

SPECTRAL MULTIPLIER THEOREM FOR H^1 SPACES ASSOCIATED WITH SOME SCHRÖDINGER OPERATORS

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(Communicated by Christopher D. Sogge)

ABSTRACT. Let T_t be the semigroup of linear operators generated by a Schrödinger operator $-A = \Delta - V$, where V is a nonnegative polynomial. We say that f is an element of H_A^1 if the maximal function $\mathcal{M}f(x) = \sup_{t>0} |T_t f(x)|$ belongs to L^1 . A criterion on functions F which implies boundedness of the operators $F(A)$ on H_A^1 is given.

1. INTRODUCTION

Let $A = -\Delta + V$ be a Schrödinger operator on \mathbb{R}^d , where $V(x) = \sum_{\beta \leq \alpha} a_\beta x^\beta$ is a nonnegative polynomial, $V \not\equiv 0$. Let $\{T_t\}_{t>0}$ be the semigroup of linear operators (e.g. on $L^2(\mathbb{R}^d)$) generated by $-A$. Since V is nonnegative, the Feynman-Kac formula asserts that the integral kernels $T_t(x, y)$ of the operators T_t satisfy

$$(1.1) \quad 0 \leq T_t(x, y) \leq (4\pi t)^{-d/2} \exp(-|x - y|^2/(4t)).$$

We say that a function f is an element of the space H_A^1 associated to the operator A if the maximal function

$$(1.2) \quad \mathcal{M}f(x) = \sup_{t>0} |T_t f(x)| = \sup_{t>0} \left| \int_{\mathbb{R}^d} T_t(x, y) f(y) dy \right|$$

belongs to $L^1(\mathbb{R}^d)$. We put

$$\|f\|_{H_A^1} = \|\mathcal{M}f\|_{L^1(\mathbb{R}^d)}.$$

The spaces H_A^1 were studied in [6]. It was proved there that the elements H_A^1 can be characterized in terms of special atoms. The definition of atoms for H_A^1 is as follows. We first define an auxiliary function $m(x, V)$ (see [13]) by

$$(1.3) \quad m(x, V) = \sum_{\beta \leq \alpha} |D^\beta V(x)|^{1/(|\beta|+2)}.$$

Since V is a nonzero polynomial, there is $c > 0$ such that $c \leq m(x, V) < \infty$. We set $\mathcal{B}_0 = \{x \in \mathbb{R}^d : c \leq m(x, V) < 1\}$, $\mathcal{B}_n = \{x \in \mathbb{R}^d : 2^{(n-1)/2} \leq m(x, V) < 2^{n/2}\}$

Received by the editors February 17, 1998.

1991 *Mathematics Subject Classification*. Primary 42B30, 35J10; Secondary 42B15, 42B25, 43A80.

This research was partially supported by the European Commission via TMR network “Harmonic Analysis”, and by grant 2 P03A 058 14 from KBN, Poland.

for $n = 1, 2, 3, \dots$. We say that a function a is an atom for H_A^1 associated to a ball $B(y_0, r)$ with the center at y_0 and radius r if

$$(1.4) \quad \text{supp } a \subset B(y_0, r),$$

$$(1.5) \quad \|a\|_{L^\infty} \leq |B(y_0, r)|^{-1},$$

$$(1.6) \quad \text{if } y_0 \in \mathcal{B}_n, \text{ then } r \leq 2^{1-n/2},$$

$$(1.7) \quad \text{if } y_0 \in \mathcal{B}_n \text{ and } r \leq 2^{-1-n/2}, \text{ then } \int a(x) dx = 0.$$

The atomic norm in the space H_A^1 is defined by

$$(1.8) \quad \|f\|_{H_A^1\text{-atom}} = \inf \left\{ \sum_j |c_j| \right\},$$

where the infimum is taken over all decompositions $f = \sum_j c_j a_j$, with a_j being H_A^1 atoms and c_j being scalars.

