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AN UNCERTAINTY INEQUALITY INVOLVING L^1 NORMS

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ABSTRACT. We derive a sharp uncertainty inequality of the form

$$\|x^2 f\|_1 \, \|\xi \hat{f}\|_2^2 \geq \frac{\Lambda_0}{4\pi^2} \, \|f\|_1 \, \|f\|_2^2,$$

with $\Lambda_0 = 0.428368\ldots$. As a consequence of this inequality we derive an upper bound for the so-called *Laue constant*, that is, the infimum λ_0 of the functional $\lambda(p) = 4\pi^2 \|x^2 p\|_1 \|x^2 \hat{p}\|_1 / (p(0)\hat{p}(0))$, taken over all $p \geq 0$ with $\hat{p} \geq 0$ $(p \not\equiv 0)$. Precisely, we obtain that $\lambda_0 \leq 2\Lambda_0 = 0.85673673\ldots$, which improves a previous bound of T. Gneiting.

1. INTRODUCTION

Denote by H the space of real functions $f \in L^2 \cap AC(\boldsymbol{R})$ with $f' \in L^2(\boldsymbol{R})$, $f \not\equiv 0$, and having finite variance: $\int_{\boldsymbol{R}} x^2 |f(x)| dx < \infty$. For $f \in H$ the following functional is well-defined:

$$\Lambda(f) := \frac{\|x^2 f\|_1 \|f'\|_2^2}{\|f\|_1 \|f\|_2^2}. \tag{1}$$

We now state the main result of this paper:

Theorem 1. *The functional $\Lambda(f)$ attains its minimum value at (and only at) the functions $f(x) = \alpha f_0(\beta x)$, where $\alpha, \beta \in \boldsymbol{R} \setminus 0$ and where f_0 is the following non-negative $C^1(\boldsymbol{R})$ function:*

$$f_0(x) = \begin{cases} \cos x - \cos m_0 + \dfrac{\sin m_0}{2m_0}(x^2 - m_0^2) & \text{for } |x| < m_0, \\ 0 & \text{for } |x| \geq m_0, \end{cases} \tag{2}$$

the number $m_0 = 1.750246\ldots$ being the only positive root of the equation

$$m(5 - 2m^2)\tan^2 m + 5(3 - 2m^2)\tan m - 15m = 0. \tag{3}$$

In particular, for all $f \in L^1 \cap L^2$

$$\|f\|_1 \|f\|_2^2 \leq \Lambda_0^{-1} \, \|x^2 f\|_1 \|f'\|_2^2 \tag{4}$$

with $\Lambda_0 = \Lambda(f_0) = 0.428368\ldots$, and with equality if and only if $f(x) = \alpha f_0(\beta x)$, $\alpha, \beta \in \boldsymbol{R}$, $\beta \neq 0$.

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A few comments about the properties of Λ: 1) $\Lambda(f)$ is invariant under two-sided dilations (normalization and scaling):

$$f_\beta(x) := f(\beta x) \Longrightarrow \Lambda(\alpha f_\beta) = \Lambda(f), \qquad \forall \alpha, \beta \neq 0; \tag{5}$$

2) one only needs to consider nonnegative f; 3) the functional is decreased by replacing $f \geq 0$ with its symmetric decreasing rearrangement f^*; 4) if $f \geq 0$ and symmetric decreasing, then, after suitable normalizations, a global decrease of f' implies an increase of the variance, and vice versa; 5) we can make the functional as large as we please, for example with a local increase of $|f'|$.

The above estimate can be formulated in terms of uncertainty inequalities as follows. Let $\hat{f}(\xi) = \int_{\boldsymbol{R}} e^{-2\pi i \xi x} f(x) dx$ be the Fourier transform of $f \in L^1(\boldsymbol{R})$; then (4) can be written as

$$\|x^2 f\|_1 \, \|\xi \hat{f}\|_2^2 \geq \frac{\Lambda_0}{4\pi^2} \, \|f\|_1 \, \|f\|_2^2, \qquad f \in L^1 \cap L^2, \tag{6}$$

which is then a variation of the classical Heisenberg inequality $\|xf\|_2 \|\xi \hat{f}\|_2 \geq (4\pi)^{-1} \|f\|_2^2$. Other such variations have appeared in the literature (see the recent survey paper by Folland and Sitaram [FS] for an extensive discussion) mostly with mixed L^p norms on the left and a lower bound in terms of the L^2 norm on the right. In this version we have a mixed L^1, L^2 lower bound, and the inequality is sharp.

