SPECTRAL MULTIPLIER THEOREM FOR $H^1$ SPACES ASSOCIATED WITH SOME SCHRÖDINGER OPERATORS

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(Communicated by Christopher D. Sogge)

Abstract. Let $T_t$ be the semigroup of linear operators generated by a Schrödinger operator $-A = \Delta - V$, where $V$ is a nonnegative polynomial. We say that $f$ is an element of $H^1_A$ if the maximal function $Mf(x) = \sup_{t>0} |T_t f(x)|$ belongs to $L^1$. A criterion on functions $F$ which implies boundedness of the operators $F(A)$ on $H^1_A$ is given.

1. Introduction

Let $A = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, where $V(x) = \sum_{\beta \leq \alpha} a_{\beta} x^{\beta}$ is a nonnegative polynomial, $V \not\equiv 0$. Let $\{T_t\}_{t>0}$ be the semigroup of linear operators (e.g. on $L^2(\mathbb{R}^d)$) generated by $-A$. Since $V$ is nonnegative, the Feynman-Kac formula asserts that the integral kernels $T_t(x, y)$ of the operators $T_t$ satisfy

$$0 \leq T_t(x, y) \leq (4\pi t)^{-d/2} \exp(-|x-y|^2/(4t)).$$

(1.1)

We say that a function $f$ is an element of the space $H^1_A$ associated to the operator $A$ if the maximal function

$$Mf(x) = \sup_{t>0} |T_t f(x)| = \sup_{t>0} \int_{\mathbb{R}^d} T_t(x, y) f(y) \, dy$$

belongs to $L^1(\mathbb{R}^d)$. We put

$$\|f\|_{H^1_A} = \|Mf\|_{L^1(\mathbb{R}^d)}.$$

The spaces $H^1_A$ were studied in [6]. It was proved there that the elements $H^1_A$ can be characterized in terms of special atoms. The definition of atoms for $H^1_A$ is as follows. We first define an auxiliary function $m(x, V)$ (see [13]) by

$$m(x, V) = \sum_{\beta \leq \alpha} |D^\beta V(x)|^{1/(|\beta|+2)}.$$

(1.3)

Since $V$ is a nonzero polynomial, there is $c > 0$ such that $c \leq m(x, V) < \infty$. We set $B_0 = \{x \in \mathbb{R}^d : c \leq m(x, V) < 1\}$, $B_n = \{x \in \mathbb{R}^d : 2^{(n-1)/2} \leq m(x, V) < 2^n/2\}$

Received by the editors February 17, 1998.

1991 Mathematics Subject Classification. Primary 42B30, 35J10; Secondary 42B15, 42B25, 43A80.

This research was partially supported by the European Commission via TMR network “Harmonic Analysis”, and by grant 2 P03A 058 14 from KBN, Poland.
for $n = 1, 2, 3, \ldots$. We say that a function $a$ is an atom for $H^1_A$ associated to a ball $B(y_0, r)$ with the center at $y_0$ and radius $r$ if

\begin{equation}
\text{supp } a \subset B(y_0, r),
\end{equation}

\begin{equation}
\|a\|_{L^\infty} \leq |B(y_0, r)|^{-1},
\end{equation}

\begin{equation}
\text{if } y_0 \in B_n, \text{ then } r \leq 2^{1-n/2},
\end{equation}

\begin{equation}
\text{if } y_0 \in B_n \text{ and } r \leq 2^{1-n/2}, \text{ then } \int a(x) \, dx = 0.
\end{equation}

The atomic norm in the space $H^1_A$ is defined by

\begin{equation}
\|f\|_{H^1_A, \text{atom}} = \inf \left\{ \sum_j |c_j| \right\},
\end{equation}

where the infimum is taken over all decompositions $f = \sum_j c_j a_j$, with $a_j$ being $H^1_A$ atoms and $c_j$ being scalars.

The theorem below was proved in [6].