The theorem below was proved in [6].

Theorem 1.9. *There exists a constant $C > 0$ such that*

$$(1.10) \quad \frac{1}{C} \|f\|_{H_A^1} \leq \|f\|_{H_A^1\text{-atom}} \leq C \|f\|_{H_A^1}.$$

Let E_A be the spectral decomposition of the operator A . For a bounded function F on \mathbb{R}^+ we define the operator $F(A)$ (bounded at least on $L^2(\mathbb{R}^d)$) by

$$F(A)f = \int_0^\infty F(\lambda) dE_A(\lambda)f.$$

We say that a function F on \mathbb{R} belongs to the space $C(s)$, $s \geq 0$, if

$$\|F\|_{C(s)} = \begin{cases} \sum_{k=0}^s \sup |F^{(k)}(\lambda)| & \text{if } s \in \mathbb{Z}, \\ \|F^{([s])}\|_{\text{Lip}(s-[s])} + \sum_{k=0}^{[s]} \sup |F^{(k)}(\lambda)| & \text{if } s \notin \mathbb{Z} \end{cases}$$

is finite.

Our aim is to prove

Theorem 1.11. *Let F be a bounded continuous function on $(0, \infty)$. If for some $\varepsilon > 0$ and a nonzero function $\varphi \in C_c^\infty(0, \infty)$ there exists a constant $C > 0$ such that for every $t > 0$*

$$(1.12) \quad \|\varphi(\cdot)F(t\cdot)\|_{C(\frac{d}{2}+\varepsilon)} \leq C,$$

then the operator $F(A)$ is bounded on H_A^1 .

Spectral multiplier theorems on L^p spaces attracted attention of many authors (cf. e.g. [1], [2], [7], [9], [11], [12], and references there). A. Hulanicki and E. Stein (cf. [10]) observed that if \mathcal{L} is a sublaplacian on a stratified Lie group, then the convolution kernel $F(-\mathcal{L})(x)$, which corresponds to a spectral multiplier F , satisfies Calderón-Zygmund type estimates. This combined with the fact that the atoms for Hardy spaces on homogeneous groups satisfy moment conditions implies boundedness of the operator $F(-\mathcal{L})$ on these spaces (see also [3]). Our H_A^1 spaces are of a different nature than the classical Hardy spaces or the Hardy spaces on

homogeneous groups. The main difference is that for some H_A^1 atoms no mean-value zero is required. Therefore some additional decay of kernels associated with the multiplier F is needed (see Proposition 2.2).

Our proof of the theorem is based on the following characterization of H_A^1 ; cf. [6]. Let $\psi \in C_c^\infty(1/2, 2)$ be such that $|\psi(\lambda)| \geq c > 0$ for $\lambda \in [\frac{3}{4}, \frac{7}{4}]$. We define a Littlewood-Paley square function Sf by

$$(1.13) \quad Sf(x) = \left(\sum_{\mu \in \mathbb{Z}} |\Psi_\mu f(x)|^2 \right)^{1/2},$$

where

$$(1.14) \quad \Psi_\mu f = \psi(2^{-\mu}A)f = \int_0^\infty \psi(2^{-\mu}\lambda) dE_A(\lambda)f.$$

We have

Proposition 1.15 (cf. [6]). *There exists a constant $C > 0$ such that*

$$(1.16) \quad \frac{1}{C} \|f\|_{H_A^1} \leq \|Sf\|_{L^1} + \|f\|_{L^1} \leq C \|f\|_{H_A^1}.$$

2. PROOF OF THEOREM 1.11

Let us note that under the assumption of Theorem 1.11 the inequality (1.12) holds for every $\varphi \in C_c^\infty(0, \infty)$.

