PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 12, Pages 3513–3516
S 0002-9939(99)05081-9
Article electronically published on May 13, 1999

A MORPHISM OF INTERSECTION HOMOLOGY INDUCED BY AN ALGEBRAIC MAP

ANDRZEJ WEBER

(Communicated by Leslie Saper)

Abstract. Let $f : X \to Y$ be a map of algebraic varieties. Barthel, Brasselet, Fieseler, Gabber and Kaup have shown that there exists a homomorphism of intersection homology groups $f^* : IH^*(Y) \to IH^*(X)$ compatible with the induced homomorphism on cohomology. The crucial point in the argument is reduction to the finite characteristic. We give an alternative and short proof of the existence of a homomorphism f^*. Our construction is an easy application of the Decomposition Theorem.

Let X be an algebraic variety, $IH^*(X) = H^*(X; IC_X)$ its rational intersection homology group with respect to the middle perversity and IC_X the intersection homology sheaf which is an object of derived category of sheaves over X [GM1]. We have the homomorphism $\omega_X : H^*(X;\mathbb{Q}) \longrightarrow IH^*(X)$ induced by the canonical morphism of the sheaves $\omega_X : \mathbb{Q}_X \longrightarrow IC_X$.

Let $f : X \longrightarrow Y$ be a map of algebraic varieties. It induces a homomorphism of the cohomology groups. The natural question arises: Does there exist an induced homomorphism for intersection homology compatible with f^* ?

$$\begin{array}{ccc} IH^*(Y) & \xrightarrow{?} & IH^*(X) \\ \uparrow{\scriptstyle \omega_Y} & & \uparrow{\scriptstyle \omega_X} \\ H^*(Y;\mathbb{Q}) & \xrightarrow{f^*} & H^*(X;\mathbb{Q}) . \end{array}$$

The answer is positive. For topological reasons the map in question exists for normally nonsingular maps [GM1, §5.4.3] and for placid maps [GM3, §4]. The authors of [BBFGK] proved the following:

Theorem 1. *Let $f : X \longrightarrow Y$ be an algebraic map of algebraic varieties. Then there exists a morphism $\lambda_f : IC_Y \longrightarrow Rf_*IC_X$ such that the following diagram with the canonical morphisms commutes:*

$$\begin{array}{ccc} IC_Y & \xrightarrow{\lambda_f} & Rf_*IC_X \\ \uparrow{\scriptstyle \omega_Y} & & \uparrow{\scriptstyle Rf_*(\omega_X)} \\ \mathbb{Q}_Y & \xrightarrow{\alpha_f} & Rf_*\mathbb{Q}_X . \end{array}$$

Received by the editors February 24, 1998.
1991 *Mathematics Subject Classification.* Primary 14F32, 32S60; Secondary 14B05, 14C25.
Key words and phrases. Intersection homology, algebraic varieties, morphism.
The author was partially supported by KBN 2 P03A 01113 grant.

In fact, [BBFGK] proves the existence of a morphism $\mu_f : f^*IC_Y \to IC_X$, which is adjoint to λ_f.

The sheaf language can be translated to the following: an induced homomorphism of intersection homology exists in a functorial way with respect to the open subsets of Y. This means that there exists a compatible family of induced homomorphisms

$$f^*_{\lambda,U} : IH^*(f^{-1}U) \longrightarrow IH^*(U)\,,$$

which is also compatible with the family

$$f^*_{|U} : H^*(f^{-1}U;\mathbb{Q}) \longrightarrow H^*(U;\mathbb{Q})\,.$$

As shown in [BBFGK] the morphism λ_f (and μ_f) is not unique. It is not possible to choose the morphisms λ_f (or μ_f) in a functorial way with respect to all algebraic maps (p.160). The simplest counterexample is the inclusion $\{(0,0)\} \hookrightarrow \{(x_1,x_2) : x_1x_2 = 0\}$, which can be factored through the inclusions $\{(0,0)\} \hookrightarrow \{(x_1,x_2) : x_i = 0\}$ for $i = 1$ or 2.

