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# AN INFINITE FAMILY OF MANIFOLDS WITH BOUNDED TOTAL CURVATURE

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ABSTRACT. The negative answer to the following problem of V. I. Arnold is given: Is the number of topologically different $k$-manifolds of bounded total curvature finite?

## §1. INTRODUCTION

In [Ar] V. I. Arnold defined the total curvature $\alpha(g)$ of an immersion $g : M^k \to \mathbf{R}^m$ of a $k$-manifold in the $m$-dimensional euclidean space as the $k$-volume $vol_k(g\Delta\bar{g})(M^k)$ of a submanifold $(g\Delta\bar{g})(M^k) \subset \mathbf{R}^m \times G_{m,k}$ where $\bar{g} : M^k \to G_{m,k}$ is the induced map to the Grassmann manifold. He proved that for any two submanifolds $X, Y \subset \mathbf{R}^m$ and for every diffeomorphism $A : \mathbf{R}^m \to \mathbf{R}^m$ generically the total curvature $\alpha(A^n(X) \cap Y)$ is bounded by $Ce^{\lambda n}$ where $A^n = A \circ \ldots \circ A$ and $C$ and $\lambda$ do not depend on $n$. Then the following question of Arnold appeared naturally: *Is the number of topologically different $k$-manifolds $M$ lying in $\mathbf{R}^m$ with the total curvature $\alpha(M)$ bounded from above finite?*

We note that Arnold's total curvature $\alpha(g)$ differs from the classic Chern-Lashof total curvature $\tau(g)$ [N-K]. Recall that $\tau(g) = \frac{1}{vol(S^{m-1})} \int_{SM^k} \nu^*(d\sigma)$ where $SM^k$ is the unit sphere bundle of the normal bundle of an immersion $g : M^k \to \mathbf{R}^m$, $\nu : SM^k \to S^{m-1}$ is the Gauss map and $d\sigma$ is the volume element on $S^{m-1}$. Arnold's total curvature is always greater than Chern-Lashof's and the last can be estimated from below by the Morse number $m(M^k)$= the minimal number of critical points of Morse functions.

It is easy to show that the answer to Arnold's question is affirmative for 2-manifolds. The purpose of this paper is to show that for 3-manifolds the answer is negative.

**Theorem 1.** *There exist a number $C > 0$ and an infinite family $\{M_i\}$ of pairwise nonhomeomorphic 3-dimensional compact submanifolds of $\mathbf{R}^4$ such that the total curvature $\alpha(M_i) < C$ for all $i$.*

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**Theorem 2.** *There exist a number $C > 0$ and an infinite family $\{\Sigma_i\}$ of pairwise nonhomeomorphic homology 3-spheres with embeddings $\{\eta_i : \Sigma_i \to \mathbf{R}^7\}$ such that $\alpha(\eta_i) < C$ for all $i$.*

The construction of both families of manifolds is basically due to the presence of the fundamental group. This situation suggests the following:

**Conjecture.** *The answer to Arnold's problem is positive for simply connected manifolds.*

## §2. The construction of the first family

Let $K(n,m) \subset S^3$ be an $(n,m)$-torus knot in $S^3$ and let $N(n,m)$ be its regular neighborhood. Let $\{p_i\}$ be a sequence of all odd numbers. For every $i$ let $M_i$ denote the result of doubling the compact 3-manifold $S^3 - IntN(2,p_i)$ along its boundary $\partial N(2,p_i)$.

**Lemma 1.** *For every $i \neq j$ manifolds $M_i$ and $M_j$ are not homeomorphic.*

*Proof.* An $S^1$-action on $S^3 = S^1 * S^1$ defined by the formula

$$(\theta, (\phi, \psi)) \to (\phi + 2\theta, \psi + p_i\theta)$$

has two exceptional orbits with the orbit space homeomorphic to $S^2$. We may assume that $N(2,p_i)$ is a preimage of a 2-disk under the projection to the orbit space. Then it easy to see that each $M_i$ is a Seifert manifold [Or] with the orbit space $S^2$ with 4 exceptional orbits of types $(2,1),(2,1),(p_i,2),(p_i,2)$. Since these orbit invariants are different for $i \neq j$, Seifert bundles on $M_i$ and $M_j$ are different. Note that $M_i$ are large Seifert manifolds for all $i$. Hence $M_i$ and $M_j$ are not homeomorphic for $i \neq j$ [Or, Theorem 6]. □

