ON SEMISIMPLE HOPF ALGEBRAS OF DIMENSION \( pq \)

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Abstract. We consider the problem of the classification of semisimple Hopf algebras \( A \) of dimension \( pq \) where \( p < q \) are two prime numbers. First we prove that the order of the group of grouplike elements of \( A \) is not \( q \), and that if it is \( p \), then \( q = 1 \mod p \). We use it to prove that if \( A \) and its dual Hopf algebra \( A^* \) are of Frobenius type, then \( A \) is either a group algebra or a dual of a group algebra. Finally, we give a complete classification in dimension \( 3p \), and a partial classification in dimensions \( 5p \) and \( 7p \).

In this paper we consider semisimple Hopf algebras of dimension \( pq \) over an algebraically closed field \( k \) of characteristic 0, where \( p \) and \( q \) are distinct prime numbers. Masuoka has proved that a semisimple Hopf algebra of dimension \( 2p \) over \( k \), where \( p \) is an odd prime, is trivial (i.e. is either a group algebra or a dual of a group algebra) [Ma1]. Izumi and Kasaki have proved that Kac algebras (i.e. semisimple Hopf algebras over the field of complex numbers, with an additional condition on the existence of an involution), of dimension \( 3p \) over \( k \), where \( p \) is prime, are trivial [IK]. Thus, a natural conjecture is:

Conjecture 1. Any semisimple Hopf algebra of dimension \( pq \) over \( k \), where \( p \) and \( q \) are distinct prime numbers, is trivial.

A well known property of \( A \), a finite dimensional semisimple group algebra or a dual of a group algebra, is that it is of Frobenius type; that is, the dimension of any irreducible representation of \( A \) divides the dimension of \( A \) (the definition is due to Montgomery [Mo]). A special case of Kaplansky’s 6th conjecture [K] is:

Conjecture 2. Any semisimple Hopf algebra of dimension \( pq \) over \( k \), where \( p \) and \( q \) are distinct prime numbers, is of Frobenius type.

In this paper we prove among the rest that Conjecture 1 is equivalent to Conjecture 2 (see Theorem 3.5).

A major role in the analysis is played by \( G(A) \) (where \( G(A) \) denotes the group of grouplike elements of \( A \)). By [NZ], \( |G(A)| \) is either 1, \( p, q \) or \( pq \). We prove in Theorem 2.1 that if \( p < q \), then \( |G(A)| \neq q \), and if \( |G(A)| = p \), then \( q = 1 \mod p \).
Consequently, we prove in Theorem 2.2 that if \( |G(A)| \neq 1 \) and \( q \neq 1 \mod p \), then \( A \) is a commutative group algebra.

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Thus Theorem 2.2 suggests the following question: When is $|G(A)| \neq 1$? In Proposition 3.1 we prove that this is guaranteed when $A^*$ is of Frobenius type, and in Theorem 3.2 we prove that if moreover $q \neq 1 \pmod{p}$, then $A$ is a commutative group algebra. In Theorem 3.4 we prove that if $|G(A^*)| \neq 1$ and $A^*$ is of Frobenius type, then $A$ is trivial, and $|G(A)| = p < q$ or $pq$. The equivalence of Conjectures 1 and 2 is thus a consequence of Proposition 3.1 and Theorem 3.4.

A complete classification of semisimple Hopf algebras of dimension $3p$ is then given in Theorem 4.3. Indeed, they are all trivial.

We conclude by using Theorem 2.2 to prove in Theorem 4.5 that if $A$ is a semisimple Hopf algebra of dimension $5p$, $p$ an odd prime, and if $p = 2$ or $4 \pmod{5}$ or $p \in \{13, 23\}$, then $A$ is a commutative group algebra. Moreover, we obtain in Theorem 4.6 the same result for semisimple $A$ of dimension $7p$, $p$ a prime, and $p = 6 \pmod{7}$ or $p \in \{17, 31\}$.