We will give a short proof of the main theorem from [BBFGK]. We will derive it from the Decomposition Theorem. The reference to the Decomposition Theorem is [BBD, 6.2.8] (see also [GM2]) and in a slightly different context [Sa].

We will use only the following corollary from the Decomposition Theorem:

Corollary from the Decomposition Theorem. *Let $\pi : X \longrightarrow Y$ be a proper surjective map of algebraic varieties. Then IC_Y is a direct summand in $R\pi_*IC_X$.*

The idea of the proof of our theorem is simple; the essence is the argument similar to [BBFGK, Remarque pp.172–174]. We take a resolution $\pi_Y : \widetilde{Y} \longrightarrow Y$ and enlarge the space X to obtain a map $\tilde{f} : \widetilde{X} \longrightarrow \widetilde{Y}$. There exists the induced morphism of intersection homology $\lambda_{\tilde{f}}$ for $\tilde{f}$. By the Decomposition Theorem the intersection homology of X (and Y) is a direct summand of intersection homology of $\widetilde{X}$ (resp. $\widetilde{Y}$). We compose $\lambda_{\tilde{f}}$ with the projection and inclusion in the direct sums to obtain the desired morphism λ_f.

Remark. If we insisted, then $\widetilde{X}$ might be even smooth of the same dimension as X with the map $\pi_X : \widetilde{X} \longrightarrow X$ generically finite; compare [BBFGK, p.173].

Proof of Theorem 1. We may assume that X and Y are irreducible. Let $\pi_Y : \widetilde{Y} \longrightarrow Y$ be a resolution of Y. Denote by $\widetilde{X}$ the fiber product (pull-back) $X \times_Y \widetilde{Y}$. Note that it is a variety, which may be singular and not equidimensional. We have a commutative diagram of algebraic maps (π_X and π_Y proper):

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

and the associated diagram of sheaves over Y:

$$\begin{array}{ccccccc} R\pi_{Y*}IC_{\widetilde{Y}} & = & R\pi_{Y*}\mathbb{Q}_{\widetilde{Y}} & \xrightarrow{R\pi_{Y*}(\alpha_{\tilde{f}})} & Rf_*R\pi_{X*}\mathbb{Q}_{\widetilde{X}} & \xrightarrow{Rf_*R\pi_{X*}(\omega_{\widetilde{X}})} & Rf_*R\pi_{X*}IC_{\widetilde{X}} \\ ?\uparrow & & \uparrow{\scriptstyle \alpha_{\pi_Y}} & & \uparrow{\scriptstyle Rf_*(\alpha_{\pi_X})} & & \downarrow ? \\ IC_Y & \xleftarrow{\omega_Y} & \mathbb{Q}_Y & \xrightarrow{\alpha_f} & Rf_*\mathbb{Q}_X & \xrightarrow{Rf_*(\omega_X)} & Rf_*IC_X\,. \end{array}$$

To prove the existence of a morphism $\lambda_f : IC_Y \longrightarrow Rf_*IC_X$, we will show that the arrows with question marks exist in a way that the diagram remains commutative. The existence of such morphisms follows from the Decomposition Theorem for π_Y and π_X (see the corollary). The sheaf IC_Y is a direct summand in $R\pi_{Y*}IC_{\widetilde{Y}}$:

$$i : IC_Y \hookrightarrow R\pi_{Y*}IC_{\widetilde{Y}} .$$

We also have a projection:

$$p : R\pi_{X*}IC_{\widetilde{X}} \twoheadrightarrow IC_X ,$$

which induces

$$Rf_*(p) : Rf_*R\pi_{X*}IC_{\widetilde{X}} \twoheadrightarrow Rf_*IC_X .$$

It remains to prove the commutativity of the diagram. We compare the morphisms over Y:

$$\mathbb{Q}_Y \xrightarrow{\omega_Y} IC_Y \xrightarrow{i} R\pi_{Y*}IC_{\widetilde{Y}} = R\pi_{Y*}\mathbb{Q}_{\widetilde{Y}}$$

and the natural one

$$\mathbb{Q}_Y \xrightarrow{\alpha_{\pi_Y}} R\pi_{Y*}\mathbb{Q}_{\widetilde{Y}} .$$