We fix $\varphi \in C_c^\infty(\frac{1}{2}, 2)$ such that $|\varphi(\lambda)| > c > 0$ for $\lambda \in [\frac{3}{4}, \frac{7}{4}]$ and

$$(2.1) \quad \sum_{\mu \in \mathbb{Z}} \varphi(2^{-\mu}\lambda) = 1 \quad \text{for } \lambda > 0.$$

Set $Q_\mu(\lambda) = \varphi(2^{-\mu}\lambda)F(\lambda)$. We shall denote by $Q_\mu(A)(x, y) = Q_\mu(x, y)$ the integral kernel of the operator

$$Q_\mu(A) = \int_0^\infty Q_\mu(\lambda) dE_A(\lambda),$$

that is,

$$Q_\mu(A)f(x) = \int_{\mathbb{R}^d} Q_\mu(x, y)f(y) dy.$$

The following two propositions are crucial in our proof of Theorem 1.11.

Proposition 2.2. *Assume that F satisfies (1.12). Then there exist a constant $\delta > 0$ and kernels $K_\mu(x, y) \geq 0$ which satisfy*

$$(2.3) \quad \sup_x \int_{\mathbb{R}^d} K_\mu(x, y) dy \leq 1 \quad \text{and} \quad \sup_y \int_{\mathbb{R}^d} K_\mu(x, y) dx \leq 1$$

such that for every $b > 0$ there is a constant $C_b > 0$ such that

$$(2.4) \quad |Q_\mu(x, y)| \leq C_b K_\mu(x, y) (1 + 2^{\mu/2}|x - y|)^{-\delta} (1 + 2^{-\mu/2}m(x, V))^{-b}.$$

Proposition 2.5. *There exists a constant $C > 0$ such that for every $\mu \in \mathbb{Z}$ we have*

$$(2.6) \quad \int_{\mathbb{R}^d} |Q_\mu(x, y) - Q_\mu(x, y_0)| dx \leq C 2^{\mu/2} |y - y_0|.$$

Proofs of Propositions 2.2 and 2.5 will be presented in Section 3.

Lemma 2.7. *Let a be an H_A^1 atom associated to a ball $B(y_0, r)$. Then*

$$(2.8) \quad \int_{B(y_0, 4r)} |F(A)a(x)| dx \leq C,$$

where C is independent of a .

Proof. Obviously the operator $F(A)$ is bounded on $L^2(\mathbb{R}^d)$. Therefore, by the Schwarz inequality,

$$\begin{aligned} \int_{B(y_0, 4r)} |F(A)a(x)| dx &\leq \|F(A)a\|_{L^2(\mathbb{R}^d)} |B(y_0, 4r)|^{1/2} \\ &\leq Cr^{d/2} \|a\|_{L^2(\mathbb{R}^d)} \leq C. \end{aligned}$$

□

Since $\|Sf\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)}$, the same argument as above leads to

Lemma 2.9. *There exists a constant $C > 0$ such that if a is an H_A^1 atom associated to a ball $B(y_0, r)$, then*

$$(2.10) \quad \int_{B(y_0, 4r)} S(F(A)a)(x) dx \leq C.$$

Lemma 2.11. *Assume that a is an H_A^1 atom associated to a ball $B(y_0, r)$. Let l be a nonnegative integer such that $2^{-(l+1)/2} < r \leq 2^{-l/2}$. Then*

$$\int_{|x-y_0|>4r} \sum_{\mu=l}^{\infty} |Q_{\mu}a(x)| dx \leq C,$$

with a constant C independent of a .

Proof. Applying Proposition 2.2, we get

$$\begin{aligned} \sum_{\mu=l}^{\infty} \int_{|x-y_0|>4r} |Q_{\mu}a(x)| dx &\leq \sum_{\mu=l}^{\infty} \sup_{y \in B(y_0, r)} \int_{|x-y_0|>4r} |Q_{\mu}(x, y)| dx \\ &\leq C_0 \sum_{\mu=l}^{\infty} \sup_{y \in B(y_0, r)} \int_{|x-y_0|>4r} K_{\mu}(x, y) (1 + 2^{\mu/2}|x-y|)^{-\delta} dx \\ &\leq C_0 \sum_{\mu=l}^{\infty} 2^{-\delta(\mu-l)/2} \leq C. \end{aligned}$$

□

Let us denote by σ the largest integer such that spectrum of the operator $A \geq 2^{\sigma}$. Since the bottom of the spectrum of A is strictly positive, the number σ is well defined.

