PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 12, Pages 3671–3676
S 0002-9939(99)05397-6
Article electronically published on July 12, 1999

THE CONNECTED STABLE RANK OF THE PURELY INFINITE SIMPLE C^*-ALGEBRAS

YIFENG XUE

(Communicated by David R. Larson)

ABSTRACT. Suppose that $\mathcal{A}$ is a unital purely infinite simple C^*-algebra. If the class [1] of the unit 1 in $K_0(\mathcal{A})$ has torsion, then $\mathrm{csr}(\mathcal{A}) = \infty$; if [1] is torsion-free in $K_0(\mathcal{A})$, then $\mathrm{csr}(\mathcal{A}) = 2$. If $\mathcal{A}$ is a non-unital purely infinite simple C^*-algebra, then $\mathrm{csr}(\mathcal{A}) = 2$.

Before considering the connected stable rank of the purely infinite simple C^*-algebra, we need to introduce some notation as follows. For the C^*-algebra $\mathcal{A}$ with unit 1, we denote by $\mathcal{U}_n(\mathcal{A})$ the group of unitary elements in the matrix algebra $\mathrm{M}_n(\mathcal{A})$ and also denote by $\mathcal{U}_n^0(\mathcal{A})$ the connected component of 1_n in $\mathcal{U}_n(\mathcal{A})$. We view $\mathcal{A}^n = \{(a_1, \dots, a_n)^T | a_i \in \mathcal{A}\}$ as the set of all $n \times 1$ matrices over $\mathcal{A}$. For $a = (a_1, \dots, a_n)^T$, we set $a^* = (a_1^*, \dots, a_n^*)$—the $1 \times n$ over $\mathcal{A}$. Put

$$\mathrm{S}_n(\mathcal{A}) = \left\{ (a_1, \dots, a_n)^T \in \mathcal{A}^n | a^* a = \sum_{i=1}^n a_i^* a_i = 1 \right\}.$$

$\mathrm{S}_n(\mathcal{A})$ has the base point $e_n = (1, 0, \dots, 0)^T \in \mathcal{A}^n$. Let $\pi_0(\mathrm{S}_n(\mathcal{A}), e_n)$ denote the set of all path connected components of $\mathrm{S}_n(\mathcal{A})$. We identify the path connected component containing e_n with zero element "0". Thus from [Sr], we have

$$\mathrm{csr}(\mathcal{A}) = \min\{n | \pi_0(\mathrm{S}_m(\mathcal{A}), e_m) = 0, \ \forall m \geq n\}$$

where

$$\mathrm{csr}(\mathcal{A}) = \min\{n | \mathcal{U}_m^0(\mathcal{A}) \text{ acts transitively on } \mathrm{S}_m(\mathcal{A}), \ \forall m \geq n\}$$

is the connected stable rank of $\mathcal{A}$ defined in [Rf]. If no such integer exists, we set $\mathrm{csr}(\mathcal{A}) = \infty$. If $\mathcal{A}$ is not unital, we set $\mathrm{csr}(\mathcal{A}) = \mathrm{csr}(\mathcal{A}^+)$, where $\mathcal{A}^+$ is the C^*-algebra obtained from $\mathcal{A}$ by adjoining the unit 1.

Let p, q be two projections in the C^*-algebra $\mathcal{A}$. We write $p \sim q$ if there is $u \in \mathcal{A}$ such that $p = u^* u, q = u u^*$ and denote by $[p]$ the equivalence class of p with respect to "$\sim$" and we also put

$$D(\mathcal{A}) = \{[p] | p \text{ is a non-zero projection in } \mathcal{A}\}.$$

Recall that a projection p in $\mathcal{A}$ is called infinite if there is a projection q in $\mathcal{A}$ such that $p \sim q < p$. We say that $\mathcal{A}$ is purely infinite if the closure of $a\mathcal{A}a$ contains an infinite projection for any positive element a in $\mathcal{A}$ (cf. [Cu]).

Received by the editors August 1, 1997 and, in revised form, February 24, 1998.
1991 *Mathematics Subject Classification.* Primary 46L05.
Key words and phrases. Purely infinite simple C^*-algebras, connected stable rank, K-group of the C^*-algebras.