**Theorem 1.9.** There exists a constant $C > 0$ such that

\begin{equation}
\frac{1}{C} \|f\|_{H^1_A} \leq \|f\|_{H^1_A, \text{atom}} \leq C \|f\|_{H^1_A}.
\end{equation}

Let $E_A$ be the spectral decomposition of the operator $A$. For a bounded function $F$ on $\mathbb{R}^+$ we define the operator $F(A)$ (bounded at least on $L^2(\mathbb{R}^d)$) by

\[ F(A)f = \int_0^\infty F(\lambda) \, dE_A(\lambda)f. \]

We say that a function $F$ on $\mathbb{R}$ belongs to the space $C(s)$, $s \geq 0$, if

\[ \|F\|_{C(s)} = \begin{cases} 
\sum_{k=0}^{s} \sup |F^{(k)}(\lambda)| & \text{if } s \in \mathbb{Z}, \\
\|F^{(s)}\|_{\text{Lip } (s-[s])} + \sum_{k=0}^{[s]} \sup |F^{(k)}(\lambda)| & \text{if } s \notin \mathbb{Z}
\end{cases} \]

is finite.

Our aim is to prove

**Theorem 1.11.** Let $F$ be a bounded continuous function on $(0, \infty)$. If for some $\varepsilon > 0$ and a nonzero function $\varphi \in C^\infty_c(0, \infty)$ there exists a constant $C > 0$ such that for every $t > 0$

\begin{equation}
\|\varphi(\cdot) F(t \cdot)\|_{C(\frac{3}{4} + \varepsilon)} \leq C,
\end{equation}

then the operator $F(A)$ is bounded on $H^1_A$.

Spectral multiplier theorems on $L^p$ spaces attracted attention of many authors (cf. e.g. [1], [2], [7], [9], [11], [12], and references there). A. Hulanicki and E. Stein (cf. [10]) observed that if $\mathcal{L}$ is a sublaplacian on a stratified Lie group, then the convolution kernel $F(-\mathcal{L})(x)$, which corresponds to a spectral multiplier $F$, satisfies Calderón-Zygmund type estimates. This combined with the fact that the atoms for Hardy spaces on homogeneous groups satisfy moment conditions implies boundedness of the operator $F(-\mathcal{L})$ on these spaces (see also [3]). Our $H^1_A$ spaces are of a different nature than the classical Hardy spaces or the Hardy spaces on
homogeneous groups. The main difference is that for some $H^1_A$ atoms no mean-
value zero is required. Therefore some additional decay of kernels associated with
the multiplier $F$ is needed (see Proposition 2.2).

Our proof of the theorem is based on the following characterization of $H^1_A$; cf.
[6]. Let $\psi \in C_\infty(1/2,2)$ be such that $|\psi(\lambda)| \geq c > 0$ for $\lambda \in [\frac{3}{4}, \frac{7}{4}]$. We define a
Littlewood-Paley square function $Sf$ by

$$Sf(x) = \left(\sum_{\mu \in Z} |\Psi_\mu f(x)|^2\right)^{1/2},$$

where

$$\Psi_\mu f = \psi(2^{-\mu}A)f = \int_0^\infty \psi(2^{-\mu}\lambda) dE_A(\lambda)f.$$  

We have

**Proposition 1.15** (cf. [6]). There exists a constant $C > 0$ such that

$$\frac{1}{C} \|f\|_{H^1_A} \leq \|Sf\|_{L^1} + \|f\|_{L^1} \leq C \|f\|_{H^1_A}.$$  

2. **Proof of Theorem 1.11**

Let us note that under the assumption of Theorem 1.11 the inequality (1.12)
holds for every $\varphi \in C_\infty(0,\infty)$.

