NO SUBMAXIMAL TOPOLOGY ON A COUNTABLE SET IS $T_1$-COMPLEMENTARY

MIKHAIL G. TKAČENKO, VLADIMIR V. TKACHUK, RICHARD G. WILSON, AND IVAN V. YASCHENKO

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Abstract. Two $T_1$-topologies $\tau$ and $\mu$ given on the same set $X$, are called transversal if their union generates the discrete topology on $X$. The topologies $\tau$ and $\mu$ are $T_1$-complementary if they are transversal and their intersection is the cofinite topology on $X$. We establish that for any connected Tychonoff topology there exists a connected Tychonoff transversal one. Another result is that no $T_1$-complementary topology exists for the maximal topology constructed by van Douwen on the rational numbers. This gives a negative answer to Problem 162 from Open Problems in Topology (1990).

0. Introduction

The lattice $\mathcal{L}_1(X)$ of all $T_1$-topologies on a given set $X$ has been under intensive study since 1966 when A.K. Steiner [St] showed that for any infinite set $X$ there exist $T_1$-topologies on $X$ which do not have a complement in the lattice $\mathcal{L}_1(X)$. Recall that a topology $\mu$ on $X$ is a complement of $\tau$ in $\mathcal{L}_1(X)$ if $\tau \cup \mu$ is a subbase of the discrete topology and $\tau \cap \mu$ coincides with the cofinite topology on $X$.

The papers [An], [AnSt], [StSt1] and [StSt2] contain positive results on the existence of complements in the lattice $\mathcal{L}_1(X)$ which are also called $T_1$-complements. It was proved, in particular, that every Hausdorff locally compact or Frechet-Urysohn topology has a $T_1$-complement [An]. In the same paper, Anderson constructed an example of an irresolvable ($\equiv$ not representable as a union of two disjoint dense subsets) dense in itself space which has a $T_1$-complement [An]. In the same paper, Anderson constructed an example of an irresolvable ($\equiv$ not representable as a union of two disjoint dense subsets) dense in itself space which has a $T_1$-complement and asked whether every MI-space has one [An, Question 1]. Recall that $(X, \tau)$ is a MI-space if it is dense in itself and every dense subset of $X$ is open. It is easy to see that every MI-space is irresolvable.

Later, S. Watson asked whether each Hausdorff space has a $T_1$-complement. This question is published as Problem 162 (Problem 94 in the internal enumeration of Watson’s paper) in Open Problems in Topology [Wa1]. The second part of Problem received by the editors January 15, 1998 and, in revised form, March 19, 1998.

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162 of [Wa1] is an inquiry whether every completely regular $T_1$-topology has a $T_1$-complement. The same question is repeated in [Wa2] (Problem 6.6).

In this paper we work only with $T_1$-spaces. We use the modern term “submaximal space” instead of “$MI$-space”. Our main result is Theorem 3.6 which implies that no submaximal Hausdorff topology on a countable set is $T_1$-complementary. A narrower class than submaximal topologies is formed by the maximal ones. A topology $\tau$ is maximal if it is dense in itself but any strictly stonger one is not. As there exists in ZFC a Tychonoff countable maximal space [vD], Theorem 3.6 and Corollary 3.8 provide the negative answer to the respective questions from [An].

Theorem 3.6 and [Wa1] and [Wa2].

Given a set $X$ and $\tau, \mu \in \mathcal{L}_1(X)$ we say that $\tau$ and $\mu$ are transversal if $\tau \cup \mu$ is a subbase of the discrete topology on $X$. It is immediate that if $\tau$ and $\mu$ are $T_1$-complementary, then they are transversal.

We prove in particular, that every connected Tychonoff topology has a connected Tychonoff transversal topology. Examples of the non-existence of transversal connected topologies are given.