Theorem 1 came about in our efforts to solve a problem posed by H.J. Rossberg [R]: *Among all positive definite densities $p \geq 0$, find the minimum of the functional*

$$\lambda(p) = 4\pi^2 \, \frac{\operatorname{Var}(p) \operatorname{Var}(\hat{p})}{\hat{p}(0) p(0)}, \tag{7}$$

where, of course, $\operatorname{Var}(p) = \int_{\boldsymbol{R}} x^2 p(x) dx$. Recall that a positive definite density p satisfies $\hat{p} \geq 0$, and is continuous and symmetric. Clearly $p/\hat{p}(0)$ and $\hat{p}/p(0)$ are both probability densities (the normalization constant $4\pi^2$ is there to match the original definition of $\lambda(p)$, given in [R]). The *Laue constant* is defined as

$$\lambda_0 = \inf \lambda(p)$$

and the best known bounds for it are

$$0.5276 \cdots < \lambda_0 < 0.8570750 \ldots, \tag{8}$$

the left one obtained by Drier [D], and the right one by Gneiting [G], who worked with explicit positive definite functions given by a convolution $p = f * f$, where f is positive, even, and supported in an interval. This last approach is, in effect, a natural way of producing positive definite densities: $p \in L^1$ is a positive definite density if and only if $p = f * f$, with $f \in L^2$ and even. It thus seemed natural to us to try and formulate this problem in terms of the self-convolved function f, indeed, if we define

$$\widetilde{\Lambda}(f) = \frac{2 \int_{\boldsymbol{R}} x^2 f \int_{\boldsymbol{R}} (f')^2}{\int_{\boldsymbol{R}} f \int_{\boldsymbol{R}} f^2} \tag{9}$$

for $f \in H$, then a straightforward calculation shows that

$$\lambda(f * f) = \widetilde{\Lambda}(f) \tag{10}$$

provided $f * f \geq 0$ and not a.e. 0. Vice versa, since every positive definite p with $\mathrm{Var}(p), \mathrm{Var}(\hat{p}) < \infty$ can be written as above we obtain that

$$\lambda_0 = \inf \left\{ \widetilde{\Lambda}(f) : f \in H, \quad f \text{ even }, \quad f * f \geq 0 \right\}. \tag{11}$$

Noticing that $2\Lambda(f) = \widetilde{\Lambda}(f)$ for nonnegative f, and that $f_0 \geq 0$ we obtain an improvement on the upper bound of (8):

Corollary 1.

$$\lambda_0 \leq 2\Lambda_0 = 0.85673673 \ldots .$$

It is important to observe that inequality (4) without the sharp constant can be obtained from the one-dimensional Nash inequality

$$\|f\|_2^6 \leq \frac{27}{16\pi^2} \|f'\|_2^2 \|f\|_1^4. \tag{14}$$

In the present form Nash's inequality is sharp, and the extremals are obtained by translation, normalization and scaling of the function $(1 + \cos \pi x)\, \chi_{[-1,1]}(x)$. This result is due to Nagy [N]; much later Carlen and Loss [CL] derived the n-dimensional sharp version. The link with (4) is provided by the following Carlson's type inequality:

$$\|f\|_1^5 \leq \frac{125}{9} \, \|x^2 f\|_1 \, \|f\|_2^4 \tag{15}$$

with equality if and only if $f = (1 - x^2)\, \chi_{[-1,1]}(x)$, modulo normalization and scaling. For the proof of this and a more general inequality due to Levin see the encyclopedic book by Mitrinović, Pečarić and Fink [MPF], chapt. VIII, which also contains various other references (we point out, however, that the same technique used in this paper can be used to prove (15)). Multiplying (15) by Nash's inequality (14) gives

$$\|f\|_1 \|f\|_2^2 \leq \frac{3 \cdot 125}{16\pi^2} \, \|x^2 f\|_1 \|f'\|_2^2,$$

which is (4) but with the bigger constant $\frac{3 \cdot 125}{16\pi^2} = 2.3747 \cdots > 2.3344 \cdots = \Lambda_0^{-1}$. Observe that the extremal of (4) (up to normalization and scaling) is obtained as a linear combination of a specific scaling of the extremals of (14) and (15). Our proof of (4) is variational, but it would be interesting to try to obtain a more direct proof by somehow combining the proofs of (14) and (15). This putative proof could perhaps clarify the meaning of the constant Λ_0 and also provide a sharp form of the n-dimensional version of (4), i.e.

$$\|f\|_1 \|f\|_2^2 \leq C \big\| \, |x|^2 f \big\|_1 \|\nabla f\|_2^2, \tag{16}$$

which seems otherwise difficult to achieve using the present method. Regarding this last point, our proof of the existence of radially symmetric extremals of (4) can be easily generalized to n dimensions; it is also not hard to show that the extremals of (16) must be a linear combination of the extremals of the n-dimensional Nash inequality (essentially the Bessel functions $J_{(n-2)/2}$ suitably normalized and truncated, see [CL]) and the function $(1 - |x|^2)\, \chi_{\{|x| \leq 1\}}$ (up to scaling), but it seems hard to show uniqueness of such extremals.