Respectively over X we compare the morphisms:

$$\mathbb{Q}_X \xrightarrow{\alpha_{\pi_X}} R\pi_{X*}\mathbb{Q}_{\widetilde{X}} \xrightarrow{R\pi_{X*}(\omega_{\widetilde{X}})} R\pi_{X*}IC_{\widetilde{X}} \xrightarrow{p} IC_X$$

and the canonical one

$$\mathbb{Q}_X \xrightarrow{\omega_X} IC_X .$$

Let U (resp. V) be the regular part of Y (resp. X). After multiplication by a constant if necessary, these morphisms are equal on U (resp. on V). We will show that an equality of morphisms over an open set implies the equality over the whole space. We have the restriction morphism

$$\begin{array}{ccc} Hom(\mathbb{Q}_Y, R\pi_{Y*}\mathbb{Q}_{\widetilde{Y}}) & \xrightarrow{\rho_U} & Hom((\mathbb{Q}_Y)_{|U}, (R\pi_{Y*}\mathbb{Q}_{\widetilde{Y}})_{|U}) \\ \| & & \| \\ H^0(\widetilde{Y}) & & H^0(\pi_Y^{-1}(U)) . \end{array}$$

The kernel of ρ_U is $H^0(\widetilde{Y}, \pi_Y^{-1}(U))$, which is trivial. We have the same for the morphisms over X:

$$\begin{array}{ccc} Hom(\mathbb{Q}_X, IC_X) & \xrightarrow{\rho_V} & Hom((\mathbb{Q}_X)_{|V}, (IC_X)_{|V}) \\ \| & & \| \\ IH^0(X) & & IH^0(V) . \end{array}$$

The kernel is $IH^0(X, V) = 0$. □

Remark. The restriction morphisms ρ_U and ρ_V are in fact isomorphisms. The cokernel of ρ_U is contained in $H^1(\widetilde{Y}, \pi_Y^{-1}(U)) = H^{cl}_{2(\dim \widetilde{Y})-1}(\widetilde{Y} \setminus \pi_Y^{-1}(U))$ which is trivial for dimensional reasons. The second follows from [Bo, V.9.2 p.144] as noticed in [BBFGK, p.178].

Acknowledgement

I would like to acknowledge the hospitality of the Institut de Mathèmatiques de Luminy and to thank Prof. J. P. Brasselet personally for the encouragement to find a simpler proof of the main theorem from [BBFGK]. I would also like to thank the authors of [BBFGK] for correcting this note.

References

[Bo] A. Borel (ed.), *Intersection cohomology,* Progress in Mathematics Vol. 50 (A. Borel, ed.), Birkhäuser, 1984. MR **88d:**32024

[BBD] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux Pervers*, Astérisque **100** (1983). MR **86g:**32015

[BBFGK] G. Barthel, J.-P. Brasselet, K.-H. Fieseler, O. Gabber, L. Kaup, *Relèvement de cycles algébriques et homomorphismes associés en homologie d'intersection*, Ann. Math **141** (1995), 147-179. MR **96a:**14027

[GM1] M. Goresky, R. MacPherson, *Intersection homology II*, Invent. Math. **72** (1983), 77-130. MR **84i:**57012

[GM2] M. Goresky, R. MacPherson, *On the topology of complex algebraic maps*, Geometry La Rabida, Lecture Notes in Mathematics, vol. 961, Springer Verlag, N. Y., 1982, pp. 119-129. MR **85f:**32019

[GM3] M. Goresky, R. MacPherson, *Lefschetz fixed point theorem for intersection homology*, Comm. Math. Helv. **60** (1985), 366-391. MR **87f:**32030

[Sa] M. Saito, *Decomposition theorem for proper Kähler morphisms*, Tôhoku Math. J. **42** (1990), 127-148. MR **91j:**32042

Institute of Mathematics, Warsaw University, ul. Banacha 2, 02–097 Warszawa, Poland

E-mail address: `aweber@mimuw.edu.pl`