

## AN ANALYTIC INVARIANT OF ORDINARY CURVE SINGULARITIES

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ABSTRACT. It is well known that analytically equivalent ordinary plane curve singularities have projectively equivalent tangent cones. In this note we introduce an analytic invariant in order to show two non analytically equivalent ordinary 5-fold points with projectively equivalent (or equal) tangent cones.

Let  $\mathcal{C}$  be a plane curve, having an ordinary singularity of multiplicity 5 at the point  $O$ . Choose analytic local coordinates  $x, y$  in a neighborhood of  $O$  and analytic parametrisations  $\gamma_i(t) = (x_i(t), y_i(t))$  of the branches of the curve, with  $\gamma_i(0) = O$ ,  $\gamma'_i(0) \neq 0$  for  $i = 1, 2, 3, 4, 5$ . We fix the following notations:

$$a_i = \frac{dx_i}{dt}(0), \quad b_i = \frac{dy_i}{dt}(0),$$

$$\kappa_i = b_i \frac{d^2x_i}{dt^2}(0) - a_i \frac{d^2y_i}{dt^2}(0).$$

We could say that  $b_i x - a_i y = 0$  is the tangent line at  $O$  to the  $i$ -th branch and  $\kappa_i$  its curvature, were it not for the fact that the words *line* and *curvature* have no meaning in an analytic setting. Nevertheless we do have the following.

**Proposition.** *The expression*

$$\mathcal{I} = \det \begin{pmatrix} \kappa_1 & a_1^3 & a_1^2 b_1 & a_1 b_1^2 & b_1^3 \\ \kappa_2 & a_2^3 & a_2^2 b_2 & a_2 b_2^2 & b_2^3 \\ \kappa_3 & a_3^3 & a_3^2 b_3 & a_3 b_3^2 & b_3^3 \\ \kappa_4 & a_4^3 & a_4^2 b_4 & a_4 b_4^2 & b_4^3 \\ \kappa_5 & a_5^3 & a_5^2 b_5 & a_5 b_5^2 & b_5^3 \end{pmatrix}$$

*is an analytic relative invariant of  $\mathcal{C}$ , i.e. its vanishing does not depend on the analytic coordinates or parametrisation chosen, nor is it affected by the action of the analytic group on the plane.*

*Proof.* Let us consider the scroll  $S_{1,4} \subset \mathbf{P}^6$  which Semple showed in [3] to be the variety that contains the points infinitely near  $O$  in its second neighborhood.

$$S_{1,4} = \left\{ [z_0, z_1, \dots, z_6] \in \mathbf{P}^6 \mid \text{rank} \begin{pmatrix} z_0 & z_2 & z_3 & z_4 & z_5 \\ z_1 & z_3 & z_4 & z_5 & z_6 \end{pmatrix} = 1 \right\}.$$

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Each one of our five branches has a single point in the second neighborhood of  $O$ . This point is

$$p_i = (\kappa_i a_i, \kappa_i b_i, a_i^4, a_i^3 b_i, a_i^2 b_i^2, a_i b_i^3, b_i^4) \in S_{1,4}.$$

Semple also proved that *transformations of the plane which are analytic at  $O$  and leave  $O$  invariant induce self-collineations on  $S_{1,4}$* . This reduces the problem to seeing that  $\mathcal{I}$  is a projective invariant of  $\{p_1, \dots, p_5\}$ .

It is well known that in  $S_{1,4}$  there is a unique family of rational normal quartic curves cut out by the 4-planes

$$\Pi_\alpha : \begin{cases} \alpha_0 z_0 + \alpha_1 z_2 + \alpha_2 z_3 + \alpha_3 z_4 + \alpha_4 z_5 = 0, \\ \alpha_0 z_1 + \alpha_1 z_3 + \alpha_2 z_4 + \alpha_3 z_5 + \alpha_4 z_6 = 0 \end{cases}$$

with  $\alpha = [\alpha_0, \dots, \alpha_4] \in \mathbf{P}^4$  (see [2] for generalities on rational normal scrolls). Substituting the coordinates of the points, we find that  $p_i \in \Pi_\alpha$  if and only if

$$\omega_i(\alpha) = \alpha_0 \kappa_i + \alpha_1 a_i^3 + \alpha_2 a_i^2 b_i + \alpha_3 a_i b_i^2 + \alpha_4 b_i^3 = 0.$$

Thus there exists a single  $\Pi_\alpha$  containing the five points if and only if the equations  $\omega_i(\alpha) = 0$ ,  $i = 1, \dots, 5$ , have a common solution, that is, if the determinant  $\mathcal{I}$  vanishes. As the property of the five points belonging to one of the rational normal quartics is clearly projectively invariant, the proof is complete.  $\square$

*Remark 1.* If  $\kappa_i = 0 \forall i$  (that is, if  $O$  is an inflection point of every branch), the invariant is obviously zero. Consequently, if for a curve  $C$  the invariant is nonzero, *no analytic transformation* can make its five branches into inflections. Although curvature has no analytic meaning in a smooth point of a curve, it is meaningful to say whether five smooth branches through a point are “curved with respect to one another” or not, and this is told by the invariant  $\mathcal{I}$ .

*Remark 2.* Looking at the proof, we can say further that any number of branches whose points in the second neighborhood lie on a rational normal quartic of  $S_{1,4}$  are *not* curved with respect to one another. One may also note that given 4 or less smooth branches with distinct tangent directions there is a  $\Pi_\alpha$  containing their points of the second neighborhood, so five is the minimum number for which such an invariant is possible.

**Example.** The singularities  $(x^4 - y^4)y = 0$  and  $(x^4 - y^4)(y - x^2) = 0$  are not analytically isomorphic.

#### REFERENCES

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