OPENNESS AND MONOTONEITY OF INDUCED MAPPINGS

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ABSTRACT. It is shown that for locally connected continuum $X$ if the induced mapping $C(f) : C(X) \to C(Y)$ is open, then $f$ is monotone. As a corollary it follows that if the continuum $X$ is hereditarily locally connected and $C(f)$ is open, then $f$ is a homeomorphism. An example is given to show that local connectedness is essential in the result.

All spaces considered in this paper are assumed to be metric. A mapping means a continuous function. We denote by $\mathbb{N}$ the set of all positive integers, and by $\mathbb{C}$ the complex plane. Given a space $S$, a point $c \in S$ and a number $\varepsilon > 0$, we denote by $B_S(c, \varepsilon)$ the open ball in $S$ with center $c$ and radius $\varepsilon$.

A continuum means a compact connected space. Given a continuum $X$ with a metric $d$, we let $2^X$ denote the hyperspace of all nonempty closed subsets of $X$ equipped with the Hausdorff metric $H$ defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}$$

(see, e.g., [5, (0.1), p. 1 and (0.12), p. 10]). Further, we denote by $C(X)$ the hyperspace of all subcontinua of $X$, i.e., of all connected elements of $2^X$, and by $\mathcal{F}_1(X)$ the hyperspace of singletons. The reader is referred to Nadler’s book [5] for needed information on the structure of hyperspaces.

Given a mapping $f : X \to Y$ between continua $X$ and $Y$, we consider mappings (called the induced ones)

$$2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y)$$

defined by

$$2^f(A) = f(A) \text{ for every } A \in 2^X \quad \text{and} \quad C(f)(A) = f(A) \text{ for every } A \in C(X).$$

A mapping between continua is said to be:

— open provided the image of an open subset of the domain is open in the range;
— monotone provided the point-inverses are connected;
— light provided the point-inverses are zero-dimensional.

The following theorem is the main result of this paper.

1. Theorem. Let a continuum $X$ be locally connected, and a mapping $f : X \to Y$ be such that the induced mapping $C(f) : C(X) \to C(Y)$ is open. Then $f$ is monotone.
Proof. Assume $f$ satisfies the assumptions of the theorem and that it is not monotone. Let $p$ and $q$ be two points of $X$ such that $f(p) = f(q)$ that belong to different components of $f^{-1}(f(p))$. By continuity of $f$ there is a positive $\varepsilon$ such that for every continuum $L \subset Y$ such that $f(p) \in L$ and $H(L, \{f(p)\}) < \varepsilon$ the components of $f^{-1}(L)$ containing $p$ and $q$ respectively are distinct. By local connectedness of $Y$ there is a continuum $V$ such that $f(p) \in \text{int } V$ and $H(V, \{f(p)\}) < \varepsilon$, i.e., $V \subset B_1(f(p), \varepsilon)$. Let $U_p$ and $U_q$ be components of $f^{-1}(V)$ containing $p$ and $q$ respectively. Since in locally connected continua components of open sets are open [4, §49, II, Theorem 4, p. 230], we conclude that $p \in \text{int } U_p$ and $q \in \text{int } U_q$. Let $\delta > 0$ be such that $B_X(p, \delta) \subset U_p$ and $B_X(q, \delta) \subset U_q$.

Let $\mathcal{B}$ be an order arc in $C(Y)$ from $\{f(p)\}$ to $Y$ through $V$. Define $A$ as a subset of $\mathcal{B}$ composed of all elements $L \in \mathcal{B}$ such that the component of $f^{-1}(L)$ containing $p$ is distinct from the component of $f^{-1}(L)$ containing $q$. Note that $V \in A$ and that if $L, L' \in \mathcal{B}$, $L \in A$ and $L' \subset L$, then $L' \in A$. Thus $A$ is a connected subset of $\mathcal{B}$ containing $\{f(p)\}$ and $V$. Since $\mathcal{B} \setminus A$ is closed, we see that $A$ is an open subset of $\mathcal{B}$. Let $Q = \sup A = \inf (\mathcal{B} \setminus A)$. Then $Q \in \text{cl } A \setminus A$. Denote by $P$ the component of $f^{-1}(Q)$ containing both $p$ and $q$. Openness of $C(f)$ implies that $f$ is open (see [3, Theorem 4.3, p. 243]; compare also [2, Theorem 3.2]), so $f(P) = Q$ [6, (7.5), p. 148]. We will show that $C(f)(B_{C(X)}(P, \delta))$ is not open in $C(Y)$. So, assume the contrary. Then there is a continuum $K \in B_{C(X)}(P, \delta)$ with $f(K) \in A$. Since $p, q \in P$ and $H(P, K) < \delta$, we have $K \cap U_p \neq \emptyset \neq K \cap U_q$. Then $U_p \cup K \cup U_q$ is a continuum containing both $p$ and $q$, whose image $f(U_p \cup K \cup U_q) = f(K)$ is in $A$, contrary to the definition of $A$. The proof is finished.

