \times n$ matrix $\mathbf{M}$ with integer entries satisfies $\lim_{k\to\infty} \mathbf{M}^k U = \mathbb{R}^n$ and $C \subset \mathbf{M}U \cap \mathbb{Z}^n$, then the representation of $\mathcal{O}_N$, $N < \infty$, on $\ell^2(\mathbb{Z}^n)$ defined by* (3.1) *using the digit set $D := \mathbf{M}y + (\mathbf{M}U \cap \mathbb{Z}^n)$, $y \in \mathbb{Z}^n$, is an irreducible representation.*

Proof. We first prove the case $y = 0 = (0, \dots, 0)$ in $\mathbb{Z}^n$. To this end, we construct a number system in $\mathbb{Z}^n$. Let $D_0^* := \mathbf{M}U \cap \mathbb{Z}^n$ and $D_1^* = \bigcup_{x \in D_0^*} \{\mathbf{M}x + d \mid d \in D_0^*\}$, and inductively

$$D_k^* = \bigcup_{x \in D_{k-1}^*} \{\mathbf{M}x + d \mid d \in D_0^*\}$$

for $k = 2, 3, \dots$. We have a strictly increasing sequence $\{D_k^*\}_{k=0}^{\infty}$, and furthermore $\mathbf{M}^k U \cap \mathbb{Z}^n$ is a subset of D_{k+1}^* for every $k = 0, 1, 2, \dots$. Since $\lim_{k\to\infty} \mathbf{M}^k U = \mathbb{R}^n$, we have $\lim_{k\to\infty} \mathbf{M}^k U \cap \mathbb{Z}^n = \mathbb{Z}^n$ and $\lim_{k\to\infty} D_k^* = \mathbb{Z}^n$. Now, for any given $x \in \mathbb{Z}^n$, there exists a unique positive integer l such that $x \in D_l^*$ and $x \notin D_{l-1}^*$. From the construction of the sequence $\{D_k^*\}_{k=0}^{\infty}$, there exist a unique $x_1 \in D_{l-1}^*$ and $d_l \in D_0^*$ such that $x = \mathbf{M}x_1 + d_l$. By induction on l, we can establish that

$$x = \mathbf{M}^l d_l + \mathbf{M}^{l-1} d_{l-1} + \dots + \mathbf{M}d_1 + d_0,$$

with $d_0, d_1, \dots, d_l \in D_0^*$. Thus every $x \in \mathbb{Z}^n$ has a polynomial form in base $\mathbf{M}$, where $\mathbf{M}$ is an $n \times n$ integer matrix. Note that $\sigma(x) = \mathbf{M}^{l-1} d_l + \dots + \mathbf{M}d_2 + d_1$, and by induction, $\sigma^l(x) = d_l$. Therefore $\sigma^k(x) = 0$ for $k \geq l + 1$, where $0 = (0, \dots, 0)$ in $\mathbb{Z}^n$, i.e., every $x \in \mathbb{Z}^n$ is $\sim$-equivalent to 0. By Theorem 2.1, the representation of $\mathcal{O}_N$, $N < \infty$, on $\ell^2(\mathbb{Z}^n)$ is irreducible if $y = 0$.

Now we assume $y \neq 0$ and $\mathbf{M}y = \mathbf{M}^t a_t + \cdots + \mathbf{M}a_1$, so that every $d_i \in D$ can be expressed as

$$d_i = \mathbf{M}^t a_t + \cdots + \mathbf{M}a_1 + a_{0i},$$

where $a_{0i} \in D_0^*$, $i = 0, 1, \ldots, N-1$. Since

$$M = m = t$$

with

$$M = \max\{\deg(d_i) \mid d_i \in D,\ i = 0, 1, \ldots, N-1\},$$
$$m = \min\{\deg(d_i) \mid d_i \in D,\ i = 0, 1, \ldots, N-1\},$$