From [Cu] we have known that if $\mathcal{A}$ is the unital purely infinite simple C^*-algebra, then $D(\mathcal{A})$ becomes a group with the natural addition $[p]+[q]=[p'+q']$ and $D(\mathcal{A})\cong K_0(\mathcal{A})$ where $p'\sim p, q'\sim q, p'q'=0$ and $K_0(\mathcal{A})$ is the K_0-group of $\mathcal{A}$ defined in [Bk].

Let p be a non-zero projection of the purely infinite simple C^*-algebra $\mathcal{A}$ with unit 1. Set

$$S\mathcal{U}(\mathcal{A})=\{u\in\mathcal{A}|u^*u=1, uu^*<1\},\qquad S\mathcal{U}_p(\mathcal{A})=\{u\in S\mathcal{U}(\mathcal{A})|uu^*\le p\}.$$

If $p=1$, then $S\mathcal{U}_1(\mathcal{A})=\mathcal{U}(\mathcal{A})\cup S\mathcal{U}(\mathcal{A})$. In this situation, $S\mathcal{U}_1(\mathcal{A})$ is not connected for $\mathcal{U}(\mathcal{A})\cap S\mathcal{U}(\mathcal{A})=\varnothing$ and $\mathcal{U}(\mathcal{A}), S\mathcal{U}(\mathcal{A})$ are all closed in $S\mathcal{U}_1(\mathcal{A})$.

It is known from [Rf, Proposition 6.5] that the topological stable rank of $\mathcal{A}$ is ∞ if $\mathcal{A}$ is a purely infinite simple C^*-algebra. But what is the $\mathrm{csr}(\mathcal{A})$? In this paper, we will show that $\mathrm{csr}(\mathcal{A})\in\{2,\infty\}$. First we have the following known lemma which could be deduced directly from [Cu, Lemma 1.8]:

Lemma 1. *Suppose that $\mathcal{A}$ is a purely infinite simple C^*-algebra with unit 1. Then for $k\ge 2$ there are isometries $s_1,\dots,s_k$ in $\mathcal{A}$ such that $\sum_{i=1}^k s_is_i^*=p$ is a projection in $\mathcal{A}$. Furthermore, if $[1]$ is torsion-free in $K_0(\mathcal{A})$ or if $k\not\equiv 1 \bmod n$ when $[1]$ has order n $(1\le n<\infty)$ in $K_0(\mathcal{A})$, then $[p]\ne[1]$ and if $k\equiv 1\bmod n$ for above n, then p can be chosen as $p=1$.*

From Lemma 1, we can choose k isometries $s_1,\dots,s_k$ in the unital purely infinite simple C^*-algebra $\mathcal{A}$ such that $\sum_{i=1}^k s_is_i^*=p$ is a projection in $\mathcal{A}$. Now define a map ϕ_k of $\mathrm{M}_k(\mathcal{A})$ to $\mathcal{A}$ by

$$\phi_k((a_{ij})_{k\times k})=\sum_{i,j=1}^k s_ia_{ij}s_j^*+1-p.$$

It is easy to check that ϕ_k is a $*$-homomorphism with $\phi_k(1_k)=1$.

The following corollary somewhat enhances a partial result of [Cu, Lemma 1.8].

Corollary. *Let $\mathcal{A}$ be a purely infinite simple C^*-algebra with unit 1. Then for any $u\in\mathcal{U}_k(\mathcal{A})$ $(k\ge 2)$, there is $u_0\in\mathcal{U}_k^0(\mathcal{A})$ such that $u=u_0\,\mathrm{diag}(\phi_k(u),1_{k-1})$.*

Proof. Let $s_1,\dots,s_k$ be as above and put

$$X=\begin{pmatrix} s_1 & s_2 & \dots & s_k\\ 0 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 0\end{pmatrix},\qquad Y=\begin{pmatrix} X & 1_k-XX^*\\ 0 & X^*\end{pmatrix}.$$