2. Corollary. Let a continuum $X$ be hereditarily locally connected, and a mapping $f : X \to Y$ be such that the induced mapping $C(f) : C(X) \to C(Y)$ is open. Then $f$ is a homeomorphism.

Proof. It is enough to show that monotone open mappings on hereditarily locally connected continua are homeomorphisms. Assume the contrary, and let $y \in Y$ be such that $f^{-1}(y)$ is a nondegenerate continuum in $X$. Let $\{y_n\}$ be an arbitrary sequence converging to $y$. Then continua $f^{-1}(y_n)$ tend to $f^{-1}(y)$, so $f^{-1}(y)$ is a nondegenerate continuum of convergence, contrary to hereditary local connectedness of $X$.

3. Example. There are a continuum $X$ and a mapping $f : X \to X$ such that $C(f) : C(X) \to C(X)$ is light and open, but not monotone.

Proof. Let $S = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. For $n \in \mathbb{N}$ put $X_n = S$, and let $\varphi_n : X_{n+1} \to X_n$ be defined by $\varphi_n(z) = z^3$. Then $X = \lim (X_n, \varphi_n)$ is the triodic solenoid. Define $f : X \to X$ by $f(\{z_1, z_2, \ldots\}) = \{z_1^2, z_2^2, \ldots\}$, and note that $f$ is well-defined. It has been proved in [1, Example 4.5] that the restriction $C(f)(C(X) \setminus \{X\})$ is two-to-one and $C(f)^{-1}(X)$ is a singleton. Thus $C(f)$ is light and it is not a homeomorphism. We will prove that $C(f)$ is open. To this aim it is enough to show that the mapping is interior at each point of its domain [6, p. 149], i.e., that for each $P \in C(X)$ and for each open neighborhood $U$ of $P$ in $C(X)$ we have $C(f)(P) \in \text{int } C(f)(U)$. For each $n \in \mathbb{N}$ let $f_n : X_n \to X_n$ be defined by $f_n(z) = z^2$ (and thus $f = \lim f_n$), and let $\pi_n : X \to X_n$ be the projection. Let $P \in C(X)$ be a proper subcontinuum of $X$. Then there exists an index $n \in \mathbb{N}$ such that $\pi_{n-1}(P)$ is a proper subcontinuum of $X_{n-1}$, so $\pi_n(P)$ is an arc of length less than $2\pi/3$. Let $U_n$ be an open arc in $X_n$ containing $\pi_n(P)$ and having its length still less
than $2\pi/3$. Then the set $V = \{ A \in C(X) : \pi_n(A) \in U_n \}$ is an open neighborhood of $P$ in $X$ such that the restriction $C(f)|V : V \rightarrow C(f)(V)$ is a homeomorphism onto the open set $C(f)(U) = \{ A \in C(X) : \pi_n(A) \in f(U_n) \}$ containing $C(f)(P)$. So interiority of $C(f)$ at $P$ is shown in the case $P \neq X$. To prove that $C(f)$ is interior at $X$ consider, for $n \in \mathbb{N}$, the sets $V_n = \{ A \in C(X) : \pi_n(A) = X_n \}$ and note that the family $\{ V_n : n \in \mathbb{N} \}$ is a local base of (closed) neighborhoods of $X$ on $C(X)$. So, it is enough to prove that $C(f)(V_n) \supset V_{n+1}$. To this end take $A \in V_{n+1}$, and let $B \in X$ be such that $f(B) = A$. Since $f_{n+1}(\pi_{n+1}(B)) = \pi_{n+1}(f(B)) = \pi_{n+1}(A) = X_{n+1}$, we see that $\pi_{n+1}(B)$ is an arc in $X_{n+1}$ of length at least $\pi$. Thus $\pi_n(B) = \varphi_n(\pi_{n+1}(B)) = X_n$, i.e., $B \in V_n$, whence it follows that $A = f(B) \in C(f)(V_n)$. The proof is then complete.

In connection with Theorem 1 and Example 3 it would be interesting to know if a stronger result is true, namely whether or not the conclusion of Theorem 1 can be deduced from local connectedness of $Y$ only (without assuming local connectedness of $X$). In other words we have the following question.

4. Question. Can the assumption of local connectedness of the domain continuum $X$ be relaxed to that of the range continuum $Y$ in Theorem 1?

REFERENCES


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