Proposition 2.12. *There exists a constant $C > 0$ such that if a is an H_A^1 atom associated to a ball $B(y_0, r)$, where $y_0 \in \mathcal{B}_n$, $2^{-(n+2)/2} < r \leq 2^{1-n/2}$, then*

$$(2.13) \quad \int_{\mathbb{R}^d} \sum_{\mu=\sigma}^{n+2} |Q_{\mu}a(x)| dx \leq C.$$

Proof. By Proposition 2.2, we obtain

$$\begin{aligned} |Q_\mu(x, y)| &\leq \sum_{j=2}^{\infty} b_j K_\mu(x, y) (1 + 2^{\mu/2} |x - y|)^{-\delta} \chi_{[0, j]} (1 + 2^{-\mu/2} m(x, V)) \\ &= \sum_{j=2}^{\infty} b_j P_\mu^j(x, y), \end{aligned}$$

where b_j is a sequence of rapidly decaying positive numbers, that is,

$$(2.14) \quad \sum_{j=2}^{\infty} j^N b_j = C_N < \infty \quad \text{for every } N > 0.$$

We have

$$\begin{aligned} P_\mu^j(x, y) &= K_\mu(x, y) (1 + 2^{\mu/2} |x - y|)^{-\delta} \chi_{[0, j]} (1 + 2^{-\mu/2} m(x, V)) \\ &= K_\mu(x, y) (1 + 2^{\mu/2} |x - y|)^{-\delta} \chi_{[0, j]} (1 + 2^{-\mu/2} m(x, V)) \chi_{[0, 1]} (2^{\mu/2} |x - y|) \\ &+ \sum_{k=1}^{\infty} K_\mu(x, y) (1 + 2^{\mu/2} |x - y|)^{-\delta} \chi_{[0, j]} (1 + 2^{-\mu/2} m(x, V)) \chi_{(2^{(k-1)/2}, 2^{k/2}]} (2^{\mu/2} |x - y|) \\ &= P_\mu^{j, 0}(x, y) + \sum_{k=1}^{\infty} P_\mu^{j, k}(x, y). \end{aligned}$$

Obviously,

$$\sum_{\mu=\sigma}^{n+2} |Q_\mu a(x)| \leq \sum_{\mu=\sigma}^{n+2} \sum_{j=2}^{\infty} \sum_{k=0}^{\infty} b_j P_\mu^{j, k} |a|(x),$$

where

$$P_\mu^{j, k} |a|(x) = \int P_\mu^{j, k}(x, y) |a(y)| dy.$$

Let a be an H_A^1 atom associated to a ball $B(y_0, r)$, where $y_0 \in \mathcal{B}_n$ and $2^{-(n+2)/2} < r \leq 2^{1-n/2}$. Let us note that in this case no moment condition on a is required.

If $P_\mu^{j, 0} |a| \neq 0$, then there exist $y \in B(y_0, r)$ and $x \in \mathbb{R}^d$ such that $P_\mu^{j, 0}(x, y) \neq 0$. This implies $2^{\mu/2} |x - y| \leq 1$, $|y - y_0| \leq 2^{1-n/2}$, and $m(x, V) \leq 2^{\mu/2} j$. Since $D^\beta V(y) = \sum_{\gamma \leq \alpha} \frac{1}{\gamma!} D^{\gamma+\beta} V(x) (y - x)^\gamma$, we have $|D^\beta V(y)| \leq C 2^{(|\beta|+2)\mu/2} j^{2|\alpha|+2}$ for every $\beta \leq \alpha$. On the other hand there exists $\beta \leq \alpha$ such that $|D^\beta V(y)| \geq \frac{1}{C} 2^{(|\beta|+2)n/2}$. Thus $n \leq \mu + C \log_2 j$. This gives

$$(2.15) \quad \int \sum_{\mu=\sigma}^{n+2} \sum_{j=2}^{\infty} b_j P_\mu^{j, 0} |a|(x) dx \leq C \sum_{j=2}^{\infty} b_j \log_2 j \leq C.$$