2. Proof of Theorem 1

The proof follows a classical variational scheme: 1. characterize all solutions of the Euler equation, and 2. show that extremizers exist by a weak compactness argument. Prior to that, we are going to make some preliminary reductions. As remarked in the introduction it is enough to consider functions f which are nonnegative and symmetrically decreasing. The first assertion is obvious and the second follows from the inequality

$$\Lambda(f) \geq \Lambda(f^*) \tag{17}$$

where f^* is the symmetric decreasing rearrangement of a nonnegative f. This inequality in turn is a consequence of the following standard facts:

$$\int_{\boldsymbol{R}} x^2 f \geq \int_{\boldsymbol{R}} x^2 f^*, \qquad \int_{\boldsymbol{R}} (f')^2 \geq \int_{\boldsymbol{R}} [(f^*)']^2, \qquad \int_{\boldsymbol{R}} f^p = \int_{\boldsymbol{R}} (f^*)^p.$$

Now assume that f is a minimum for (1), and is nonnegative, even, and in the class H. We also assume that f is not identically zero. For simplicity we will adopt the following notation:

$$A = \int_{\boldsymbol{R}} x^2 f dx, \quad B = \int_{\boldsymbol{R}} (f')^2 dx, \quad C = \int_{\boldsymbol{R}} f dx, \quad D = \int_{\boldsymbol{R}} f^2 dx. \tag{18}$$

Consider perturbations of the form $f_\epsilon = f + \epsilon\phi$, where ϕ is a fixed, smooth, compactly supported function, with $\operatorname{supp}\phi \subset \operatorname{supp} f$. Observe that $\operatorname{supp} f$ is an interval (possibly unbounded) centered at the origin.

Clearly, for all ϵ sufficiently small f_ϵ is nonnegative and belongs to H. Since f is a minimum, it must be that $\frac{d}{d\epsilon}\big|_{\epsilon=0}\Lambda(f_\epsilon) = 0$, i.e.

$$\frac{1}{A}\int_{\boldsymbol{R}} x^2\phi dx + \frac{1}{B}\int_{\boldsymbol{R}} 2\phi' f' dx = \frac{1}{C}\int_{\boldsymbol{R}} \phi dx + \frac{1}{D}\int_{\boldsymbol{R}} 2\phi f dx, \tag{19}$$

all terms being finite by Hölder's inequality. But this means that f must be the weak solution, and therefore classical solution, of the differential equation

$$\frac{2}{B} f''(x) + \frac{2}{D} f(x) = \frac{1}{A}x^2 - \frac{1}{C}, \qquad x \in \operatorname{supp} f. \tag{20}$$

Since Λ is invariant under dilation we may assume that

$$B = D \tag{21}$$

(namely, consider the function $f(x\sqrt{D/B})$). Assuming (21) and the fact that f is even we get that f must satisfy

$$f(x) = \beta\cos x + \frac{B}{2A}x^2 - \frac{B}{2C} - \frac{B}{A}, \qquad x \in \operatorname{supp} f, \tag{22}$$

for some constant $\beta \neq 0$ (if $\beta = 0$ (20) cannot be satisfied). We may also assume that $\beta = 1$, since f satisfies (20),(21) if and only if f/β does. From this last equation we deduce immediately that $\operatorname{supp} f = [-m, m]$ for some $m \in \boldsymbol{R}$, $m > 0$, since f is symmetrically decreasing and integrable. Thus, if a minimum exists, it is to be found among 2-sided dilations of functions of the form

$$f(x) = \begin{cases} (\cos x - \cos m) + \alpha(x^2 - m^2) & \text{for } |x| < m, \\ 0 & \text{for } |x| \geq m, \end{cases} \tag{23}$$

where $\alpha, m \in \boldsymbol{R}$ and $m > 0$ are such that (20) and (21) hold. This means that the following system needs to be satisfied:

$$\begin{cases} E_1 := 2\alpha A - B = 0, \\ E_2 := 2C(\alpha m^2 - 2\alpha + \cos m) - B = 0, \\ E_3 := D - B = 0, \end{cases} \tag{24}$$