**Lemma 2.** *Suppose that $g_i : M \to N$ is a sequence of embeddings of a Riemannian manifold $M$ into a Riemannian manifold $N$, converging in the $C^1$-topology to an embedding $g : M \to N$. Then $\lim vol(g_i(M)) = vol(g(M))$, where $vol(L)$ for $L \subset N$ means the volume of submanifold $L$ in $N$.*

The proof follows for instance from [D-N-F], v.1.

Let $D^m$ denote the unit ball in $\mathbf{R}^m$ and let $D^m_r$ be a concentric ball of radius $r$. The *normal regular neighborhood of radius* $r$ of immersed $k$-manifold $f : M^k \to \mathbf{R}^m$ is an immersion $F : M^k \times D^{m-k}_r \to \mathbf{R}^m$ such that 1) $F\mid_{M^k \times \{0\}} = f$ where 0 is the center of the disk $D^{m-k}_r$, 2) for every $t \in M^k$ the restriction $F\mid_{\{t\}\times D^{n-k}_r}$ is an isometrical embedding and the disk $F(\{t\} \times D^{m-k}_r)$ is orthogonal to the manifold $f(M^k)$ at the point $f(t)$.

Every $C^1$-manifold has locally a normal regular neighborhood for some $r$ [H], and hence, every closed manifold with trivial normal bundle has a normal regular neighborhood. Therefore every smooth enough immersion of the circle in $\mathbf{R}^3$ has a normal regular neighborhood. Here is a parameterized version of that statement.

**Lemma 3.** *Suppose that $\{f_i : S^1 \to \mathbf{R}^3\}$ is a sequence of immersions of $S^1$ into $\mathbf{R}^3$, converging in $C^2$-metric to an immersion $f_\infty$. Then there exist $r > 0$ and normal regular neighborhoods $F_i : S^1 \times D_r \to \mathbf{R}^3$ , $i = 1, 2, ..., \infty$, such that $\{F_i\}$ converges in $C^2$-metric to $F_\infty$.*

Let $C_k(\ ,\ )$ be a space of mappings in $C^k$-topology. Denote by $S^1$ the boundary of the unit disk $D^2$ and denote by $w_r : S^1 \to \partial D_r$ the natural projection. In this notation the center of $D^2$ is $\partial D_0$. Suppose that $f : S^1 \to \mathbf{R}^3$ is a $C^2$-immersion with a normal regular neighborhood $F$ of radius $r$. We define a map $R_f : [0,r] \to C_2(S^1 \times S^1, \mathbf{R}^3)$ by setting $R_f(x) = F \circ (id_{S^1} \times w_r) : S^1 \times S^1 \to \mathbf{R}^3$.

**Lemma 4.** *Suppose that a sequence $\{f_i : S^1 \to \mathbf{R}^3\}$ is converging to $f_\infty$ in $C^2$-metric and assume that $r$ is given by Lemma 3. Then $R_{f_i}$ uniformly converges to $R_{f_\infty}$.*

We omit the proof since it is straightforward.

We define the 'Gauss map' $G_f : [0,r] \to C_1(S^1 \times S^1, G_{3,2})$ for $R_f$ in the following way. For $x > 0$ the value $G_f(x)$ is the tangent bundle map $\nu : S^1 \times S^1 \to G_{3,2}$ for $S^1 \times \partial D_x$. For $x = 0$ for every $(t,\theta) \in S^1 \times S^1$ the value $G_f(0)(t,\theta)$ is the span of the derivative $f'(t)$ and the vector orthogonal to $F(t,(r,\theta)) - F(t,0) = b(t,\theta)$. Here $(r,\theta) \in D_r$ means a point written in polar coordinates. Note that for all $x \in [0,r]$ the Gauss map is defined as $G_f(x)(t,\theta) = span\{\frac{\partial F(t,(x,\theta))}{\partial t}, b(t,\theta)^\perp\}$.