We fix $\varphi \in C_\infty(\frac{1}{2},2)$ such that $|\varphi(\lambda)| > c > 0$ for $\lambda \in [\frac{3}{4}, \frac{7}{4}]$ and

$$\sum_{\mu \in Z} \varphi(2^{-\mu}\lambda) = 1 \quad \text{for} \quad \lambda > 0.$$  

Set $Q_\mu(\lambda) = \varphi(2^{-\mu}\lambda)F(\lambda)$. We shall denote by $Q_\mu(A)(x,y) = Q_\mu(x,y)$ the integral
kernel of the operator

$$Q_\mu(A) = \int_0^\infty Q_\mu(\lambda) dE_A(\lambda),$$

that is,

$$Q_\mu(A)f(x) = \int_{\mathbb{R}^d} Q_\mu(x,y)f(y)dy.$$  

The following two propositions are crucial in our proof of Theorem 1.11.

**Proposition 2.2.** Assume that $F$ satisfies (1.12). Then there exist a constant
$\delta > 0$ and kernels $K_\mu(x,y) \geq 0$ which satisfy

$$\sup_x \int_{\mathbb{R}^d} K_\mu(x,y) dy \leq 1 \quad \text{and} \quad \sup_y \int_{\mathbb{R}^d} K_\mu(x,y) dx \leq 1$$

such that for every $b > 0$ there is a constant $C_b > 0$ such that

$$|Q_\mu(x,y)| \leq C_b K_\mu(x,y)(1 + 2^{n/2}|x-y|)^{-\delta}(1 + 2^{-n/2}m(x,V))^{-b}.$$  

**Proposition 2.5.** There exists a constant $C > 0$ such that for every $\mu \in \mathbb{Z}$ we have

$$\int_{\mathbb{R}^d} |Q_\mu(x,y) - Q_\mu(x,y_0)| dx \leq C 2^{\mu/2}|y-y_0|.$$  

Proofs of Propositions 2.2 and 2.5 will be presented in Section 3.
Lemma 2.7. Let $a$ be an $H^1_A$ atom associated to a ball $B(y_0, r)$. Then
\begin{equation}
\int_{B(y_0, 4r)} |F(A)a(x)| \, dx \leq C,
\end{equation}
where $C$ is independent of $a$.

Proof. Obviously the operator $F(A)$ is bounded on $L^2(\mathbb{R}^d)$. Therefore, by the Schwarz inequality,
\begin{align*}
\int_{B(y_0, 4r)} |F(A)a(x)| \, dx & \leq \|F(A)a\|_{L^2(\mathbb{R}^d)} |B(y_0, 4r)|^{1/2} \\
& \leq C r^{d/2} \|a\|_{L^2(\mathbb{R}^d)} \leq C.
\end{align*}
Since $\|Sf\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)}$, the same argument as above leads to

Lemma 2.9. There exists a constant $C > 0$ such that if $a$ is an $H^1_A$ atom associated to a ball $B(y_0, r)$, then
\begin{equation}
\int_{B(y_0, 4r)} S(F(A)a)(x) \, dx \leq C.
\end{equation}

Lemma 2.11. Assume that $a$ is an $H^1_A$ atom associated to a ball $B(y_0, r)$. Let $l$ be a nonnegative integer such that $2^{-(l+1)/2} < r \leq 2^{-l/2}$. Then
\begin{equation}
\int_{|x-y_0| > 4r} \sum_{\mu = l}^{\infty} |Q_{\mu}a(x)| \, dx \leq C,
\end{equation}
with a constant $C$ independent of $a$.

Proof. Applying Proposition 2.2, we get
\begin{align*}
\sum_{\mu = l}^{\infty} \int_{|x-y_0| > 4r} |Q_{\mu}a(x)| \, dx & \leq \sum_{\mu = l}^{\infty} \sup_{y \in B(y_0, r)} \int_{|x-y_0| > 4r} |Q_{\mu}(x, y)| \, dx \\
& \leq C_0 \sum_{\mu = l}^{\infty} \sup_{y \in B(y_0, r)} \int_{|x-y_0| > 4r} K_{\mu}(x, y)(1 + 2^{\mu/2}|x - y|)^{-\delta} \, dx \\
& \leq C_0 \sum_{\mu = l}^{\infty} 2^{-\delta(\mu-1)/2} \leq C.
\end{align*}

Let us denote by $\sigma$ the largest integer such that spectrum of the operator $A \geq 2^\sigma$. Since the bottom of the spectrum of $A$ is strictly positive, the number $\sigma$ is well defined.