We now consider $P_\mu^{j, k} |a|$ for $k \geq 1$. If $P_\mu^{j, k} |a| \neq 0$, then there exist $x \in \mathbb{R}^d$ and $y \in B(y_0, r)$ such that $P_\mu^{j, k}(x, y) \neq 0$. Therefore $|y - y_0| \leq 2^{1-n/2}$, $m(x, V) \leq 2^{\mu/2} j$, $2^{(k-1)/2} \leq 2^{\mu/2} |x - y| \leq 2^{k/2}$. This leads to $n \leq \mu + C \log_2 j + k|\alpha|$. Let us note

that $\int P_\mu^{j,k}|a|(x) dx \leq C2^{-\delta(k-1)/2}$. Thus, by (2.14),

$$\begin{aligned} & \sum_{\mu=\sigma}^{n+2} \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} \int b_j P_\mu^{j,k}|a|(x) dx \\ & \leq C \sum_{j=2}^{\infty} b_j \sum_{\mu=\sigma}^{n+2} \sum_{k \geq (n-\mu-C \log_2 j)/|\alpha|} 2^{-\delta(k-1)/2} \leq C < \infty. \end{aligned}$$

□

Proposition 2.16. *There exists a constant $C > 0$ such that if a is an H_A^1 atom associated to a ball $B(y_0, r)$, $y \in \mathcal{B}_n$, $2^{-(l+1)/2} < r \leq 2^{-l/2} \leq 2^{-(n+2)/2}$, then*

$$\sum_{\mu=\sigma}^l \int_{\mathbb{R}^d} |Q_\mu(a)| dx \leq C.$$

Proof. Let a be an H_A^1 atom associated to a ball $B(y_0, r)$, such that $y \in \mathcal{B}_n$, $2^{-(l+1)/2} < r \leq 2^{-l/2} \leq 2^{-(n+2)/2}$. Then, by the definition of H_A^1 atoms, a has mean-value 0. Therefore, applying Proposition 2.5, we obtain

$$\begin{aligned} \sum_{\mu=\sigma}^l \int_{\mathbb{R}^d} |Q_\mu a(x)| dx &= \sum_{\mu=\sigma}^l \int_{\mathbb{R}^d} \left| \int_{B(y_0, r)} [Q_\mu(x, y) - Q_\mu(x, y_0)] a(y) dy \right| dx \\ &\leq \sum_{\mu=\sigma}^l C 2^{\mu/2} \int_{B(y_0, r)} |y - y_0| |a(y)| dy \leq C \sum_{\mu=\sigma}^l 2^{\mu/2} r \leq C. \end{aligned}$$

□

Proof of Theorem 1.11. We first note that $\varphi(2^{-\mu}A) = 0$ for every $\mu \leq \sigma - 1$. By Theorem 1.9 and Proposition 1.15 it suffices to show that there exists a constant $C > 0$ such that

$$(2.17) \quad \|F(A)a\|_{L^1} \leq C$$

and

$$(2.18) \quad \left\| \left(\sum_{\mu=\sigma}^{\infty} |\varphi(2^{-\mu}A)F(A)a(x)|^2 \right)^{1/2} \right\|_{L^1(dx)} \leq C$$

for every H_A^1 atom a . Let a be an H_A^1 atom associated to a ball $B(y_0, r)$. In order to prove (2.17) it is sufficient, by Lemma 2.7, to show that

$$(2.19) \quad \int_{B(y_0, 4r)^c} |F(A)a(x)| dx \leq C,$$

with a constant C independent of a . But this is a consequence of the equality $F(A)a(x) = \sum_{\mu=\sigma}^{\infty} Q_\mu(A)a(x)$, Lemma 2.11 and Propositions 2.12 and 2.16.