**Lemma 5.** *Suppose that a sequence $\{f_i : S^1 \to \mathbf{R}^3\}$ is converging to $f_\infty$ in $C^2$-metric and assume that the number $r$ is provided by Lemma 3. Then $G_{f_i}$ converges uniformly to $G_{f_\infty}$.*

*Proof.* According to Lemma 3 a sequence $F_i$ converges uniformly to $F_\infty$ in $C^2$-metric. It implies a uniform convergence of $\frac{\partial F_i}{\partial t}$ to $\frac{\partial F_\infty}{\partial t}$ in $C^1$-metric. The second vector $b_i(t,\theta)^\perp$ does not depend on $x$, and therefore by virtue of Lemma 3 the sequence $b_i(t,\theta)^\perp$ converges to $b_\infty(t,\theta)^\perp$ in $C^1$-metric as a sequence of functions in $t$ and $\theta$. This implies the lemma. □

*The proof of Theorem 1.* We construct manifolds $M_i$ in $\mathbf{R}^4$ symmetrically with respect to the hyperplane $\{0\} \times \mathbf{R}^3$. Let us consider two symmetric 3-dimensional spheres $S^3_+ \subset [1,\infty) \times \mathbf{R}^3$ and $S^3_- \subset (-\infty,-1] \times \mathbf{R}^3$ such that the intersection $S^3_+ \cap (\{1\} \times \mathbf{R}^3)$ is a 3-disk $D^3_+$ and similarly $S^3_- \cap (\{-1\} \times \mathbf{R}^3) = D^3_-$. Let $S^1 \subset D^3_+$ be a smoothly embedded circle and let $f : S^1 \to S^1 \subset D^3_+$ be a double winding around that circle ($f \in C^2$). For every $i$ we consider the boundary of the normal regular neighborhood of $S^1$ of radius $\epsilon_i$ and realize a torus knot $K(2,p_i)$ on it. For small enough $\epsilon_i$ it is possible to realize the knot $K(2,p_i)$ by a map $f : S^1 \to \mathbf{R}^3$ which is $1/i$-close to $f$ in $C^2$-metric. By virtue of Lemma 3 there exist a number $r$ and the normal regular neighborhoods $F_i$ of radius $r$. For every $i$ there exists $r_i < r$ such that the restriction of $F_i$ on $S^1 \times D^2_{r_i}$ is an embedding. We denote $N(2,p_i) = F_i(S^1 \times D^2_{r_i})$. Let $C_i$ be the cylinder over the boundary $\partial N(2,p_i)$ that connects $S^3_+$ with $S^3_-$. We define $M_i = (S^3_+ - IntN(2,p_i)) \cup C_i \cup (S^3_- - IntN(2,p_i))$.

We will write $\alpha(M)$ if the immersion of $M$ is fixed. Since

$$\alpha(M_i) \leq \alpha(S^3_+) + \alpha(S^3_-) + \alpha(\partial N(2,p_i) \times [-1,1]),$$

it suffices to find a common estimate for $\alpha(\partial N(2,p_i) \times [-1,1])$. By an obvious version of the Kuiper theorem [N-K] for the Arnold total curvature it follows that $\alpha(\partial N(2,p_i) \times [-1,1]) = 2\alpha(\partial N(2,p_i))$.

Lemmas 4, 5 imply that $\lim_{i\to\infty} R_{f_i}(r_i) = R_{f_\infty}(0)$ and $\lim_{i\to\infty} G_{f_i}(r_i) = G_{f_\infty}(0)$. Denote by $A_i$ the diagonal product of $R_i$ and $G_i$, $i = 1,2,\ldots,\infty$. By Lemma 2 it follows that $\lim_{i\to\infty} vol(A_i(r_i)(S^1 \times S^1)) = vol(A_\infty(0)(S^1 \times S^1))$. Since $vol(A_i(r_i)(S^1 \times S^1)) = \alpha(\partial N(2,p_i))$, it follows that the sequence $\alpha(\partial N(2,p_i))$