Then it is clear that $X^*X=1_k$, $Y\in\mathcal{U}_{2k}(\mathcal{A})$ and moreover, by the definitions of ϕ_k and Y, we have the following identity:

$$Y\,\mathrm{diag}(u,1_k)Y^*=\mathrm{diag}(\phi_k(u),1_{2k-1}). \tag{1}$$

Noting that Y can be decomposed as the form

$$Y=\begin{pmatrix}1_k & X\\ 0 & 1_k\end{pmatrix}\begin{pmatrix}0 & 1_k-2XX^*\\ 1_k & 0\end{pmatrix}\begin{pmatrix}1_k & X^*\\ 0 & 1_k\end{pmatrix}\begin{pmatrix}1_k & 0\\ -X & 1_k\end{pmatrix},$$

we obtain that $Y\in\mathcal{U}_{2k}^0(\mathcal{A})$. Therefore applying [LZ, Condition (ii)] and [Lin, Lemma 2.2] to (1), we get the assertion. □

Let $s_1,\dots,s_k$ be as above. Then $(s_1^*u,\dots,s_k^*u)\in\mathrm{S}_k(\mathcal{A})$ for any $u\in S\mathcal{U}_p(\mathcal{A})$ because $uu^*\le p$ iff $pu=u$ for $u\in S\mathcal{U}_p(\mathcal{A})$. This leads to the following lemma.

Lemma 2. *The map $\beta\colon (S\mathcal{U}_p(\mathcal{A}), s_1) \to (\mathrm{S}_k(\mathcal{A}), e_k)$ given by $\beta(u) = (s_1^*u, \dots, s_k^*u)$ is homeomorphic.*

Proof. Obviously, β is continuous and $\beta(s_1) = e_k$. Now for any $(a_1, \dots, a_k) \in \mathrm{S}_k(\mathcal{A})$, put $u = \sum_{i=1}^k s_i a_i$. Since $s_i^* s_j = \delta_{ij} 1$ and $ps_i = s_i$, $i, j = 1, \dots, k$, it follows that $pu = u$ and

$$u^*u = \sum_{i,j} a_i^* s_i^* s_j a_j = \sum_{i=1}^k a_k^* a_k = 1,$$

i.e., $u \in S\mathcal{U}_p(\mathcal{A})$. Therefore we can define a continuous map $\gamma\colon (\mathrm{S}_k(\mathcal{A}), e_k) \to (S\mathcal{U}_p(\mathcal{A}), s_1)$ by $\gamma((a_1, \dots, a_k)) = \sum_{i=1}^k s_i a_i$. By the definitions of β and γ, the assertion follows. □

Using Lemma 1 and Lemma 2, we then establish the theorem in the following.

Theorem 1. *Suppose that $\mathcal{A}$ is a unital purely infinite simple C^*-algebra. If the class $[1]$ of the unit 1 in $K_0(\mathcal{A})$ has torsion in $K_0(\mathcal{A})$, then $\mathrm{csr}(\mathcal{A}) = \infty$; if $[1]$ is torsion-free in $K_0(\mathcal{A})$, then $\mathrm{csr}(\mathcal{A}) = 2$.*

Proof. We assume that n $(1 \le n < \infty)$ is the order of $[1]$ in $K_0(\mathcal{A})$, i.e., n is the least positive integer such that $n[1] = 0$ in $K_0(\mathcal{A})$.

If $k \equiv 1 \bmod n$, then $S\mathcal{U}_1(\mathcal{A})$ is not connected by Lemma 1 and consequently $\pi_0(\mathrm{S}_k(\mathcal{A}), e_k) \neq 0$ by Lemma 2. If $k \not\equiv 1 \bmod n$, then there are k isometries $s_1, \dots, s_k$ in $\mathcal{A}$ such that $\sum_{i=1}^k s_i s_i^* = p$ is a projection in $\mathcal{A}$ with $[p] \neq [1]$ by Lemma 1. In this case, we have $uu^* \neq p$ for all $u \in S\mathcal{U}_p(\mathcal{A})$.

Since $D(\mathcal{A})$ is a group and

$$[p] = [p - uu^*] + [uu^*] = [p - s_1 s_1^*] + [s_1 s_1^*]$$

in $D(\mathcal{A})$ for any $u \in S\mathcal{U}_p(\mathcal{A})$, it follows that $[p - uu^*] = [p - s_1 s_1^*]$ in $D(\mathcal{A})$. Noting that $uu^* \le p$, $s_1 s_1^* \le p$, we have $p - uu^*$, $p - s_1 s_1^* \in p\mathcal{A}p$ and $p - uu^* \sim p - s_1 s_1^*$ in $p\mathcal{A}p$. Since $us_1^* \in p\mathcal{A}p$ and $uu^* = (us_1^*)(us_1^*)^*$, $s_1 s_1^* = (us_1^*)^*(us_1^*)$ in $p\mathcal{A}p$, it follows that there is $w_0 \in \mathcal{U}_1(p\mathcal{A}p)$ such that $uu^* = w_0^* s_1 s_1^* w_0$. Put $a = s_1^* w_0^* u$ and $w = (s_i^* w_0 s_j)_{k \times k}$. Then

$$u = w_0 s_1 a, \qquad a \in \mathcal{U}_1(\mathcal{A}) \text{ and } w \in \mathcal{U}_k(\mathcal{A}). \tag{2}$$