1. Preliminaries

In this paper $k$ will always denote an algebraically closed field of characteristic 0.

Recall that a finite dimensional Hopf algebra over $k$ is semisimple if and only if it is cosemisimple [LR].

Let $A$ be semisimple Hopf algebra over $k$, and let $\rho_V : A \to \text{End}_k(V)$ be a finite dimensional representation of $A$. The associated character $\chi_V$ is given by $\chi_V(a) = tr(\rho_V(a))$ for all $a \in A$. A character $\chi_V$ is called irreducible if the representation $V$ is irreducible. Let $R(A)$ denote the character ring of $A$; that is, the $k$-subalgebra of $A^*$ generated by the characters $\chi_V$ of finite dimensional $A$-modules $V$. The set of all irreducible characters forms a basis of $R(A)$ [La]. Zhu has proved that $R(A)$ is semisimple and if $e_{A^*}, e_1, \ldots, e_k$ are the primitive idempotents of $R(A)$, where $e_{A^*}$ is an integral of $A^*$, then

$$
(1) \quad \dim A = 1 + \sum_{i=1}^{k} \dim(e_i A^*)
$$

and the dimension of each $e_i A^*$ divides the dimension of $A [Z]$. Note that $\dim R(A) \geq k + 1$, and equality holds if and only if $R(A)$ is commutative.

Let $f : A \to A^*$ be the map given by $f(a) = a \mapsto \lambda = \sum(a, \lambda_{(2)})\lambda_{(1)}$ for all $a \in A$, where $\lambda$ is a non-zero integral of $A^*$. Recall that $f$ gives a linear isomorphism between $kG(A)$ and the sum of the 1-dimensional ideals of $A^*$, and a linear isomorphism between the center $Z(A)$ of $A$ and $R(A)$. Therefore, using the notation of (1), $\dim(e_i A^*) = 1$ for some $i$ if and only if $G(A) \cap Z(A) \neq \{1\}$.

Let $A$ be a semisimple Hopf algebra over $k$. Any simple subcoalgebra $C_l$ of $A^*$ has a basis $\{x_{ij}^l | 1 \leq i, j \leq n_l\}$, where $\Delta(x_{ij}^l) = \sum_{k=1}^{n_l} x_{ik}^l \otimes x_{kj}^l$ and $\varepsilon(x_{ij}^l) = \delta_{i,j}$. Note that $n_l = 1$ if and only of $C_l = \{g\}$ for some $g \in G(A)$. Nichols and Richmond have proved that if $\dim A$ is odd, then $A$ does not have a 2-dimensional irreducible module [NR], hence

$$
(2) \quad \dim A = |G(A)| + \sum_{l} n_l^2, \quad n_l \geq 3.
$$

Now, $L$ is an irreducible left coideal of $C_l$ if and only if

$$
(3) \quad L = L_j^k = sp\{x_{kj}^l | 1 \leq k \leq n_l\}
$$
for some $1 \leq j \leq n_l$. Similarly, $R$ is an irreducible right coideal of $C_l$ if and only if

$$R = R_k^j = sp\{x_{kj}^j|1 \leq j \leq n_l\}$$

for some $1 \leq k \leq n_l$. Note that

$$\dim(L_j^i \cap R_k^j) = 1$$

for any $1 \leq j, k \leq n_l$.

In what follows we recall some of the properties of a Hopf algebra with a projection, which we shall use in the sequel.

**Theorem 1.1 ([R]).** If $H \xrightarrow{i} A \xrightarrow{\pi} H$ is a sequence of finite dimensional Hopf algebra maps where $i$ is injective, $\pi$ is surjective and $\pi \circ i = id_H$, then there exists $B \subseteq A$ so that:

(i) $B$ is a left $H$-module algebra and coalgebra via the adjoint action.

(ii) $B$ is a left $H$-comodule algebra and coalgebra via $\rho(b) = \sum b^{(1)} \otimes b^{(2)} = \sum \pi(b_{(1)}) \otimes b_{(2)}$.