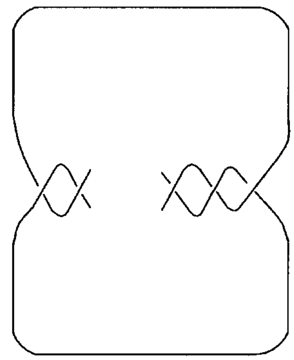

FIGURE 1

has an upper bound. So, it remains to note that by Lemma 1 the family $\{M_i\}$ is pairwise nonhomeomorphic. $\square$

*Remark.* Manifolds $M_i$ are not smoothly embedded in $\mathbf{R}^4$ because at the points where spheres $S^3_+$ and $S^3_-$ are attached to the cylinder $C_i$ we have singularities. If we smooth these angles by a small perturbation near $\partial N(2, p_i)$, then $\alpha(M_i)$ will increase by approximately $vol(\partial N(2, p_i)) \times \pi/2$. Since $vol(\partial N(2, p_i)) \to 0$, then the sequence $\alpha(M_i)$ for resulting smooth manifolds $M_i$ also will be bounded.

## §3. THE CONSTRUCTION OF THE SECOND FAMILY

For any knot $K \subset S^3$ we denote by $K_1$ the 3-manifold obtained from $S^3$ by Dehn surgery on the knot $K$ with the homology class $(1, 1)$ [R]. Let $K^m$ be the following knot with $2m + 1$-crossings shown in Figure 1:

**Lemma 6.** *The manifolds $K^m_1$ and $K^n_1$ are homeomorphic if and only if $m = n$.*

*Proof.* The direct computation of the Casson invariant $\lambda(K^k_1)$ shows that $\lambda(K^m_1)$ and $\lambda(K^n_1)$ are different for $m \neq n$ [A-M]. $\square$

By the definition a normal regular neighborhood induces a framing of the normal vector bundle. Let $\xi : S^1 \to \mathbf{R}^3$ be an immersion of the circle in $\mathbf{R}^3$ and let $F^k : S^1 \times D^2_r \to \mathbf{R}^3$ be a normal regular neighborhood of $\xi$ of radius $r$ such that the induced framing has Hopf invariant equal to $k$. We consider the embedding of $\mathbf{R}^3$ in $\mathbf{R}^7$ as a factor $\mathbf{R}^3 \times 0 \times ... \times 0 \subset \mathbf{R}^7$. A map $\phi : (M, N) \to (\mathbf{R}^7, \mathbf{R}^3)$ is said to be a relative embedding if the restriction of $\phi$ on $N$ is an immersion into $\mathbf{R}^3$ and the restriction of $\phi$ on $M - N$ is an embedding in $\mathbf{R}^7 - \mathbf{R}^3$, transversal to $\mathbf{R}^3$.

**Lemma 7.** *Let $f : (D^2, \partial D^2) \to (\mathbf{R}^7, \mathbf{R}^3)$ be a relative embedding of class $C^2$. Then there exist $r > 0$ and a relative embedding of class $C^2$, $F : (D^2 \times D^2_r, \partial D^2 \times D^2_r) \to (\mathbf{R}^7, \mathbf{R}^3)$, such that*

*1) $F|_{D^2 \times \{0\}} = f$.*

*2) $f|_{\partial D^2 \times D^2_r} = F^1$ is a normal regular neighborhood in $\mathbf{R}^3$, with Hopf invariant equal to one.*

*3) For every $x \in IntD^2$ the restriction $F|_{\{x\} \times D^2_r}$ imbeds the disk $D^2_r$ isometrically into the 5-plane, orthogonal to $f(D^2)$ at the point $f(x)$.*