Proposition 2.12. There exists a constant $C > 0$ such that if $a$ is an $H^1_A$ atom associated to a ball $B(y_0, r)$, where $y_0 \in B_n$, $2^{-(n+2)/2} < r \leq 2^{1-n/2}$, then
\begin{equation}
\int_{\mathbb{R}^d} \sum_{\mu = \sigma}^{n+2} |Q_{\mu}a(x)| \, dx \leq C.
\end{equation}
Proof. By Proposition 2.2, we obtain
\[ |Q_\mu(x, y)| \leq \sum_{j=2}^{\infty} b_j K_\mu(x, y)(1 + 2^{\mu/2}|x - y|)^{-\delta} \chi_{[0,j]}(1 + 2^{-\mu/2}m(x, V)) \]
\[ = \sum_{j=2}^{\infty} b_j P_\mu^j(x, y), \]
where \( b_j \) is a sequence of rapidly decaying positive numbers, that is,
\[ \sum_{j=2}^{\infty} j^N b_j = C_N < \infty \quad \text{for every} \quad N > 0. \]

We have
\[ P_\mu^j(x, y) = K_\mu(x, y)(1 + 2^{\mu/2}|x - y|)^{-\delta} \chi_{[0,j]}(1 + 2^{-\mu/2}m(x, V)) \]
\[ = K_\mu(x, y)(1 + 2^{\mu/2}|x - y|)^{-\delta} \chi_{[0,j]}(1 + 2^{-\mu/2}m(x, V)) \chi_{[2(k-1)/2, 2k/2]}(2^{\mu/2}|x - y|) \]
\[ + \sum_{k=1}^{\infty} K_\mu(x, y)(1 + 2^{\mu/2}|x - y|)^{-\delta} \chi_{[0,j]}(1 + 2^{-\mu/2}m(x, V)) \chi_{[2(k-1)/2, 2k/2]}(2^{\mu/2}|x - y|) \]
\[ = P_\mu^{j,0}(x, y) + \sum_{k=1}^{\infty} P_\mu^{j,k}(x, y). \]

Obviously,
\[ \sum_{\mu=\sigma}^{n+2} |Q_\mu a(x)| \leq \sum_{\mu=\sigma}^{n+2} \sum_{j=2}^{\infty} \sum_{k=0}^{\infty} b_j P_\mu^{j,k} |a|(x), \]
where
\[ P_\mu^{j,k} |a|(x) = \int P_\mu^{j,k}(x, y)|a(y)| \, dy. \]

Let \( a \) be an \( H^1_A \) atom associated to a ball \( B(y_0, r) \), where \( y_0 \in B_n \) and \( 2^{-(n+2)/2} < r \leq 2^{1-n/2} \). Let us note that in this case no moment condition on \( a \) is required.

If \( P_\mu^{j,0} |a| \neq 0 \), then there exist \( y \in B(y_0, r) \) and \( x \in \mathbb{R}^d \) such that \( P_\mu^{j,0}(x, y) \neq 0 \).

This implies \( 2^{\mu/2}|x - y| \leq 1, \ |y - y_0| \leq 2^{1-n/2} \), and \( m(x, V) \leq 2^{\mu/2}j \). Since \( D^\beta V(y) = \sum_{|\gamma| \leq \alpha} \frac{1}{\gamma!} D^\gamma + \beta V(x) (y - x)^\gamma \), we have \( |D^\beta V(y)| \leq C 2^{(\beta+2)|\alpha|+2} \) for every \( \beta \leq \alpha \). On the other hand there exists \( \beta \leq \alpha \) such that \( |D^\beta V(y)| \geq C 2^{(\beta+2)n/2} \). Thus \( n \leq \mu + C \log_2 j \). This gives
\[ \int \sum_{\mu=\sigma}^{n+2} \sum_{j=2}^{\infty} \sum_{k=0}^{\infty} b_j P_\mu^{j,k} |a|(x) \, dx \leq C \sum_{j=2}^{\infty} b_j \log_2 j \leq C. \]