(iii) $B \cong A/AH^+$ as a coalgebra, via the map $b \times h \mapsto be(h)$.

(iv) $B$ is a left coideal subalgebra of $A$.

(v) As an algebra $A = B \times H$ is a smash product.

(vi) As a coalgebra $A = B \times H$ is a smash coproduct, that is: $\Delta(b \times h) = \sum b_{(1)} \times h_{(1)} b_{(2)} h_{(2)}$.

(vii) The map $B \times H \to A(b \times h \mapsto bi(h))$ is an isomorphism of bialgebras.

2. On the order of $G(A)$ and $G(A^*)$

In this section we prove some results concerning the group of grouplike elements of semisimple Hopf algebras of dimension $pq$.

**Theorem 2.1.** Let $A$ be a semisimple Hopf algebra of dimension $pq$ over $k$, where $p < q$ are two prime numbers. Then:

1. $|G(A)| \neq q$.

2. If $|G(A)| = p$, then $q = 1 \pmod{p}$.

**Proof.** 1. Suppose to the contrary that $|G(A)| = q$. If $G(A) \cap Z(A) = G(A)$, then $H = kG(A)$ is central in $A$, hence is a normal sub-Hopf algebra of $A$. Since $A/AH^+$ is a Hopf algebra of dimension $p$ it follows by [Z] that $A/AH^+ \cong kC_p$. An elementary argument which follows from [Ma2, Section 2], shows that $A$ is isomorphic as an algebra to the twisted group ring $kC_p^t[C_p]$ of the cyclic group $C_p$ over the commutative algebra $kC_q$, and hence must be commutative. Thus, $A^*$ is a group algebra and hence of Frobenius type. By (2), $pq = q + ap^2 + bq^2$ for some integers $a, b \geq 0$. But $p < q$, hence $b = 0$, which yields a contradiction. We conclude that $G(A) \cap Z(A) = \{1\}$. Therefore, using the notation of (1), $\dim(\varepsilon_i A^*) \in \{p, q\}$ for all $i$. Let $E_0$ be the integral of $H = kG(A)$ with $\varepsilon(E_0) = 1$. Since $\dim(A/AH^+) = p$, it follows that $AH^+ = A(1 - E_0)$ has dimension $(q - 1)p$ and thus $\dim(AE_0) = p$. Moreover, $E_0 e_A = e_A$, hence $E_0 = e_A + \sum_j e_{ij}$. But $p < q$, hence counting dimensions yields a contradiction and the result follows.

2. If $G(A) \cap Z(A) = G(A)$, then $H = kG(A)$ is central in $A$, hence $A$ is commutative. Therefore, $A^*$ is a group algebra and hence of Frobenius type. By (2), $pq = p + ap^2 + bq^2$ for some integers $a, b \geq 0$. Clearly, $b = 0$ and hence $q = 1 + ap$. If $G(A) \cap Z(A) = \{1\}$, then using the notation of (1), $\dim(\varepsilon_i A^*) \in \{p, q\}$ for all $i$. Since $\dim(A/AH^+) = q$, it follows that $AH^+ = A(1 - E_0)$ has dimension $(p - 1)q$. 


and thus dim(\(AE_0\)) = q. Hence, \(E_0 = e_A + \sum_j e_{ij}\). But, counting dimensions yields that dim(\(e_j A^*\)) = p for all \(j\), and the result follows in this case as well. 

As a direct consequence of Theorem 2.1 we have:

**Theorem 2.2.** Let \(A\) be a semisimple Hopf algebra of dimension \(pq\) over \(k\), where \(p < q\) are two prime numbers satisfying \(q \not\equiv 1 \pmod{p}\). If \(|G(A)| \neq 1\), then \(A\) is a commutative group algebra.

### 3. The main result

In this section we consider semisimple Hopf algebras \(A\) of dimension \(pq\) such that \(A^*\) is of Frobenius type. First we find out when \(|G(A)| \neq 1\) is guaranteed.