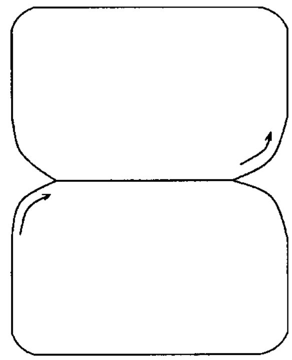

FIGURE 2

*Proof.* The map $f$ induces a map $\nu(f) : D^2 \to G_{7,5}$ via the normal bundle. Let us consider a fibration $\eta : E \to G_{7,5}$ with the Stiefel manifold $V_{5,2}$ as a fiber. Here $E$ is a subset of $G_{7,5} \times V_{7,2}$ defined in the following way: $L \times (a, b) \in E$ if and only if $a, b \in L$ and $\eta(L \times (a, b)) = L$. First we choose a normal regular neighborhood $F^1$ of $f \mid_{\partial D^2}$ of some radius $r$ and with Hopf invariant equal to 1. The map $F^1$ induces a lifting $\beta : \partial D^2 \to E$ of a map $\nu(f) \mid_{\partial D^2} : \partial D^2 \to G_{7,5}$. Since the fundamental group $\pi_1(V_{5,2})$ is trivial, there is a lifting $\gamma : D^2 \to E$ extending $\beta$. The map $\gamma$ defines a map $\Phi : D^2 \times \mathbf{R}^2 \to \mathbf{R}^7$ such that (1) $\Phi \mid_{D^2 \times \{0\}} = f$ , (2) $\Phi \mid_{\partial D^2 \times D^2_{r_0}} = F^1$ and (3) for every $x \in IntD^2$, the restriction $\Phi \mid_{\{x\} \times \mathbf{R}^2}$ is an isometric imbedding into a normal plane $\nu(f)(x)$ at the point $f(x) \in \mathbf{R}^7$. In order to complete the proof we choose $r < r_0$ such that the restriction $F = \Phi \mid_{D^2 \times D^2_r}$ is a relative embedding. □

We note that the following parameterized version of Lemma 7 is valid.

**Lemma 8.** *Suppose that a sequence $f_i : (D^2, \partial D^2) \to (\mathbf{R}^7, \mathbf{R}^3)$ converges in $C^2$-metric to $f$. Then there exists $r > 0$ and a sequence of maps $F_i : (D^2 \times D^2_r, \partial D^2 \times D^2_r) \to (\mathbf{R}^7, \mathbf{R}^3)$ with the properties (1)-(3) of Lemma 6 for $f_i$ for every $i = 1, 2, ..., \infty$, such that $F_i$ converges in $C^2$-metric to $F_\infty$.*

*Proof of Theorem 2.* Consider the 3-sphere $S^3$ in $\mathbf{R}^7$ with a flat part in $\mathbf{R}^3 \subset \mathbf{R}^7$. We define a sequence of embeddings $f_i : (D^2, \partial D) \to (\mathbf{R}^7, \mathbf{R}^3)$, $C^2$-converging to a relative embedding $f_\infty$ with $f_i(\partial D^2) \subset S^3$ and $f_i(IntD^2) \cap S^3 = \emptyset$ for $i = 1, 2, ..., \infty$. Let $g : S^1 \to \mathbf{R}^3 \cap S^3$ be an immersion of the circle with the self-intersection along the interval as shown in Figure 2.

Let $f : D^2 \to \mathbf{R}^7$ be an extension of $g$ to a relative $C^2$-embedding with $f(IntD^2) \cap (S^3) = \emptyset$. For every $i$ we realize the knot $K^i$ by a map $g_i : S^1 \to \mathbf{R}^3 \cap S^3$ such that the distance between $g$ and $g_i : S^1 \to \mathbf{R}^3$ in $C^2$-metric is less than $1/i$. For every $i$ there is an extension $f_i$ of $g_i$ which is $2/i$-close to $f$ and moreover $f_i$ is an embedding. Apply Lemma 8 to the sequence $f_i$ to obtain $F_i$, $i = 1, 2, ..., \infty$. For every $i < \infty$ there exists a small number $\epsilon_i > 0$ such that the restriction $R_i = F_i \mid_{D^2 \times \partial D^2_{\epsilon_i}}$ is an embedding. We define $\Sigma_i = S^3 - (S^1 \times D^2_{\epsilon_i}) \cup ImR_i$ for all $i$. It easy to check that $\Sigma_i$ is homeomorphic to $K^i_1$.

In order to complete the proof it is sufficient to find a common upper bound for $\alpha(ImR_i)$. For large $i$ there is a rough estimate for $\alpha(ImR_i)$ as $2\pi\alpha(f(D^2))$. That implies an upper bound for the sequence $\{\alpha(ImR_i)\}$. Thus, Lemma 6 completes the proof. □

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