Now we turn to our proof of (2.18). By Lemma 2.9 we only need to show that

$$\int_{B(y_0, 4r)^c} S(F(A)a)(x) dx \leq C,$$

with C independent of a .

By virtue of Lemma 2.11 and Propositions 2.12 and 2.16, we get

$$\begin{aligned} \int_{B(y_0, 4r)^c} S(F(A)a)(x) dx &= \int_{B(y_0, 4r)^c} \left(\sum_{\mu=\sigma}^{\infty} |\varphi(2^{-\mu}A)F(A)a(x)|^2 \right)^{1/2} dx \\ &\leq \int_{B(y_0, 4r)^c} \sum_{\mu=\sigma}^{\infty} |Q_{\mu}(A)a(x)| dx \leq C. \end{aligned}$$

□

3. PROOFS OF PROPOSITIONS 2.2 AND 2.5

For an integral kernel $K(x, y)$ and a number $\delta > 0$ we write

$$\|K\|_{\omega(\delta)} = \max\left\{ \sup_y \int |K(x, y)|(1 + |x - y|)^{\delta} dx, \sup_x \int |K(x, y)|(1 + |x - y|)^{\delta} dy \right\}.$$

For $t > 0$ we set $A^{[t]} = -\Delta + V^{[t]}$, where $V^{[t]}(x) = tV(t^{1/2}x)$. If F is a bounded function on $[0, \infty)$, we denote by $F(A^{[t]}) = F(-\Delta + V^{[t]})$ the operator $\int_0^{\infty} F(\lambda) dE_{A^{[t]}}(\lambda)$, where $E_{A^{[t]}}$ is the spectral resolution for $A^{[t]}$.

The following two lemmas follow from [7], [8].

Lemma 3.1. *For every $\varepsilon > 0$ and $0 < a < b < \infty$ there exist $\delta > 0$ and $C > 0$ such that if $F \in C(\frac{d}{2} + \varepsilon)$, $\text{supp } F \subset (a, b)$, then the kernel $F(A^{[t]})(x, y)$ of the operator $F(A^{[t]})$ satisfies*

$$(3.2) \quad \|F(A^{[t]})\|_{\omega(\delta)} \leq C \|F\|_{C(\frac{d}{2} + \varepsilon)}.$$

The constant C in (3.2) depends on d , ε , δ , a , b , but it is independent of F and $A^{[t]}$.

Lemma 3.3. *If F is a continuous function on \mathbb{R}^+ supported on (a, b) , $0 < a < b < \infty$, then*

$$F(tA)(x, y) = t^{-d/2} F(A^{[t]})\left(\frac{x}{t^{1/2}}, \frac{y}{t^{1/2}}\right).$$

It is shown in [5] that for every multi-index $\alpha \in \mathbb{Z}_+^d$ there exist a homogeneous Lie group G and a regular symmetric kernel P of order 2 such that for every nonnegative polynomial $W(x) = \sum_{\beta \leq \alpha} b_{\beta} x^{\beta}$ there is a unitary representation Π^W that acts on $L^2(\mathbb{R}^d)$ such that

$$\Pi_P^W = -\Delta + W.$$

The Lie algebra of the group G is generated, as a Lie algebra, by X_1, X_2, \dots, X_d, Y ; cf. [4], [5]. The elements X_1, \dots, X_d are homogeneous of degree 1, whereas Y is homogeneous of degree 2. Moreover,

$$(3.4) \quad \Pi_{X_j}^W = \frac{\partial}{\partial x_j}, \quad \Pi_Y^W = iW(x).$$

The kernel P satisfies maximal subelliptic estimates, that is, for every left-invariant homogeneous differential operator ∂ on G there exists a constant $C > 0$ such that

$$(3.5) \quad \|\partial f\|_{L^2(G)} \leq C \|P^{s/2} f\|_{L^2(G)} = C \left\| \int_0^{\infty} \lambda^{s/2} dE_P(\lambda) f \right\|_{L^2(G)},$$

where E_P is the spectral resolution of the operator $P : f \mapsto f * P$ and s is the degree of homogeneity of ∂ (cf. [5]).