**Proposition 3.1.** Let \(A\) be a semisimple Hopf algebra of dimension \(pq\) over \(k\), where \(p < q\) are two prime numbers. If \(A^*\) is of Frobenius type, then either \(|G(A)| = p\) and \(q = 1 \pmod{p}\), or \(|G(A)| = pq\).

**Proof.** If \(A\) is cocommutative, then \(|G(A)| = pq\). Otherwise, \(|G(A)| \neq pq\), and by Theorem 2.1, \(|G(A)| \neq q\). If \(|G(A)| = 1\), then by (1), \(pq = 1 + ap^2 + bq^2\) for some integers \(a, b \geq 0\), as \(A^*\) is of Frobenius type. But, \(q^2 > pq\) hence \(b = 0\) which yields a contradiction. 

As a corollary of Proposition 3.1 we have:

**Theorem 3.2.** Let \(A\) be a semisimple Hopf algebra of dimension \(pq\) over \(k\), where \(p < q\) are two prime numbers satisfying \(q \not\equiv 1 \pmod{p}\). If \(A^*\) is of Frobenius type, then \(A\) is a commutative group algebra.

In the following proposition we determine the coalgebra structure of \(A\).

**Proposition 3.3.** Let \(A\) be a non-cocommutative and non-commutative semisimple Hopf algebra of dimension \(pq\) over \(k\), where \(p < q\) are prime numbers. Let \(R(A^*)\) be the character ring of \(A^*\). If \(A^*\) is of Frobenius type, then:

1. \(R(A^*)\) is commutative.
2. As a coalgebra \(A = k1 \oplus kg \oplus \cdots \oplus kg^{p-1} \oplus C_1 \oplus \cdots \oplus C_a\), where \(a = \frac{q-1}{p}\), \(g\) is a grouplike element and \(C_i\) is a simple subcoalgebra of \(A\) of dimension \(p^2\) for all \(1 \leq i \leq a\).
3. \(gC_i = C_i g\) for all \(1 \leq i \leq a\).

**Proof.** Set \(H = kG(A)\). By Theorem 2.1 and Proposition 3.1, \(\dim H = p\). If \(H\) is central in \(A\), then (as in the proof of Theorem 2.1) \(A\) must be commutative. Therefore, we conclude that \(G(A) \cap Z(A) = \{1\}\).

Set \(n = \dim R(A^*) - 1\). Then, by (1) there exist two natural numbers \(a\) and \(b\) such that:

(6) \hspace{1cm} \(pq = 1 + ap + bq\)

and

(7) \hspace{1cm} \(n = a + b\).

Clearly, \(a \geq 1\) and \(b < p\). Moreover, \(A^*\) is of Frobenius type and \(p < q\), hence by (2)

(8) \hspace{1cm} \(pq = p^2(n + 1 - p) + p\).
Proof. Let $A$ be a semisimple Hopf algebra of dimension $pq$ over $k$, where $p < q$ are prime numbers. If $A^*$ is of Frobenius type and $|G(A^*)| \neq 1$, then $A$ is trivial, and $|G(A)| = p < q$ or $pq$.

Proof. If $A$ is either cocommutative or commutative, then $A$ is either a group algebra or a dual of a group algebra respectively. In any event $A^*$ is of Frobenius type, hence by Proposition 3.1, $|G(A)| = p < q$ or $pq$ and we are done.