We now consider \( P_\mu^{j,k} |a| \) for \( k \geq 1 \). If \( P_\mu^{j,k} |a| \neq 0 \), then there exist \( x \in \mathbb{R}^d \) and \( y \in B(y_0, r) \) such that \( P_\mu^{j,k}(x, y) \neq 0 \). Therefore \( |y - y_0| \leq 2^{1-n/2}, m(x, V) \leq 2^{\mu/2}j, \ 2^{(k-1)/2} \leq 2^{\mu/2}|x - y| \leq 2^{k/2} \). This leads to \( n \leq \mu + C \log_2 j + k|\alpha| \). Let us note
that $\int P_\mu^{j,k}\cdot |a(x)| \, dx \leq C 2^{-\delta(k-1)/2}$. Thus, by (2.14),
\[
\sum_{j=2}^{n} \sum_{\mu=1}^{\infty} \sum_{k=1}^{\infty} b_j P_\mu^{j,k} |a(x)| \, dx
\]
\[
\leq C \sum_{j=2}^{n} \sum_{\mu=1}^{\infty} \sum_{k=1}^{\infty} \frac{2^{-\delta(k-1)/2}}{C \log(j)/|a|}
\]
\[
\leq C \sum_{j=2}^{n} b_j \sum_{\mu=1}^{\infty} \sum_{k=1}^{\infty} 2^{-\delta(k-1)/2} \leq C < \infty.
\]

Proposition 2.16. There exists a constant $C > 0$ such that if $a$ is an $H^1_A$ atom associated to a ball $B(y_0, r)$, $y \in B_n$, $2^{-(l+1)/2} < r \leq 2^{-l/2} \leq 2^{-(n+2)/2}$, then
\[
\sum_{\mu=1}^{l} \int_{B_r} |Q_\mu(a)| \, dx \leq C.
\]

Proof. Let $a$ be an $H^1_A$ atom associated to a ball $B(y_0, r)$, such that $y \in B_n$, $2^{-(l+1)/2} < r \leq 2^{-l/2} \leq 2^{-(n+2)/2}$. Then, by the definition of $H^1_A$ atoms, $a$ has mean-value 0. Therefore, applying Proposition 2.5, we obtain
\[
\sum_{\mu=1}^{l} \int_{B_r} |Q_\mu(a(x))| \, dx = \sum_{\mu=1}^{l} \int_{B_r} \int_{B(y_0, r)} |Q_\mu(x, y) - Q_\mu(x, y_0)| a(y) \, dy \, dx
\]
\[
\leq \sum_{\mu=1}^{l} C 2^{l/2} \int_{B(y_0, r)} |y - y_0| |a(y)| \, dy \leq C \sum_{\mu=1}^{l} 2^{l/2} r \leq C.
\]

Proof of Theorem 1.11. We first note that $\varphi(2^{-\mu} A) = 0$ for every $\mu \leq \sigma - 1$. By Theorem 1.9 and Proposition 1.15 it suffices to show that there exists a constant
\[
C > 0
\]
\begin{equation}
\|F(A) a\|_{L^1} \leq C
\end{equation}

and
\begin{equation}
\|\left( \sum_{\mu=1}^{\infty} \varphi(2^{-\mu} A) F(A) a(x) \right)^2 \right)^{1/2} \|_{L^1(dx)} \leq C
\end{equation}

for every $H^1_A$ atom $a$. Let $a$ be an $H^1_A$ atom associated to a ball $B(y_0, r)$. In order to prove (2.17) it is sufficient, by Lemma 2.7, to show that
\begin{equation}
\int_{B(y_0, 4r)} |F(A) a(x)| \, dx \leq C,
\end{equation}

with a constant $C$ independent of $a$. But this is a consequence of the equality $F(A) a(x) = \sum_{\mu=1}^{\infty} Q_\mu(A) a(x)$, Lemma 2.11 and Propositions 2.12 and 2.16.