1. Notations and terminology

All spaces are assumed to be $T_1$. If $X$ is a space, then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X) \setminus \{\emptyset\}$. Analogously, if $\tau$ is a topology, then $\tau^* = \tau \setminus \{\emptyset\}$. A topology $\tau$ on a set $X$ is called cofinite (and is denoted by $\mathcal{CF}(X)$) if every non-empty element of $\tau$ is a complement of a finite set. If $\tau$ is a topology on a set $X$ and $x \in X$, then $\tau(x) = \{U \in \tau : x \in U\}$. We will write $\tau(x, X)$ instead of $\tau(X)(x)$. Given a space $(X, \tau)$ and a subset $A \subset X$ we denote by $\text{cl}_\tau(A)$ and $\text{Int}_\tau(A)$ the $\tau$-closure and $\tau$-interior of $A$ respectively. A space $X$ is called submaximal if it is dense in itself and every dense subset of $X$ is open. A topology $\tau$ on a set $X$ is maximal if $(X, \tau)$ has no isolated points, but $(X, \mu)$ has an isolated point if $\mu$ is a topology strictly stronger than $\tau$.

All other notions are standard and can be found in [En].

2. Transversal topologies

Let us start with the main definitions and some auxiliary results.

2.1. Definition. Two topologies $\tau$ and $\mu$ on the same set $X$ are called transversal if their join $\tau \vee \mu$ (\(\equiv\) the smallest topology that contains $\tau \cup \mu$) is discrete. The topology $\mu$ will be referred to as a $\tau$-transversal one.

2.2. Proposition. If $\tau$ and $\mu$ are topologies on the same set $X$, then the following conditions are equivalent:

1. $\tau$ and $\mu$ are transversal;
2. for each point $x \in X$ there exist $U \in \tau$ and $V \in \mu$ such that $U \cap V = \{x\}$;
3. there is a $\tau$-open cover $\gamma$ of $X$ such that $X$ is a union of $\mu$-isolated points of elements of $\gamma$.

Proof. It is clear that the family $\mathcal{B}(\tau, \mu) = \{U \cap V : U \in \tau, V \in \mu\}$ is a base of $\tau \vee \mu$. As each singleton belongs to $\tau \vee \mu$, we have $\{x\} \in \mathcal{B}(\tau, \mu)$ for every $x \in X$. This proves $(1) \implies (2)$.

For each $x \in X$ take $U_x \in \tau(x)$ and $V_x \in \mu(x)$ such that $\{x\} = U_x \cap V_x$. Let $\gamma = \{U_x : x \in X\}$. It is clear that $\gamma$ is a $\tau$-open cover of $X$ and $x$ is $\mu$-isolated in $U_x$ for each $x \in X$. Thus (2) $\implies$ (3).
To prove (3) \(\implies\) (1) consider any \(\gamma\) as in (3). For every \(x \in X\) there is a \(U_x \in \gamma\) such that \(x\) is \(\mu\)-isolated in \(U_x\). This means that \(U_x \cap V_x = \{x\}\) for some \(V_x \in \mu\) and therefore \(\{x\}\) is open in \(\tau \cap \mu\).

2.3. Theorem. Let \((X, \tau)\) be a space of weight \(\kappa\). If \(\mu\) is a \(\tau\)-transversal topology on \(X\), then \((X, \mu)\) is a union of \(\leq \kappa\) discrete subspaces.

Proof. Take a base \(B\) for \((X, \tau)\) of cardinality \(\kappa\). For every \(x \in X\) fix a \(U_x \in B\) and \(V_x \in \mu\) with \(U_x \cap V_x = \{x\}\). For every \(U \in B\) put \(A_U = \{x \in X : U = U_x\}\). Since for each \(x \in A_U\) we have \(V_x \cap A_U = \{x\}\), the set \(A_U\) is discrete in \((X, \mu)\). Now \(X = \bigcup\{A_U : U \in B\}\) is a union of \(\kappa\) many sets discrete in \((X, \mu)\).

