PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 12, Pages 3591–3592
S 0002-9939(99)05462-3
Article electronically published on July 8, 1999

CONVOLUTION WITH AFFINE ARCLENGTH MEASURES IN THE PLANE

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(Communicated by Christopher D. Sogge)

ABSTRACT. We obtain an estimate for the $L^{3/2,1}(\mathbb{R}^2) - L^3(\mathbb{R}^2)$ norm of a certain convolution operator.

Let ϕ be real-valued and smooth on an interval $(a,b) \subseteq \mathbb{R}$. Define the measure λ on $\mathbb{R}^2$ by

$$\int_{\mathbb{R}^2} f \, d\lambda = \int_a^b f(t,\phi(t)) \, |\phi''(t)|^{1/3} dt.$$

We are interested in the $L^p(\mathbb{R}^2) - L^q(\mathbb{R}^2)$ mapping properties of the operator given by convolution with λ. The study of this operator was initiated by Drury ([D]), who used complex interpolation and certain integral estimates to obtain the optimal result

$$\|\lambda * f\|_3 \le C \, \|f\|_{3/2}$$

when ϕ satisfies certain regularity conditions. Drury's approach was refined and extended by Choi in [C]. The purpose of this note is to prove an almost-sharp estimate for a large class of functions ϕ:

Theorem. *Suppose* $\phi'' > 0$ *and* $\phi^{(3)} \ge 0$ *on* (a,b)*. Then*

$$\|\lambda * \chi_E\|_3 \le 12^{1/3} \, \|\chi_E\|_{3/2} \tag{1}$$

for any measurable $E \subseteq \mathbb{R}^2$.

Proof. Define $\gamma(s) = \big(s, \phi(s)\big)$. Inequality (1) is equivalent to

$$\int_{\mathbb{R}^2} \int_a^b \int_a^b \int_a^b \prod_1^3 \Big[\chi_E\big(x - \gamma(s_i)\big)\phi''(s_i)^{1/3}\Big] \, ds_1 \, ds_2 \, ds_3 \, dx \le 12 \, |E|^2$$

and so will follow from

$$\int_{\mathbb{R}^2} \int_a^b \int_{s_3}^b \int_{s_3}^b \prod_1^3 \Big[\chi_E\big(x - \gamma(s_i)\big)\phi''(s_i)^{1/3}\Big] \, ds_1 \, ds_2 \, ds_3 \, dx \le 4 \, |E|^2.$$

Received by the editors February 16, 1998.
1991 *Mathematics Subject Classification.* Primary 42B15.
The author was partially supported by a grant from the National Science Foundation.

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By Fubini's theorem this is equivalent to

$$\int_{\mathbb{R}^2} \chi_E(x) \int_a^b \Big(\int_t^b \chi_E\big(x+\gamma(t)-\gamma(s)\big)\ \phi''(s)^{1/3} ds\Big)^2 \phi''(t)^{1/3} dt\ dx \le 4\ |E|^2.$$

Thus, it is enough to establish the inequality

$$\int_a^b \Big(\int_t^b \chi_E\big(\gamma(t)-\gamma(s)\big)\ \phi''(s)^{1/3} ds\Big)^2 \phi''(t)^{1/3} dt \le 4\ |E|. \tag{2}$$

The convexity of the graph of ϕ shows that the change of variables

$$(s,t) \to \gamma(s)-\gamma(t) = \big(s-t, \phi(s)-\phi(t)\big)$$

is one-to-one. Thus

$$\int_a^b \int_a^b \chi_E\big(\gamma(t)-\gamma(s)\big)\ |\phi'(s)-\phi'(t)|\ ds\ dt \le |E|$$

and (2) will follow from the inequality

$$\phi''(t)^{1/3}\Big(\int_t^b \chi_A(s)\ \phi''(s)^{1/3} ds\Big)^2 \le 4 \int_t^b \chi_A(s)\big(\phi'(s)-\phi'(t)\big)\ ds \tag{3}$$

if $A \subseteq (t,b)$. To prove (3) we let $|A_u|$ stand for the (one-dimensional) Lebesgue measure of $A \cap (u,b)$ whenever $t \le u \le b$. Then

$$\int_t^b \chi_A(s)\big(\phi'(s)-\phi'(t)\big) ds = \int_t^b \chi_A(s) \int_t^s \phi''(u)\ du\ ds = \int_t^b \phi''(u)|A_u|\ du. \tag{4}$$

Also

$$\int_t^b \chi_A(s)\ \phi''(s)^{1/3} ds = \int_t^b \chi_A(s)\ \phi''(s)^{1/3} |A_s|^{1/3} |A_s|^{-1/3} ds$$

$$\le \Big(\int_t^b \chi_A(s)\ \phi''(s)\ |A_s|\ ds\Big)^{1/3} \Big(\int_t^b \chi_A(s) |A_s|^{-1/2} ds\Big)^{2/3}.$$

Thus, it follows from (4) that

(5)

$$\Big(\int_t^b \chi_A(s) \phi''(s)^{1/3} ds\Big)^3 \le \Big(\int_t^b \chi_A(s)\ \big(\phi'(s)-\phi'(t)\big) ds\Big)\Big(\int_t^b \chi_A(s)|A_s|^{-1/2} ds\Big)^2.$$

If $0 \le \rho \le |A|$, then $|\{s \in A :\ |A_s| \le \rho\}| = \rho$, and so

$$\int_t^b \chi_A(s) |A_s|^{-1/2} ds = \int_0^{|A|} y^{-1/2} dy = 2\ |A|^{1/2}.$$

With this and the fact that ϕ'' is nondecreasing, (5) yields (3) to complete the proof.

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