Suppose that $A$ is not cocommutative and not commutative. Then Proposition 3.3 is applicable. Set $H = kG(A)$. By Proposition 3.1, $|G(A)| = p$. Let $g$ be a generator of $G(A)$. By Theorem 2.1, $|G(A^*)| \neq q$, hence $|G(A^*)| = p$ too. Thus we have the following sequence of maps:

$$H \xrightarrow{i} A \xrightarrow{\pi} H$$
where $i$ is the inclusion map and $\pi$ is a surjection homomorphism of Hopf algebras. If $\pi \circ i = \varepsilon$, then $H \subseteq K = A^{\alpha H}$. Since $K$ is a left coideal of $A$, it is a direct sum of irreducible left coideals $K = k_1 \oplus k_2 \oplus \cdots \oplus k_{p-1} \oplus V_1 \oplus \cdots \oplus V_n$. Since $A^\ast$ is of Frobenius type it follows that $\dim V_i = p$ for all $i$. But, this is a contradiction since $p$ does not divide $\dim K = q$. Therefore $\pi \circ i \neq \varepsilon$ and we may assume that $\pi \circ i = id_H$.

Therefore by Theorem 1.1, there exists $B \subseteq A$ so that $A \cong B \times H$. By Theorem 1.1(iv), $B$ is a left coideal of $A$, hence a direct sum of irreducible left coideals of $A$. By Proposition 3.3(3), the dimensions of these irreducible left coideals are either 1 or $p$. Since $\dim B = q$, it follows that $B$ contains an irreducible left coideal $V$ of $A$, of dimension $p$. Since $B \subseteq C$ for some $p^2$-dimensional simple subcoalgebra $C$, it follows by Theorem 1.1(ii) and Proposition 3.3(3), that $V \times H \subseteq C$. But, $\dim(V \times H) = p^2 = \dim C$, hence $V \times H = C$. By Theorem 1.1(iii), $A/AH^+ \cong B$ as coalgebras, and $V$ is the image of $C = V \times H$ under this isomorphism, hence $V$ is a subcoalgebra of $B$.

We wish to prove that $V$ is a simple subcoalgebra of $B$ and thus to reach a contradiction. Note that since $V$ is an irreducible left coideal of $A$ it follows that $V \times g^i$ is also an irreducible left coideal of $A$ for all $0 \leq i \leq p - 1$. By Proposition 3.3(3), it follows that

$$\{V \times g^i | 0 \leq i \leq p - 1\}$$

is the set of all the irreducible left coideals of $A$ contained in $C$. Since $V$ is a left coideal of $A$, it follows from Theorem 1.1(ii) that $V$ is an $H$ subcomodule of $B$. Let $\rho : B \rightarrow H \otimes B$ be the comodule structure map, and write $V = \bigoplus_{i=0}^{p-1} V_i$, where $V_i = \rho^{-1}(g^i \otimes V)$. We claim that $\dim V_i = 1$ for all $i$. Indeed, let $\{v_0, \ldots, v_{p-1}\}$ be a basis of $V$ consisting of homogeneous elements; that is, $\rho(v_i) = g^m_i \otimes v_i$ for some $0 \leq m_i \leq p - 1$. Let $0 \neq v \in V$ and write $\Delta_B(v) = \sum_{i=0}^{p-1} b_i \otimes v_i$. Then by Theorem 1.1(vi),

$$\Delta_A(v \times 1) = \sum_{i=0}^{p-1} b_i \times g^m_i \otimes v_i \times 1.$$ 

Therefore, using Kaplansky's notation [K], $L(v \times 1) = sp\{b_i \times g^m_i | 0 \leq i \leq p - 1\} \subseteq C$ is a right coideal of $A$ of dimension $\leq p$. Since $C$ is a simple subcoalgebra of $A$ of dimension $p^2$, it follows that $\dim(L(v \times 1)) = p$ and $L(v \times 1)$ is irreducible.