Now we turn to our proof of (2.18). By Lemma 2.9 we only need to show that
\[
\int_{B(y_0, 4r)} S(F(A) a(x)) \, dx \leq C,
\]

with $C$ independent of $a$. 


By virtue of Lemma 2.11 and Propositions 2.12 and 2.16, we get
\[
\int_{B(y_0,4r)^c} S(F(A)a)(x) \, dx = \int_{B(y_0,4r)^c} \left( \sum_{\mu = \sigma} \infty \right) \frac{|\varphi(2^{-\mu}A)F(A)a(x)|^2}{2} \, dx \\
\leq \int_{B(y_0,4r)^c} \sum_{\mu = \sigma} |Q_\mu(A)a(x)| \, dx \leq C.
\]

3. Proofs of Propositions 2.2 and 2.5
For an integral kernel \( K(x, y) \) and a number \( \delta > 0 \) we write
\[
\|K\|_{\omega(\delta)} = \max \left\{ \sup_y \int |K(x, y)|(1 + |x - y|)^\delta \, dx, \sup_x \int |K(x, y)|(1 + |x - y|)^\delta \, dy \right\}.
\]
For \( t > 0 \) we set \( A^{[t]} = -\Delta + V^{[t]} \), where \( V^{[t]}(x) = tV(t^{1/2}x) \). If \( F \) is a bounded function on \([0, \infty)\), we denote by \( F(A^{[t]}) = F(-\Delta + V^{[t]}) \) the operator \( \int_0^\infty F(\lambda) \, dE_{A^{[t]}}(\lambda) \), where \( E_{A^{[t]}} \) is the spectral resolution for \( A^{[t]} \).

The following two lemmas follow from [7], [8].

**Lemma 3.1.** For every \( \varepsilon > 0 \) and \( 0 < a < b < \infty \) there exist \( \delta > 0 \) and \( C > 0 \) such that if \( F \in C(\frac{d}{2} + \varepsilon) \), supp \( F \subset (a, b) \), then the kernel \( F(A^{[t]})(x, y) \) of the operator \( F(A^{[t]}) \) satisfies
\[
\|F(A^{[t]})\|_{\omega(\delta)} \leq C\|F\|_{C(\frac{d}{2} + \varepsilon)}.
\]

The constant \( C \) in (3.2) depends on \( d, \varepsilon, \delta, a, b \), but it is independent of \( F \) and \( A^{[t]} \).

**Lemma 3.3.** If \( F \) is a continuous function on \( \mathbb{R}^+ \) supported on \((a, b), 0 < a < b < \infty \), then
\[
F(tA)(x, y) = t^{-d/2}F(A^{[t]})(\frac{x}{t^{1/2}}, \frac{y}{t^{1/2}}).
\]

It is shown in [5] that for every multi-index \( \alpha \in \mathbb{Z}_d \), there exist a homogeneous Lie group \( G \) and a regular symmetric kernel \( P \) of order 2 such that for every nonnegative polynomial \( W(x) = \sum_{\beta \leq \alpha} b_\beta x^\beta \) there is a unitary representation \( \Pi^W \) that acts on \( L^2(\mathbb{R}^d) \) such that
\[
\Pi^W_P = -\Delta + W.
\]
The Lie algebra of the group \( G \) is generated, as a Lie algebra, by \( X_1, X_2, \ldots, X_d, Y \); cf. [4], [5]. The elements \( X_1, \ldots, X_d \) are homogeneous of degree 1, whereas \( Y \) is homogeneous of degree 2. Moreover,
\[
\Pi^W_{X_j} = \frac{\partial}{\partial x_j}, \quad \Pi^W_Y = iW(x).
\]
The kernel \( P \) satisfies maximal subelliptic estimates, that is, for every left-invariant homogeneous differential operator \( \partial \) on \( G \) there exists a constant \( C > 0 \) such that
\[
\|\partial f\|_{L^2(G)} \leq C\|P^{s/2}f\|_{L^2(G)} = C\| \int_0^\infty \lambda^{s/2} dE_P(\lambda)f\|_{L^2(G)},
\]
where \( E_P \) is the spectral resolution of the operator \( P : f \mapsto f * P \) and \( s \) is the degree of homogeneity of \( \partial \) (cf. [5]).
Lemma 3.6. For every left-invariant differential operator \( \partial \) on \( G \) there is a constant \( C > 0 \) such that for every function \( F \in C_c \left( \frac{1}{2}, 2 \right) \), and every unitary representation \( \Pi^W \), where \( W \) is a nonnegative polynomial which has the form \( W(x) = \sum_{\beta \leq \alpha} b_\beta x^\beta \), we have

