CONVOLUTION WITH AFFINE ARCLENGTH MEASURES IN THE PLANE

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Abstract. We obtain an estimate for the $L^{3/2,1}(\mathbb{R}^2) - L^3(\mathbb{R}^2)$ norm of a certain convolution operator.

Let $\phi$ be real-valued and smooth on an interval $(a, b) \subseteq \mathbb{R}$. Define the measure $\lambda$ on $\mathbb{R}^2$ by

$$\int_{\mathbb{R}^2} f \, d\lambda = \int_a^b f(t, \phi(t)) \left| \phi''(t) \right|^{1/3} dt.$$ 

We are interested in the $L^p(\mathbb{R}^2) - L^q(\mathbb{R}^2)$ mapping properties of the operator given by convolution with $\lambda$. The study of this operator was initiated by Drury ([D]), who used complex interpolation and certain integral estimates to obtain the optimal result

$$\|\lambda * f\|_3 \leq C \|f\|_{3/2}$$

when $\phi$ satisfies certain regularity conditions. Drury’s approach was refined and extended by Choi in [C]. The purpose of this note is to prove an almost-sharp estimate for a large class of functions $\phi$:

**Theorem.** Suppose $\phi'' > 0$ and $\phi^{(3)} \geq 0$ on $(a, b)$. Then

$$\|\lambda * \chi_E\|_3 \leq 12^{1/3} \|\chi_E\|_{3/2}$$

for any measurable $E \subseteq \mathbb{R}^2$.

**Proof.** Define $\gamma(s) = (s, \phi(s))$. Inequality (1) is equivalent to

$$\int_{\mathbb{R}^2} \int_a^b \int_a^b \prod_{i=1}^3 \left[ \chi_E(x - \gamma(s_i)) \phi''(s_i)^{1/3} \right] ds_1 ds_2 ds_3 dx \leq 12 |E|^2$$

and so will follow from

$$\int_{\mathbb{R}^2} \int_a^b \int_{s_3}^{b} \prod_{i=1}^3 \left[ \chi_E(x - \gamma(s_i)) \phi''(s_i)^{1/3} \right] ds_1 ds_2 ds_3 dx \leq 4 |E|^2.$$
By Fubini’s theorem this is equivalent to
\[
\int_{\mathbb{R}^2} \chi_E(x) \int_a^b \left( \int_t^b \chi_E(x + \gamma(t) - \gamma(s)) \phi''(s)^{1/3} \, ds \right)^2 \phi''(t)^{1/3} \, dt \, dx \leq 4 \, |E|^2.
\]

Thus, it is enough to establish the inequality
\[
\int_a^b \left( \int_t^b \chi_E(\gamma(t) - \gamma(s)) \phi''(s)^{1/3} \, ds \right)^2 \phi''(t)^{1/3} \, dt \leq 4 \, |E|.
\]

The convexity of the graph of \(\phi\) shows that the change of variables
\[(s, t) \rightarrow (s, t, \phi(s) - \phi(t))\]
is one-to-one. Thus
\[
\int_a^b \int_a^b \chi_E(\gamma(t) - \gamma(s)) |\phi'(s) - \phi'(t)| \, ds \, dt \leq |E|
\]
and (2) will follow from the inequality
\[
\phi''(t)^{1/3} \left( \int_t^b \chi_A(s) \phi''(s)^{1/3} \, ds \right)^2 \leq 4 \int_t^b \chi_A(s)(\phi'(s) - \phi'(t)) \, ds
\]
if \(A \subseteq (t, b)\). To prove (3) we let \(|A_u|\) stand for the (one-dimensional) Lebesgue measure of \(A \cap (u, b)\) whenever \(t \leq u \leq b\). Then
\[
\int_t^b \chi_A(s)(\phi'(s) - \phi'(t)) \, ds = \int_t^b \chi_A(s) \int_t^s \phi''(u) \, du \, ds = \int_t^b \phi''(u)|A_u| \, du.
\]

Also
\[
\int_t^b \chi_A(s) \phi''(s)^{1/3} \, ds = \int_t^b \chi_A(s) \phi''(s)^{1/3}|A_u|^{1/3}|A_s|^{-1/3} \, ds
\]
\[
\leq \left( \int_t^b \chi_A(s) \phi''(s)|A_s| \, ds \right)^{1/3} \left( \int_t^b \chi_A(s)|A_s|^{-1/2} \, ds \right)^{2/3}.
\]

Thus, it follows from (4) that
\[
\left( \int_t^b \chi_A(s) \phi''(s)^{1/3} \, ds \right)^3 \leq \left( \int_t^b \chi_A(s)(\phi'(s) - \phi'(t)) \, ds \right) \left( \int_t^b \chi_A(s)|A_s|^{-1/2} \, ds \right)^2.
\]

If \(0 \leq \rho \leq |A|\), then \(|\{s \in A : |A_s| \leq \rho\}| = \rho\), and so
\[
\int_t^b \chi_A(s)|A_s|^{-1/2} \, ds = \int_0^{|A|} y^{-1/2} \, dy = 2 \, |A|^{1/2}.
\]

With this and the fact that \(\phi''\) is nondecreasing, (5) yields (3) to complete the proof.

**References**


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