Since $(s_1^*u, \dots, s_k^*u)^T = w(a, 0, \dots, 0)^T$ by (2), it follows from the Corollary that there exists $u_0 \in \mathcal{U}_k^0(\mathcal{A})$ such that

$$\begin{aligned}(s_1^*u, \dots, s_k^*u)^T &= u_0 \operatorname{diag}(\phi_k(w), 1_{k-1})(a, 0, \dots, 0)^T \\ &= u_0 \operatorname{diag}(w_0 + 1 - p, 1_{k-1})(a, 0, \dots, 0)^T \\ &= u_0 \operatorname{diag}(b, b^*, 1_{k-2}) e_k,\end{aligned}$$

where $b = (w_0 + 1 - p)a \in \mathcal{U}_1(\mathcal{A})$, that is, $(s_1^*u, \dots, s_k^*u)^T$ is in the component of e_k for $u_0 \operatorname{diag}(b, b^*, 1_{k-2}) \in \mathcal{U}_k^0(\mathcal{A})$. Therefore $\pi_0(\mathrm{S}_k(\mathcal{A}), e_k) = 0$ by Lemma 2 when $k \not\equiv 1 \bmod n$.

The above shows that $\mathrm{csr}(\mathcal{A}) = \infty$ when $[1]$ has torsion.

As to the case that $[1]$ is torsion-free in $K_0(\mathcal{A})$, we can use Lemma 1 and the same method as in the proof of $\pi_0(\mathrm{S}_k(\mathcal{A}), e_k) = 0$ when $k \not\equiv 1 \bmod n$ to deduce that $\pi_0(\mathrm{S}_k(\mathcal{A}), e_k) = 0$ $\forall k \ge 2$, i.e., $\mathrm{csr}(\mathcal{A}) \le 2$. Since $\mathcal{A}$ contains an isometry s_1, $\mathrm{csr}(\mathcal{A})$ must be equal to two. □

Let $\mathcal{O}_n$ be the Cuntz algebra. Then by Theorem 1, we have $\operatorname{csr}(\mathcal{O}_n) = \infty$ $(2 \le n < \infty)$, and $\operatorname{csr}(\mathcal{O}_\infty) = 2$ since $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ $(n < \infty)$, $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and the class [1] of unit 1 is the generator of $K_0(\mathcal{O}_n)$ (cf. [Cu]).

As an end of this paper, we will consider the $\operatorname{csr}(\mathcal{A})$ if $\mathcal{A}$ is a non-unital purely infinite simple C^*-algebra. Our result is the following:

Theorem 2. *Let $\mathcal{A}$ be a non-unital purely infinite simple C^*-algebra. Then* $\operatorname{csr}(\mathcal{A}) = 2$.

In order to prove this theorem, we need the following lemmas.

Lemma 3. *Let $\mathcal{A}$ be a C^*-algebra with unit 1 and let $a = (a_1, \dots, a_n)^T$, $b = (b_1, \dots, b_n)^T \in \mathrm{S}_n(\mathcal{A})$ such that $\|a_i - b_i\| \le (\sum_{i=1}^n \|a_i\|)^{-1}$. Then a, b are in the same component of $\mathrm{S}_n(\mathcal{A})$.*

Proof. Put $c_t = ((1-t)a + tb)^*((1-t)a + tb)$ $\forall t \in [0,1]$. Since $a^*a = b^*b = 1$, it follows that

$$\|1 - c_t\| = \|(1-t)t(a^*(b-a) + (b-a)^*a)\| \le \tfrac{1}{2} \quad \forall t \in [0,1].$$

Set $G_t = ((1-t)a + tb)(c_t)^{-1/2}$. Then G_t is the path from a to b in $\mathrm{S}_n(\mathcal{A})$. □