2.4. Corollary. If \((X, \tau)\) is a second countable space and \(\mu\) is a \(\tau\)-transversal topology on \(X\), then \((X, \mu)\) is \(\sigma\)-discrete.

2.5. Corollary. Let \((X, \tau)\) be an infinite connected second countable space. Then there is no dense-in-itself compact (or even Baire) \(\tau\)-transversal topology \(\mu\) on \(X\).

Proof. Suppose that \(\mu\) is transversal for \(\tau\) and \((X, \mu)\) is a Baire space. According to Corollary 2.4 we have \(X = \bigcup\{X_n : n \in \omega\}\), where \(X_n\) is a \(\mu\)-discrete subset of \(X\). The Baire property of \((X, \mu)\) implies \(U = \text{Int}_\mu(\text{cl}_\mu(X_n)) \neq \emptyset\) for some \(n \in \omega\). Then \(X_n \cap U \neq \emptyset\) and any \(x \in X_n \cap U\) is an isolated point of \((X, \mu)\) which contradicts the fact that \((X, \mu)\) is dense-in-itself.

2.6. Theorem. For every cardinal \(\kappa \geq \omega\) there exists a space \(D_\kappa\) with the following properties:

1. \(D_\kappa\) is homeomorphic to a dense subspace of \(I^\kappa\) and hence \(D_\kappa\) is a Tychonoff space without isolated points;
2. \(D_\kappa = \bigcup\{D_\kappa^n : n \in \omega\}\), where each \(D_\kappa^n\) is closed, discrete in \(D_\kappa\) and \(|D_\kappa^n| = \kappa\);
3. if \(A \subset D_\kappa\) and \(|A| < \kappa\), then \(A\) is closed and discrete in \(D_\kappa\);
4. if \(\kappa \geq \omega\), then the space \(D_\kappa\) is connected.

Proof. It is possible to represent the set \(\kappa\) in the following form: \(\kappa = \bigcup\{K_\alpha : \alpha < \kappa\}\), where \(|K_\alpha| = \kappa\) for all \(\alpha < \kappa\) and \(K_\alpha \cap K_\beta = \emptyset\) if \(\alpha \neq \beta\).

If \(\kappa < \omega\), then for every finite \(A \subset \kappa\) pick a countable dense subspace \(C_A\) of the space \(I^A\) and let \(\mathcal{F} = \bigcup\{C_A : A \text{ is a finite subset of } \kappa\}\).

If \(\kappa \geq \omega\), then for every finite \(A \subset \kappa\) put \(C_A = I^A\) and let \(\mathcal{F} = \bigcup\{C_A : A \text{ is a finite subset of } \kappa\}\).

It is clear that in both cases \(|\mathcal{F}| = \kappa\), so it is possible to enumerate the elements of \(\mathcal{F}\) as \(\{x_\alpha : \alpha < \kappa\}\) in such a way that for each \(x \in \mathcal{F}\) the set \(\{\alpha : x_\alpha = x\}\) has cardinality \(\kappa\). Given an \(x_\alpha \in \mathcal{F}\) we denote by \(S_\alpha\) the finite set of coordinates corresponding to the face to which the point \(x_\alpha\) belongs.

For each \(\alpha < \kappa\) let

\[
d_\alpha(t) = \begin{cases} 
0, & \text{if } t \notin K_\alpha \cup S_\alpha; \\
x_\alpha(t), & \text{if } t \in S_\alpha; \\
1, & \text{if } t \in K_\alpha \setminus S_\alpha,
\end{cases}
\]

and denote by \(D_\kappa\) the subspace \(\{d_\alpha : \alpha < \kappa\}\) of \(I^\kappa\).

Let us prove that the space \(D_\kappa\) has the properties we promised. Observe first that \(D_\kappa\) is dense in \(I^\kappa\) for all \(\kappa \geq \omega\).