Therefore by (5), $\dim(L(v \times 1) \cap (V \times g^i)) = 1$ for all $i$, hence $\{m_i | 0 \leq i \leq p - 1\} = \{0, 1, \ldots, p - 1\}$. Thus $V$ has a basis $\{v_i | 0 \leq i \leq p - 1\}$, where $\rho(v_i) = g^i \otimes v_i$. Since $V$ is an $H$-comodule coalgebra it follows that $\Delta_B(v_i) = \sum_{j=0}^{p-1} \alpha_{ij} v_j \otimes v_{i-j}$, hence $\Delta_A(v_i) = \sum_{j=0}^{p-1} \alpha_{ij} v_j \otimes v_{i-j}$ for all $i$, where $\alpha_{ij} \in k$. Computing $\Delta_A(v \times 1)$ yields that $R_i = L(v_i \times 1) = sp\{\alpha_{ij} v_j \times g^{i-j} | 0 \leq j \leq p - 1\} \subseteq C$ is a right coideal of $A$ of dimension $\leq p$, for all $i$. Hence $\dim R_i = p$ and

$$R_i = sp\{v_j \times g^{i-j} | 0 \leq j \leq p - 1\}$$

is irreducible. It is straightforward to verify that $R_i \neq R_t$ for $i \neq t$, and hence the set $\{R_i | 0 \leq i \leq p - 1\}$ is the set of all the irreducible right coideals of $A$ which are contained in $C$.

Finally, let $D \subseteq V$ be a subcoalgebra. By Theorem 1.1(vi), $D \times H \subseteq C$ is a right coideal of $A$ and hence $D \times H = \bigoplus_i R_{ti}$, where $R_{ti}$ is as in (10). But, the image of $D \times H$ under the map $id \otimes \varepsilon : A \rightarrow B$ equals $D$, while the image of $\bigoplus_i R_{ti}$ under
this map equals $V$. Therefore $D = V$, and hence $V$ is a simple coalgebra. But, this is a contradiction since $\dim V = p$ is not a square.

As a corollary we obtain the following:

**Theorem 3.5.** Let $A$ be a semisimple Hopf algebra of dimension $pq$ over $k$, where $p < q$ are prime numbers. If both $A$ and $A^*$ are of Frobenius type, then $A$ is trivial.

**Proof.** Follows from Proposition 3.1 and Theorem 3.4.

### 4. The dimensions $3p, 5p$ and $7p$

We start this section with a complete classification of semisimple Hopf algebras of dimension $3p$.

**Proposition 4.1.** Let $A$ be a non-cocommutative semisimple Hopf algebra of dimension $3p$ over $k$, where $p > 3$ is prime. Then $|G(A)| = 3$.

**Proof.** By Theorem 2.1, $|G(A)| \neq p$. Since $A$ is non-cocommutative, $|G(A)| \neq 3p$. Assume $|G(A)| = 1$ and let $R(A^*) \subseteq A$ be the ring of characters of $A^*$. Set $n = \dim R(A^*) - 1$. Then by (1), there exist two natural numbers $a$ and $b$ such that:

$$3p = 1 + 3a + bp \quad \text{and} \quad n \geq a + b.$$ 

Note that $a \geq 1$ and hence $b = 1$ or $2$. Since 2 does not divide $3p$, and $A^*$ is semisimple we have by [NR] that $A^*$ does not have a 2-dimensional irreducible module and hence the following two inequalities hold:

$$n \geq a + 1 = \frac{(3-b)p + 2}{3} \quad \text{and} \quad 3p \geq 9n + 1.$$ 

But these two inequalities are incompatible since they imply that $(-6 + 3b)p \geq 7$ which is impossible. This concludes the proof of the proposition.

**Proposition 4.2.** Every semisimple Hopf algebra $A$ of dimension $3p$ over $k$, where $p > 3$ is prime, is of Frobenius type.

**Proof.** If $A$ is a group algebra or a dual of a group algebra, then it is known that $A$ is of Frobenius type. Otherwise, by Proposition 4.1, $|G(A)| = 3$. Since $A$ is non-commutative we must have $G(A) \cap Z(A) = \{1\}$.