Lemma 3.6. *For every left-invariant differential operator ∂ on G there is a constant $C > 0$ such that for every function $F \in C_c(\frac{1}{2}, 2)$, and every unitary representation Π^W , where W is a nonnegative polynomial which has the form $W(x) = \sum_{\beta \leq \alpha} b_\beta x^\beta$, we have*

$$(3.7) \quad \|\Pi_\partial^W F(-\Delta + W)f\|_{L^2(\mathbb{R}^d)} \leq C\|F\|_{C_c(1/2,2)}\|f\|_{L^2(\mathbb{R}^d)}.$$

Proof. The required estimate (3.7) is a consequence of (3.5) and a transference method. □

We are in a position to prove

Lemma 3.8. *For every $N > 0$ there exists a constant $C > 0$ such that for every $F \in C_c(\frac{1}{2}, 2)$ and every nonnegative polynomial W of the form $W(x) = \sum_{\beta \leq \alpha} b_\beta x^\beta$ the kernel $F(-\Delta + W)(x, y)$ of the operator $F(-\Delta + W)$ satisfies*

$$(3.9) \quad |F(-\Delta + W)(x, y)| \leq C\|F\|_{C_c(1/2,2)}(1 + m(x, W))^{-N}.$$

Proof. It follows from (3.4) and Lemma 3.6 that for every $q = 0, 1, 2, \dots$ and $x \in \mathbb{R}^d$ the functional $f \mapsto W^q(x)F(-\Delta + W)f(x)$ is bounded on $L^2(\mathbb{R}^d)$ and

$$(3.10) \quad |W^q(x)F(-\Delta + W)f(x)| \leq C_q\|F\|_{C_c(1/2,2)}\|f\|_{L^2(\mathbb{R}^d)}$$

with a constant $C_q > 0$ independent of $x \in \mathbb{R}^d$ and W . Therefore,

$$(3.11) \quad \int_{\mathbb{R}^d} |W^q(x)F(-\Delta + W)(x, y)|^2 dy \leq C_q\|F\|_{C_c(1/2,2)}^2,$$

where C_q is independent of W and x . Since $F(-\Delta + W) = F(-\Delta + W)\phi(-\Delta + W)^*$, where $\phi \in C_c^\infty(\frac{1}{4}, 4)$, $\phi \equiv 1$ on $[\frac{1}{2}, 2]$, we have

$$(3.12) \quad |W^q(x)F(-\Delta + W)(x, y)| \leq C_q\|F\|_{C_c(1/2,2)}.$$

Similarly, we can prove using (3.4) and Lemma (3.6) that

$$(3.13) \quad |(D^\beta W(x))^q(x)F(-\Delta + W)(x, y)| \leq C_{q,\alpha}\|F\|_{C_c(1/2,2)}.$$

By (3.12) and (3.13) we obtain (3.9). □

Proof of Proposition 2.2. It follows from Lemma 3.3 that

$$(3.14) \quad \begin{aligned} Q_\mu(x, y) &= 2^{\mu d/2}\tilde{Q}_\mu(-\Delta + V^{[2^{-\mu}]})(2^{\mu/2}x, 2^{\mu/2}y) \\ &= 2^{\mu d/2}\tilde{Q}_\mu(A^{[2^{-\mu}]})(2^{\mu/2}x, 2^{\mu/2}y), \end{aligned}$$

where $\tilde{Q}_\mu(\lambda) = Q_\mu(2^\mu\lambda) = \varphi(\lambda)F(2^\mu\lambda)$. Applying Lemma 3.1 we obtain that there exist constants $C > 0$ and $\delta' > 0$ such that

$$(3.15) \quad \|\tilde{Q}_\mu(A^{[2^{-\mu}]})\|_{\omega(\delta')} \leq C\|\tilde{Q}_\mu\|_{C(\frac{\delta}{2}+\varepsilon)} \leq C.$$