where clearly $E_j = E_j(\alpha, \beta, m)$, and A, B, C, D are constructed from (23) and (18). Explicit computations yield

$$\begin{aligned} A &= \tfrac{2}{3}(6m - m^3)\cos m + 2(m^2 - 2)\sin m - \tfrac{4}{15}\alpha m^5, \\ B &= 8\alpha m \cos m - 8\alpha \sin m - \tfrac{1}{2}\sin(2m) + \tfrac{8}{3}\alpha^2 m^3 + m, \\ C &= -2m\cos m + 2\sin m - \tfrac{4}{3}\alpha m^3, \\ D &= \tfrac{8}{3}(\alpha m^3 + 3\alpha m)\cos m + m\cos(2m) \\ &\quad - 8\alpha \sin m - \tfrac{3}{2}\sin(2m) + \tfrac{16}{15}\alpha^2 m^5 + 2m, \end{aligned} \tag{25}$$

and also

$$\begin{cases} E_1 + \tfrac{1}{2}E_3 = -m(\sin m - 2\alpha m)^2, \\ 5E_1 - E_2 = -4m(\sin m - 2\alpha m)^2. \end{cases} \tag{26}$$

This means that (24) is equivalent to

$$\begin{cases} \alpha = \dfrac{\sin m}{2m}, \\ D - B = 0 \end{cases} \tag{27}$$

(the first equation means that $f'(\pm m) = 0$). After some calculations this last system reduces to

$$G(m) := m(5 - 2m^2)\sin^2 m + 5(3 - 2m^2)\sin m \cos m - 15m\cos^2 m = 0, \tag{28}$$

and since this equation is not satisfied when $\cos m = 0$ we obtain that (24) is equivalent to (3):

$$H(m) := m(5 - 2m^2)\tan^2 m + 5(3 - 2m^2)\tan m - 15m = 0. \tag{29}$$

Now we show that (29) has only one positive root between $\pi/2$ and $(45/4)^{1/4} = 1.83142\ldots$. The solutions of (29) satisfy

$$\tan m = \frac{10m^2 - 15 \pm \sqrt{5(45 - 4m^4)}}{2m(5 - 2m^2)} \tag{30}$$

and from this we deduce that there are no real positive solutions for $m > (45/4)^{1/4}$.

Next we are going to show that there are no positive roots in the interval $(0, \pi/2)$. Consider the quantity $D - B$ computed from (25), prior to substituting $\alpha = \sin m/(2m)$:

$$D - B = \alpha^2 m^3(\tfrac{16}{15}m^2 + \tfrac{8}{3}) + \tfrac{8}{3}\alpha m^3 \cos m + m\cos(2m) - \sin(2m) + m. \tag{31}$$

Some more calculations yield that the discriminant of the above quadratic equation in α can be written as

$$-m^3 \cos m[m(-15 + m^2)\cos m + 3(5 - 2m^2)\sin m]. \tag{32}$$

This quantity is negative in $(0,\pi/2)$. To see this, we compute

(33)
$$\frac{d}{dm}(m(-15+m^2)\cos m+3(5-2m^2)\sin m)=-m(3m\cos m+(m^2-3)\sin m)$$

and

$$\frac{d}{dm}(3m\cos m+(m^2-3)\sin m)=m(m\cos m-\sin m)$$

which is negative in $(0,\pi/2)$. This implies that (33) is positive and (32) negative in $(0,\pi/2)$. Where the discriminant of (31) is negative, the system (27), and hence (29), does not have any solution.

To establish that $H(m)$ has only one root $m_0\in(\pi/2,3\pi/4)$ it is enough show that in that interval the function $G(m)$ in (28) is strictly decreasing. We compute

(34)
$$\begin{aligned} G'(m) &= -5-3m^2+(5-7m^2)\cos(2m)-2m(m^2-5)\sin(2m)\\ G''(m) &= 2m(-3+(3-2m^2)\cos(2m)+4m\sin(2m))\\ \frac{d}{dm}\left[\frac{G''(m)}{2m}\right] &= 4m\cos(2m)+2(-1+2m^2)\sin(2m). \end{aligned}$$

In the interval $(\pi/2,3\pi/4)$ the last quantity is negative so that G'' is negative (since $G''(\pi/2)<0$) and G' is negative (since $G'(\pi/2)<0$). This actually shows that m_0 is located between $\pi/2$ and $\sqrt[4]{45/4}<3\pi/4$; numerical evaluations yield $m_0=1.750246\ldots$, and $\Lambda_0(f_0)=0.428368\ldots$.