Indeed, for a finite \(S \subset \kappa\) every \(x \in C_S\) occurs \(\kappa\) times in the enumeration of \(\mathcal{F}\). Thus, there is an \(\alpha(x) < \kappa\) such that \(x_\alpha(x) = x\) and hence \(S_\alpha(x) = S\). It is clear
that $\pi_S(d_\alpha(x)) = x$. Therefore, $\pi_S(D_\kappa) \supset C_S$ for every finite $S \subseteq \kappa$. Since the set $C_S$ is dense in $I^S$, we can conclude that $\pi_S(D_\kappa)$ is dense in $I^S$ for every finite $S \subseteq \kappa$. Thus $D_\kappa$ is dense in $I^\kappa$. This shows that (1) is true for $D_\kappa$.

Now let $D_\kappa^n = \{d_\alpha \in D_\kappa : |S_\alpha| = n\}$ for every $n \in \omega$. It is evident that $|D_\kappa^n| = \kappa$ for each $n \in \omega$ and $\bigcup\{D_\kappa^n : n \in \omega\} = D_\kappa$. Given an $\alpha < \kappa$ take any distinct $t_1, \ldots, t_{n+1} \in K_\alpha \setminus S_\alpha$. Then $d_\alpha(t_i) = 1$ for every $i \leq n + 1$. Thus, $W_\alpha = \{d \in D_\kappa : d(t_i) > \frac{1}{2} \text{ for all } i \leq n + 1\}$ is an open neighbourhood of $d_\alpha$. If $\beta \neq \alpha$ and $|S_\beta| = n$, then

$$\{t_1, \ldots, t_{n+1}\} \setminus (K_\beta \cup S_\beta) \neq \emptyset,$$

which implies $d_\beta(t_i) = 0$ for some $i \leq n + 1$. Therefore, $d_\beta \notin W_\alpha$. This proves that $D_\kappa^n$ is closed and discrete in $D_\kappa$. Hence we established (2) for $D_\kappa$.

Take a subset $A$ of $D_\kappa$ with $|A| < \kappa$. The sets $B = \{\alpha < \kappa : d_\alpha \in A\}$ and $H = \bigcup\{S_\alpha : \alpha \in B\}$ have cardinality less than $\kappa$. Take any $\alpha_0 < \kappa$. It is clear that $K_{\alpha_0} \setminus H \neq \emptyset$. Pick any $t \in K_{\alpha_0} \setminus H$ and let $V_{\alpha_0} = \{d \in D_\kappa : d(t) > \frac{1}{2}\}$. Then $V_{\alpha_0}$ is an open neighbourhood of $d_{\alpha_0}$ which does not contain any point of $A$, distinct from $d_{\alpha_0}$. Thus, $A$ is closed and discrete in $D_\kappa$. This proves (3).

Finally, suppose that $\kappa \geq \varsigma$. If $D_\kappa$ is disconnected, then there is a continuous surjective function $\varphi : D_\kappa \to [0, 1]$. As $D_\kappa$ is dense in $I^K$, there is a countable subset $T \subseteq \kappa$ and a continuous function $\varphi_1 : \pi_T(D_\kappa) \to [0, 1]$ such that $\varphi_1 \circ \pi_T = \varphi [Ar]$. In particular, $\varphi_1$ is surjective. Now, if $S$ is a finite subset of $T$ and $x \in I^S$, then the set $P(S) = \{\alpha < \kappa : x_\alpha = x\}$ has cardinality $\kappa > \omega$. Therefore, $K_\alpha \cap T = \emptyset$ for some $\alpha \in P(S)$ and hence $d_\alpha|_S = x$ and $d_\alpha|_{T \setminus S} \equiv 0$. This proves that $\pi_T(D_\kappa)$ contains the set $\sigma_T = \{z \in I^T : |\{t \in T : z(t) \neq 0\}| < \omega\}$ which is connected and dense in $I^T$. Since $\pi_T(D_\kappa)$ contains a dense connected subspace, it is connected. This gives a contradiction with the continuity and surjectivity of $\varphi_1$. Consequently, $D_\kappa$ is connected.