Setting $\tilde{K}_\mu(x, y) = \tilde{Q}_\mu(A^{[2^{-\mu}]})(x, y)(1 + |x - y|)^{\delta'/2}$, we have

$$\|\tilde{K}_\mu(x, y)\|_{\omega(\frac{\delta'}{2})} \leq C.$$

This implies that there exist constants $C > 0$ and $0 < \varepsilon' < 1$ such that

$$(3.16) \quad \sup_y \int |\tilde{K}_\mu(x, y)|^{1-\varepsilon'} dx \leq C, \quad \sup_x \int |\tilde{K}_\mu(x, y)|^{1-\varepsilon'} dy \leq C.$$

Moreover, by Lemma 3.8, for every $N > 0$ there exists a constant C_N such that

$$(3.17) \quad |\tilde{Q}_\mu(A^{[2^{-\mu}]})(x, y)| \leq C_N\|\tilde{Q}_\mu\|_{C_c(1/2,2)}(1 + m(x, V^{[2^{-\mu}]})^{-N}.$$

Therefore, by (3.17), we get

$$|\tilde{Q}_\mu(A^{[2^{-\mu}]})(x, y)| \leq C_N |\tilde{K}_\mu(x, y)|^{1-\varepsilon'} (1+|x-y|)^{-\delta'(1-\varepsilon')/2} (1+m(x, V^{[2^{-\mu}]})^{-N\varepsilon'}.$$

Since $m(\frac{x}{t^{1/2}}, V^{[t]}) = t^{1/2}m(x, V)$, we obtain, by (3.14) and (3.16), the required estimates (2.3) and (2.4). \square

Proof of Proposition 2.5. It follows from (3.2) of [5] that the kernels $T_t^{[2^{-\mu}]}(x, y)$ of the semigroup $T_t^{[2^{-\mu}]} = e^{-tA^{[2^{-\mu}]}}$ satisfy

$$(3.18) \quad |\nabla_y T_t^{[2^{-\mu}]}(x, y)| \leq C t^{-(d+1)/2} \left(1 + \frac{|x-y|}{t^{1/2}}\right)^{-d-3},$$

with a constant $C > 0$ independent of μ .

Let $\tilde{Q}_\mu(\lambda)$ be as in the proof of Proposition 2.2. Since $\tilde{Q}_\mu(\lambda) = \tilde{Q}_\mu(\lambda)e^\lambda e^{-\lambda} = \tilde{R}_\mu(\lambda)e^{-\lambda}$, we have $\tilde{Q}_\mu(A^{[2^{-\mu}]}) = \tilde{R}_\mu(A^{[2^{-\mu}]})T_1^{[2^{-\mu}]}$. By virtue of Lemma 3.1 we obtain $\int_{\mathbb{R}^d} |\tilde{R}_\mu(A^{[2^{-\mu}]})(x, z)| dx \leq C$, with $C > 0$ independent of μ . Therefore, applying (3.18), we get

$$\begin{aligned} & \int_{\mathbb{R}^d} |\tilde{Q}_\mu(A^{[2^{-\mu}]})(x, y) - \tilde{Q}_\mu(A^{[2^{-\mu}]})(x, y_0)| dx \\ &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \tilde{R}_\mu(A^{[2^{-\mu}]})(x, z) \left(T_1^{[2^{-\mu}]}(z, y) - T_1^{[2^{-\mu}]}(z, y_0) \right) dz \right| dx \leq C|y - y_0|. \end{aligned}$$

The estimate (2.6) is now a consequence of (3.14). \square

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