\[
\| \Pi^W \| F(-\Delta + W) f \|_{L^2(\mathbb{R}^d)} \leq C \| F \|_{C_c(1/2, 2)} \| f \|_{L^2(\mathbb{R}^d)}.
\]

Proof. The required estimate (3.7) is a consequence of (3.5) and a transference method.

We are in a position to prove

Lemma 3.8. For every \( N > 0 \) there exists a constant \( C > 0 \) such that for every \( F \in C_c \left( \frac{1}{2}, 2 \right) \) and every nonnegative polynomial \( W \) of the form \( W(x) = \sum_{\beta \leq \alpha} b_\beta x^\beta \), the kernel \( F(-\Delta + W)(x,y) \) of the operator \( F(-\Delta + W) \) satisfies

\[
|F(-\Delta + W)(x,y)| \leq C \| F \|_{C_c(1/2, 2)} (1 + m(x,W))^{-N}.
\]

Proof. It follows from (3.4) and Lemma 3.6 that for every \( q = 0, 1, 2, \ldots \) and \( x \in \mathbb{R}^d \) the functional \( f \mapsto W^q(x)F(-\Delta + W)f(x) \) is bounded on \( L^2(\mathbb{R}^d) \) and

\[
|W^q(x)F(-\Delta + W)f(x)| \leq C_q \| F \|_{C_c(1/2, 2)} \| f \|_{L^2(\mathbb{R}^d)}
\]

with a constant \( C_q > 0 \) independent of \( x \in \mathbb{R}^d \) and \( W \). Therefore,

\[
\int_{\mathbb{R}^d} |W^q(x)F(-\Delta + W)(x,y)|^2 \, dy \leq C_q \| F \|_{C_c(1/2, 2)}^2,
\]

where \( C_q \) is independent of \( W \) and \( x \). Since \( F(-\Delta + W) = F(-\Delta + W) \phi(-\Delta + W)^* \), where \( \phi \in C_c^\infty \left( \frac{1}{4}, 4 \right) \), \( \phi \equiv 1 \) on \( \left[ \frac{1}{2}, 2 \right] \), we have

\[
|W^q(x)F(-\Delta + W)(x,y)| \leq C_q \| F \|_{C_c(1/2, 2)}.
\]

Similarly, we can prove using (3.4) and Lemma (3.6) that

\[
|(D^\beta W(x))^q(x)F(-\Delta + W)(x,y)| \leq C_{q, \alpha} \| F \|_{C_c(1/2, 2)}.
\]

By (3.12) and (3.13) we obtain (3.9).