for every $x \in \mathbb{Z}^n$, there exists a positive integer n_x such that $\deg\left(\sigma^k(x)\right) = t-1$ for every $k \geq n_x$. For $x \in \mathbb{Z}^n$ with $\deg(x) = t-1$, we may express x as $x = \mathbf{M}^{t-1} b_{t-1} + \cdots + \mathbf{M}b_1 + b_0$, where $b_0, b_1, \ldots, b_{t-1} \in D_0^*$. Then

$$\begin{aligned} \sigma(x) &= \mathbf{M}^{-1}(x - d_{i_0}) \\ &= -\mathbf{M}^{t-1} a_t + \mathbf{M}^{t-2}(b_{t-1} - a_{t-1}) + \cdots + b_1 - a_1, \end{aligned}$$

where $d_{i_0} = \mathbf{M}^t a_t + \cdots + \mathbf{M}a_1 + a_0$ with $a_0 = b_0$. Therefore $\sigma^k(x) = -\mathbf{M}^{t-1}a_t - \mathbf{M}^{t-2}a_t - \cdots - \mathbf{M}a_t - a_t$ for large k. As a consequence, every $x \in \mathbb{Z}^n$ is $\sim$-equivalent to the point $-\mathbf{M}^{t-1}a_t - \mathbf{M}^{t-2}a_t - \cdots - \mathbf{M}a_t - a_t$, which implies that every representation of $\mathcal{O}_N$, $N < \infty$, of the type described is an irreducible representation. □

Corollary 3.4. *If an $n \times n$ matrix $\mathbf{M}$ with integer entries satisfies $\lim_{k\to\infty} \mathbf{M}^k U = \mathbb{R}^n$ and $C \subset \mathbf{M}U \cap \mathbb{Z}^n$, then the representation defined by* (3.1) *using any digit set D which is a residue set modulo $\mathbf{M}\mathbb{Z}^n$ in $\mathbb{Z}^n$ decomposes into finitely many irreducible subrepresentations.*

Proof. The proof follows from Theorem 2.2 and the first half of the proof of Theorem 3.3. □

Remark 3.5. If the set C is not a subset of $\mathbf{M}U \cap \mathbb{Z}^n$, the element x_1 is not uniquely determined in the proof of Theorem 3.3, but the representation coming from (3.1) still has a finite irreducible decomposition.

Theorem 3.6. *If an $n \times n$ integer matrix $\mathbf{M}$ satisfies the condition $\lim_{k\to\infty} \mathbf{M}^k U = \mathbb{R}^n$, then the representation coming from* (3.1) *with digit set $D := \mathbf{M}U \cap \mathbb{Z}^n$ decomposes into finitely many irreducible subrepresentations. In fact, there are at most $2n+1$ such irreducible subrepresentations. Loosely speaking, if $\lim_{k\to\infty} \mathbf{M}^k U = \mathbb{R}^n$, the corresponding representation with any residue set D modulo $\mathbf{M}\mathbb{Z}^n$ as a digit set decomposes into finitely many irreducible subrepresentations.*

Proof. Let us define two sequences $\{G_k\}_{k=0}^{\infty}$ and $\{H_k\}_{k=0}^{\infty}$ of sets in $\mathbb{R}^n$ and $\mathbb{Z}^n$, respectively. Let

$$G_0 = 0 = H_0 \quad \text{in } \mathbb{R}^n,$$

$$G_1 = \left(\bigcup_{x \in D} (x + U)\right) \cup \left(\bigcup_{x \in C - D} (x + U)\right) \quad \text{and} \quad H_1 = G_1 \cap \mathbb{Z}^n,$$

and then, inductively,

$$G_k = \bigcup_{x \in H_{k-1}} \{\sigma_i(x) + U \mid i = 0, 1, \ldots, N-1\} \quad \text{and} \quad H_k = G_k \cap \mathbb{Z}^n,$$