Lemma 4. *Suppose that $\mathcal{A}$ is a purely infinite simple C^*-algebra. Then there exists a in $\mathcal{A}$ with $0 \le a < 1$ such that $\overline{a\mathcal{A}a}$ is non-unital.*

Proof. Choose a non-trivial projection q in $\mathcal{A}$. Since $\mathcal{A}$ is purely infinite simple, there is a projection q_0 in $\mathcal{A}$ such that $q_0 < q$ and $q_0 \sim q$. Using the same method as in the proof of [Cu, Lemma 1.8], we can find a sequence of pairwise orthogonal projections $r_i < q - r_0$ in $q\mathcal{A}q$ such that $r_i \sim q$, $i \ge 0$. Let $\{\beta_n\}_0^\infty$ be a sequence in $(0,1)$ with $\lim_{n\to\infty} \beta_n = 0$. Set $a = \sum_{n=0}^\infty \beta_n r_n$. Then $a \in \mathcal{A}$ and $0 \le a < 1$.

We now claim that $\overline{a\mathcal{A}a}$ is non-unital for such a. If the statement is not true, then there is a projection R in $\overline{a\mathcal{A}a}$ such that R is the unit of $\overline{a\mathcal{A}a}$. Thus we have $Ra^3 = a^3$ and $\|R = axa\| < \frac{1}{2}$ for some $x \in \mathcal{A}$. Consequently, $r_n \le R$ and

$$\|r_n - \beta_n^2 r_n x r_n\| \le \|r_n(R - axa)r_n\| < \tfrac{1}{2}.$$

This implies that $\beta_n^2\|x\| \ge \beta_n^2\|r_n x r_n\| > \|r_n\| - \frac{1}{2} = \frac{1}{2}$ which is contrary to the assumption that $\lim_{n\to\infty} \beta_n = 0$. □

Proof of Theorem 2. If $\mathcal{A}$ is a σ-unital, then by [Zh, Theorem 1.2] $\mathcal{A}$ is stable, i.e., $\mathcal{A} \cong \mathcal{A} \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of all compact operators on a separable, infinite dimensional Hilbert space over $\mathbb{C}$. In this case, $\operatorname{csr}(\mathcal{A}) \le 2$ by [Ni, Corollary 2.5].

Consider the general case. For $k \ge 2$, let $\tilde{a} = (a_1 + \lambda_1, \dots, a_k + \lambda_k)^T \in \mathrm{S}_k(\mathcal{A}^+)$ where $a_i \in \mathcal{A}$, $\lambda_i \in \mathbb{C}$. Set

$$x = \sum_{i=1}^k (a_i^* a_i + a_i a_i^*).$$

Then for $\varepsilon = \frac{1}{2}\min((6k)^{-1}, (1+12k)^{-1}(\sum_{i=1}^n \|a_i + \lambda_i\|)^{-1})$, there is a projection r in $\mathcal{A}$ such that $\|x(1-r)\| < \varepsilon^2$ by [LZ, Condition (ii)] and [BP, Theorem 2.6]. Since $(1-r)\mathcal{A}(1-r)$ is purely infinite simple ([Zh, Theorem 1.3]), there is by Lemma 4 a c in $(1-r)\mathcal{A}(1-r)$ with $0 \le c < 1 - r$ such that $\overline{c\mathcal{A}c}$ is non-unital. Put $d = r + c$.

Then $0 \le d \le 1$ and $\mathcal{B} = \overline{d\mathcal{A}d}$ is non-unital (if Q is the unit of $\mathcal{B}$, $Q - r$ must be the unit of $\overline{c\mathcal{A}c}$) and furthermore, we have

$$x^{1/2}(1-d)^2x^{1/2} \le x^{1/2}(1-d)x^{1/2} \le x^{1/2}(1-r)x^{1/2} \le \varepsilon^2. \tag{3}$$