Set $n = \dim R(A^*) - 1$. Then, by (1) there exist two natural numbers $a$ and $b$ such that:

$$3p = 1 + 3a + bp \quad \text{and} \quad n \geq a + b.$$ 

Clearly, $a \geq 1$ and hence $b = 1$ or $2$. Since 2 does not divide $3p$ we have by [NR] that $A^*$ does not have a 2-dimensional irreducible module and hence that the following two inequalities hold:

$$n \geq a + b \quad \text{and} \quad 3p \geq 9(n - 2) + 3.$$ 

Therefore, $3p \geq 9(\frac{(3-b)p - 1}{3} + b - 2) + 3$ and hence $(-2 + b)p \geq 3b - 6$. Clearly, this is possible if and only if the equalities above hold, and $b = 2$. Therefore, $3p = 1 + 3a + 2p$ and $a = \frac{p+1}{2}$. This implies that $R(A^*)$ is commutative and that

$$A = k1 \oplus kg \oplus kg^2 \oplus C_1 \oplus \cdots \oplus C_a,$$
as a coalgebra where $C_i$ is a simple subcoalgebra of $A$ of dimension 9 for all $1 \leq i \leq a$. Hence $A^*$ is of Frobenius type. Replacing $A$ by $A^*$ yields the same result for $A$.

As a corollary of the above and of Theorem 3.5 we have:

**Theorem 4.3.** A semisimple Hopf algebra of dimension $3p$ over $k$, where $p > 3$ is prime, is trivial.

We conclude the paper by considering semisimple Hopf algebras of dimensions $5p$ and $7p$.

**Lemma 4.4.** Let $A$ be a semisimple Hopf algebra of odd dimension over $k$. If $|G(A)| = 1$, then there exists an irreducible $A^*$-module $V$ with $\dim V \geq 4$.

**Proof.** Suppose on the contrary that for any non-trivial $A^*$-irreducible module $V$, $\dim V \leq 3$. Then by [NR], $\dim V = 3$. Hence, $\dim(V \otimes V^*) = 9$ and by [La], $V \otimes V^* = k \oplus V_1 \oplus \cdots \oplus V_i$, where $V_j \neq k$ is an $A^*$-irreducible module for all $j$. Since $\dim V_j = 3$, this is a contradiction.

**Theorem 4.5.** Let $A$ be a semisimple Hopf algebra over $k$. If $\dim A = 5p$, $p$ an odd prime, and if $p \equiv 2$ or $4 \pmod 5$ or $p \in \{13, 23\}$, then $A$ is a commutative group algebra.

**Proof.** We wish to show that $|G(A)| \neq 1$. Suppose on the contrary that $|G(A)| = 1$. Set $n = \dim R(A^*) - 1$. By (1), there exist two natural numbers $1 \leq a$ and $1 \leq b \leq 4$ such that

\[
5p = 1 + 5a + bp, \\
n \geq a + b \quad \text{and} \\
5p \geq 9(n - 1) + 16 + 1
\]

where the last inequality follows from (2) and Lemma 4.4. Hence $(-20 + 9b)p \geq 45b + 31$. But, if $p \equiv 2$ or $4 \pmod 5$, then $b = 2$ or $1$ respectively and if $p \in \{13, 23\}$, then $b = 3$. In any event this is impossible and we have proved that $|G(A)| \neq 1$. The result follows now from Theorem 2.2.

**Theorem 4.6.** Let $A$ be a semisimple Hopf algebra over $k$. If $\dim A = 7p$, $p$ a prime, and if $p \equiv 6 \pmod 7$ or $p \in \{17, 31\}$, then $A$ is a commutative group algebra.

**Proof.** Suppose $|G(A)| = 1$ and set $n = \dim R(A^*) - 1$. By (1), there exist two natural numbers $1 \leq a$ and $1 \leq b \leq 6$ so that

\[
7p = 1 + 7a + bp, \\
n \geq a + b \quad \text{and} \\
7p \geq 9(n - 1) + 16 + 1
\]

where the last inequality follows from (2) and Lemma 4.4. Thus, $(-14 + 9b)p \geq 63 + 47$. But, if $p \equiv 6 \pmod 7$ or $p \in \{17, 31\}$, then this is impossible. The result follows now from Theorem 2.2.

\[\square\]
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