In summary, we have shown that if there is a minimum of Λ, then it must be a dilation of the function f_0 given as in (2).

Remark. We notice that an extra derivation in (34) gives

$$\frac{d^2}{dm^2}\left[\frac{G''(m)}{2m}\right]=8m^2\cos(2m)$$

and one can easily reconstruct G as follows:

$$G(m)=\tfrac{4}{3}m^7\int_0^1 v^2(1-v^2)(1-v)^2\cos(2mv)dv,$$

which shows in particular that $G(m)\sim\frac{2}{63}m^7$, as $m\to 0$.

Existence of the minimum. Let $f_k\in H$ be a sequence of nonnegative, symmetric decreasing functions such that $\Lambda(f_k)\to\Lambda_0$, where Λ_0 denotes the infimum of Λ over H. Assume first that $\Lambda_0\neq 0$, that is, (4) holds with *some* positive finite constant; the proof that $\Lambda_0\neq 0$ will follow from the argument below. After suitable normalization and scaling we can assume that

$$\text{a) } \|f_k'\|_2=1,\qquad \text{b) } \|x^2f_k\|_1=1$$

(the first is clear, the second one is obtained by a dilation of type $\sqrt{a}f_k(x/a)$, which leaves both $\Lambda(f_k)$ and $\|f_k'\|_2^2$ invariant). Hence $\|f_k\|_1\|f_k\|_2^2\to\Lambda_0^{-1}$. Proceeding as in [B] or [L], the fact that f_k is symmetric decreasing and b) imply that

(35)
$$f_k(r)\le\tfrac{3}{2}\,r^{-3}\,,\qquad \forall r>0$$

(since $1 \geq 2\int_0^r x^2 f_k(x)dx \geq \frac{2}{3}r^3 f_k(r)$) and from Helly's theorem we can find a subsequence (again denoted by f_k) such that for $r>0$ $f_k \to f$ almost everywhere, for some symmetric decreasing f. Now write for any $r>0$

$$f_k(0) - f_k(r) = \int_0^r (-f_k')dx.$$

From a) and Hölder's inequality $f_k(0) \leq f_k(r) + r^{1/2}$ and hence, using (35), we deduce that $f_k(r) \leq f_k(0) \leq \frac{5}{2}$ for $0 < r \leq 1$. This implies that $f_k(r) \leq \frac{5}{2}\min(1, r^{-3}) \in L^1 \cap L^2$, for any $r > 0$. Thus, by the dominated convergence theorem,

$$\|f\|_1 \|f\|_2^2 = \Lambda_0^{-1}. \tag{36}$$

and f is not zero a.e. Now we take a subsequence of f_k so that $f_k' \to h \in L^2$ weakly in L^2; by lower semicontinuity of the norms $\|h\|_2 \leq \liminf \|f_k'\|_2 = 1$. It is not hard to see that the distributional derivative of f is the function h and hence $f' \in L^2$ with $\|f'\|_2 \leq 1$. By Fatou's lemma we also have that $\|x^2 f\|_1 \leq 1$, thus $f \in H$. From (36) it must hold that $\|x^2 f\|_1 = \|f'\|_2 = 1$ and hence the function f is an extremal.

To show $\Lambda_0 \neq 0$ we simply use the above estimates for a specific function $f \in H$ (nonnegative and symmetric decreasing). Namely, if $\|f'\|_2 = \|x^2 f\|_1 = 1$, then $f(r) \leq \frac{5}{2}\min(1, r^{-3})$, and hence $\|f\|_1 \|f\|_2^2 \leq C$ for some positive finite constant C.

Added in proof

1. After this paper was submitted and accepted for publication W. Ehms, T. Gneiting, and D. Richards informed us that they had independently found the function f_0, with essentially the same variational approach as ours. Furthermore, by means of a very simple trick they were able to produce the slightly better upper bound $\lambda_0 \leq 0.8567341\ldots < 2\Lambda_0$ (W. Ehms, T. Gneiting, and D. Richards, *On the uncertainty relation for positive-definite probability densities*, II, Statistics, to appear).

2. The same techniques used in this paper can be used to derive another sharp uncertainty inequality:

$$\|f\|_2 \leq C_0 \|x^2 f\|_1^{2/7} \|\xi \hat{f}\|_2^{5/7}$$

where the sharp constant C_0 is attained (up to dilations) at the function f_0 defined as in (2) but with m being the first positive zero of the equation $\tan m = 2m/(2-m^2)$. Details will appear in a forthcoming note. The above inequality (without sharp constant) appears in [FS] as a special case of a general family of uncertainty inequalities.

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