for $k = 2, 3, \ldots$. We then see that $\mathbf{M}^k U$ is a subset of G_k for every k. Since $\lim_{k\to\infty} \mathbf{M}^k U = \mathbb{R}^n$, it follows that $\lim_{k\to\infty} G_k = \mathbb{R}^n$ and $\lim_{k\to\infty} H_k = \mathbb{Z}^n$. For any $x \in \mathbb{Z}^n$, there is a positive integer k such that $x \in H_l$ for $l \geq k$ and $x \notin H_{k-1}$. Because the sequence $\{H_k\}_{k=0}^{\infty}$ has been constructed by an iteration starting with the set H_1, the sequence $\sigma^t(x)$, $x \in \mathbb{Z}^n$, eventually converges in the set H_1. However, every $x \in D \subset H_1$ is $\sim$-equivalent to 0 by construction of H_1, so D is a subset of the 0 $\sim$-equivalence class. Thus the number of irreducible subrepresentations is at most $1 + \#(C - D)$. Since $\#C = 2n + 1$ and $\#D = N$, if $D \subset C$, then $1 + \#(C - D) \leq 1 + (2n + 1) - N = 2n + 2 - N$. In general, the number of irreducible subrepresentations is at most $1 + \#(C - D) \leq 2n + 1$. The last statement follows from the proof of Theorem 2.2 using the set H_1 instead of the set $\{a_0, a_1, \ldots, a_{N-1}\}$, with the necessary changes. This completes the proof. $\square$

Example 1 (of Theorem 3.3). Let $\mathbf{M} = \left(\begin{smallmatrix} 2 & 1 \\ -1 & 2 \end{smallmatrix}\right)$ with $N = 5$ and $D := \mathbf{M}U \cap \mathbb{Z}^2 = \left\{\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right\}$. Here $D = C$ and $x \sim 0$ for every $x \in \mathbb{Z}^2$. The representation of $\mathcal{O}_5$ defined by (3.1) on $\ell^2(\mathbb{Z}^2)$ is irreducible.

Example 2 (of Theorem 3.6). Let $\mathbf{M} = \left(\begin{smallmatrix} 2 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ with $N = 3$ and $D := \mathbf{M}U \cap \mathbb{Z}^2 = \left\{\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right\}$. Note that both eigenvalues of $\mathbf{M}$ have modulus greater than 1, so $\lim_{k\to\infty} \mathbf{M}^k U = \mathbb{R}^2$, but $C \supsetneq D$. In fact, for any $x \in \mathbb{Z}^2$ only one of the following is true:

$$\text{either } x \sim 0 \text{ or } x \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ or } x \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The number of irreducible subrepresentations is $3 = 2n + 2 - N$ with $n = 2$ and $N = 3$.

Example 3 (of having infinitely many irreducible subrepresentations). Let $\mathbf{M} = \left(\begin{smallmatrix} 3 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ with $N = 4$ and $D := \mathbf{M}U \cap \mathbb{Z}^2 = \left\{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)\right\}$. Note that the eigenvalues of $\mathbf{M}$ are 4 and 1, which implies that $\lim_{k\to\infty} \mathbf{M}^k U$ is a proper subset of $\mathbb{R}^2$. Since $\left(\begin{smallmatrix} z \\ -z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ modulo $\mathbf{M}\mathbb{Z}^2$ in $\mathbb{Z}^2$, $\sigma\left(\begin{smallmatrix} z \\ -z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3 & 2 \\ 1 & 2 \end{smallmatrix}\right)^{-1}\left(\left(\begin{smallmatrix} z \\ -z \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} z \\ -z \end{smallmatrix}\right)$. Thus the $\sim$-equivalence classes coming from $\left(\begin{smallmatrix} z \\ -z \end{smallmatrix}\right)$, $z \in \mathbb{Z}$, are mutually disjoint. Thus, the representation of $\mathcal{O}_4$ defined by (3.1) on $\ell^2(\mathbb{Z}^2)$ has countably many irreducible subrepresentations.

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Department of Mathematics, College of Natural Sciences, Seoul National University, Seoul 151-742, Korea

E-mail address: jeong@cau.ac.kr