Now (3) indicates that $\|a_i(1-d)\| < \varepsilon$ and $\|(1-d)a_i\| < \varepsilon$, $1 \le i \le k$. Set $b_i = da_id + \lambda_i$, $b = \sum_{i=1}^k b_i^* b_i$. Then b_i, $b \in \mathcal{B}^+$, $\|a_i + \lambda_i - b_i\| < 2\varepsilon$, $1 \le i \le k$, and

$$\begin{aligned}\|1-b\| &= \left\| \sum_{i=1}^k (a_i+\lambda_i-b_i)^*(a_i+\lambda_i) + \sum_{i=1}^k b_i^*(a_i+\lambda_i-b_i) \right\| \\ &\le 3\sum_{i=1}^k \|a_i+\lambda_i-b_i\| < 6k\varepsilon < \frac{1}{2}\end{aligned}$$

(for $\|a_i + \lambda_i\| \le 1$, $\|\lambda_i\| \le 1$ and $\|b_i\| = \|d(a_i+\lambda_i)d + \lambda_i(1-d^2)\| \le 2$). Thus b is invertible in $\mathcal{B}^+$ with $\|b^{-1}\| < 2$. Since $b > 0$, we get that $\|b^{-1/2}\| < 2^{1/2}$ and

$$\|1 - b^{-1/2}\| = \|b^{-1/2}(1-b^{1/2})\| < 2^{1/2}\|1-b\| < 12k\varepsilon.$$

Now set $c_i = b_i b^{-1/2} \in \mathcal{B}^+$. Then $(c_1, \ldots, c_k)^T \in \mathrm{S}_k(\mathcal{B}^+)$ and

$$\begin{aligned}\|a_i+\lambda_i-c_i\| &= \|a_i+\lambda_i-b_i+b_i(1-b^{-1/2})\| \\ &< 2\varepsilon + 24k\varepsilon \le \left(\sum_{i=1}^k \|a_i+\lambda_i\|\right)^{-1}.\end{aligned} \tag{4}$$

Since $\mathcal{B}$ is a non-unital , σ-unital, hereditary C^*-subalgebra of $\mathcal{A}$, it follows from [Zh, Theorem 1.2] that $\mathcal{B}$ is stable and consequently, by means of the above argument, there exists $w \in \mathcal{U}_k^0(B^+) \subset \mathcal{U}_k^0(\mathcal{A}^+)$ such that $(c_1, \ldots, c_k)^T = we_k$. Applying Lemma 3 to (4), we obtain that $\tilde{a}$ is in the component containing e_k. Thus $\mathrm{csr}(\mathcal{A}^+) \le 2$. Now choose an isometry s in $r\mathcal{A}r$ with $s^*s = r$, $ss^* < r$. Set $T = s + 1 - r$. Then T is an isometry in $\mathcal{A}^+$. Therefore $\mathrm{csr}(\mathcal{A}) = 2$. □

Acknowledgement

The author wishes to thank Professor Huaxin Lin for his helpful suggestions to the proofs of Theorem 1 and Theorem 2.

References

[Bk] B. Blackadar, *K-Theory for Operator Algebras*, MSRI Publications No. 5, Springer-Verlag/New York/Berlin/Heidelberg/London/Paris/Tokyo, 1986. MR **88g:**46082

[BP] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149. MR **92m:**46086

[Cu] J. Cuntz, *K-Theory for Certain C^*-Algebras*, Ann. of Math. **113** (1981), 181–197. MR **84c:**46058

[Lin] H. Lin, *Approximation by normal elements with finite spectra in C^*-algebras of real rank zero*, Pacific J. Math. **173** (1995), 443–489. MR **98h:**46059

[LZ] H. Lin and S. Zhang, *On infinite simple C^*-algebras*, J. Funct. Anal. **100** (1991), 221–231. MR **92m:**46088

[Ni] V. Nistor, *Stable range for tension products of extensions of $\mathcal{K}$ by $C(X)$*, J. Operator Theory **16** (1986), 387–396. MR **88b:**46085

[Rf] M. A. Rieffel, *Dimensional and stable rank in the K-theory of C^*-Algebras*, Proc. London Math. Soc. **46**, no. (3) (1983), 301–333. MR **84g:**46085

[Sr] H. Schröder, *On the homotopy type of the regular group of a W^*-algebra*, Math. Ann. **167** (1992), 171–277.

[Zh] S. Zhang, *C^*-algebras with real rank zero and their multiplier algebras*, I, Pacific J. Math. **155** (1992), 169–197.

Institute of Fundamental Education, East China University of Science and Technology, Shanghai 200237, People's Republic of China

Current address: Department of Mathematics, East China University of Science and Technology, Shanghai 200237, People's Republic of China