Proof of Proposition 2.2. It follows from Lemma 3.3 that

\[
Q_\mu(x,y) = 2^{\mu/2} \hat{Q}_\mu(-\Delta + V^{[2^{-r}]}) (2^{\mu/2} x, 2^{\mu/2} y) = 2^{\mu/2} \hat{Q}_\mu(A^{[2^{-r}]}) (2^{\mu/2} x, 2^{\mu/2} y),
\]

where \( \hat{Q}_\mu(\lambda) = Q_\mu(2^{\mu} \lambda) = \varphi(\lambda) F(2^{\mu} \lambda) \). Applying Lemma 3.1 we obtain that there exist constants \( C > 0 \) and \( \delta' > 0 \) such that

\[
\| \hat{Q}_\mu(A^{[2^{-r}]}) \|_{L^1(\mathbb{R}^d)} \leq C \| \hat{Q}_\mu \|_{C(\frac{4}{3} + \varepsilon)} \leq C.
\]

Setting \( \tilde{K}_\mu(x,y) = \tilde{Q}_\mu(A^{[2^{-r}]}) (x,y) (1 + |x - y|)^{-\delta/2} \), we have

\[
\| \tilde{K}_\mu(x,y) \|_{L^1(\mathbb{R}^d)} \leq C.
\]

This implies that there exist constants \( C > 0 \) and \( 0 < \varepsilon < 1 \) such that

\[
\sup_y \int |\tilde{K}_\mu(x,y)|^{-\varepsilon} \, dx \leq C, \quad \sup_x \int |\tilde{K}_\mu(x,y)|^{-\varepsilon} \, dy \leq C.
\]

Moreover, by Lemma 3.8, for every \( N > 0 \) there exists a constant \( C_N \) such that

\[
|\tilde{Q}_\mu(A^{[2^{-r}]}) (x,y) | \leq C_N \| \tilde{Q}_\mu \|_{C_c(1/2, 2)} (1 + m(x, V^{[2^{-r}]})^{-N}.
\]
Therefore, by (3.17), we get
\[ |\tilde{Q}_\mu(A^{[2^{-s}]})(x, y)| \leq C_N |\tilde{K}_\mu(x, y)|^{1-\varepsilon'} (1 + |x - y|)^{-\delta'(1-\varepsilon')} (1 + m(x, V^{[2^{-s}]}))^{-N\varepsilon'}. \]
Since \( m(t^{1/2}x, V^{[2^{-s}]}) = t^{1/2}m(x, V) \), we obtain, by (3.14) and (3.16), the required estimates (2.3) and (2.4).

Proof of Proposition 2.5. It follows from (3.2) of [5] that the kernels \( \tilde{T}_t^{[2^{-s}]}(x, y) \) of the semigroup \( T_t^{[2^{-s}]} = e^{-tA^{[2^{-s}]}}, \) satisfy
\[ (3.18) \quad |\nabla_x T_t^{[2^{-s}]}(x, y)| \leq Ct^{-(d+1)/2}(1 + |x - y|)^{-d-3}, \]
with a constant \( C > 0 \) independent of \( \mu \).

Let \( \tilde{Q}_\mu(\lambda) \) be as in the proof of Proposition 2.2. Since \( \tilde{Q}_\mu(\lambda) = \tilde{Q}_\mu(\lambda)e^\lambda e^{-\lambda} = \tilde{R}_\mu(\lambda)e^\lambda \), we have \( \tilde{Q}_\mu(A^{[2^{-s}]} = \tilde{R}_\mu(A^{[2^{-s}]} T_1^{[2^{-s}]}). \) By virtue of Lemma 3.1 we obtain \( \int_{\mathbb{R}^d} |\tilde{R}_\mu(A^{[2^{-s}]})(x, z)| \leq C, \) with \( C > 0 \) independent of \( \mu \). Therefore, applying (3.18), we get
\[ \int_{\mathbb{R}^d} \left| \tilde{Q}_\mu(A^{[2^{-s}]})(x, y) - \tilde{Q}_\mu(A^{[2^{-s}]})(x, y_0) \right| dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{R}_\mu(A^{[2^{-s}]})(x, z) \left( T_1^{[2^{-s}]}(z, y) - T_1^{[2^{-s}]}(z, y_0) \right) dz \, dx \leq C|y - y_0|. \]
The estimate (2.6) is now a consequence of (3.14). \( \square \)

References

8. W. Hebisch, Functional calculus for slowly decaying kernels, preprint University of Wroclaw.