2.7. Corollary. For every cardinal $\kappa \geq \omega$ there exists a Tychonoff space $E_\kappa$ with the following properties:

(1) $E_\kappa = A_\kappa \cup B_\kappa$, where each one of the subspaces $A_\kappa, B_\kappa$ is closed in $E_\kappa$ and homeomorphic to the space $D_\kappa$ from Theorem 2.6. In particular, if $A \subseteq E_\kappa$ and $|A| < \kappa$, then $A$ is closed and discrete in $E_\kappa$;

(2) there is a point $p \in E_\kappa$ such that $A_\kappa \cap B_\kappa = \{p\}$;

(3) $A_\kappa \setminus \{p\} = \bigcup\{A_\kappa^n : n \in \omega\}$ and $B_\kappa \setminus \{p\} = \bigcup\{B_\kappa^n : n \in \omega\}$, where $A_\kappa^n$ and $B_\kappa^n$ are closed and discrete in $E_\kappa$ for all $n \in \omega$;

(4) $|A_\kappa^n| = |B_\kappa^n| = \kappa$ for all $n \in \omega$;

(5) $A_\kappa^n \cap A_\kappa^m = \emptyset$ and $B_\kappa^n \cap B_\kappa^m = \emptyset$ whenever $m \neq n$;

(6) if $\kappa \geq \varsigma$, then the space $E_\kappa$ is connected.

Proof. Let $P_\kappa$ and $Q_\kappa$ be two disjoint copies of the space $D_\kappa$ with $P_\kappa^n \subseteq P_\kappa$ and $Q_\kappa^n \subseteq Q_\kappa$ as respective copies of $D_\kappa^n$. Pick points $a \in P_\kappa$, $b \in Q_\kappa$ and identify $a$ and $b$ in the space $P_\kappa \sqcup Q_\kappa$. Denote by $E_\kappa$ the resulting quotient space and by $\varphi_\kappa : P_\kappa \sqcup Q_\kappa \to E_\kappa$ the relevant quotient map. Define the point $p$ by the equality $\{p\} = \varphi_\kappa([a, b])$ and let $A_\kappa = \varphi_\kappa(P_\kappa)$, $A_\kappa^n = \varphi_\kappa(P_\kappa^n \setminus \{a\})$ and $B_\kappa = \varphi_\kappa(Q_\kappa)$, $B_\kappa^n = \varphi_\kappa(Q_\kappa^n \setminus \{b\})$. It is straightforward that the space $E_\kappa$ satisfies (1)-(6).

The following example shows that not every Tychonoff space has a transversal connected or even dense-in-itself topology.
2.8. Example. The one-point compactification of any infinite discrete space has no transversal connected (or even dense-in-itself) topology.

Proof. Let \( X \) be the one-point compactification of an infinite discrete space. If \( a \) is the only non-isolated point of \( X \), then for every \( U \in \tau(a, X) \) the set \( X \setminus U \) is finite. Suppose that \( \mu \) is a dense-in-itself topology, transversal to \( \tau(X) \). Then there is a \( V \in \mu \) such that \( U \cap V = \{a\} \) for some \( U \in \tau(a, X) \). Hence \( V \subset (X \setminus U) \cup \{a\} \) is a non-empty finite set. Since \((X, \mu)\) is a \( T_1 \)-space, any point of \( V \) is isolated in \((X, \mu)\), which is a contradiction. \( \square \)

2.9. Example. There exists a dense-in-itself Tychonoff space \( X \) of cardinality \( \kappa \) such that \( \tau(X) \) has no transversal connected Tychonoff topology.

Proof. Consider the discrete union \( Y = \bigoplus \{Q_\alpha : \alpha < \omega \} \) of \( \omega \) copies of rationals. Take any point \( z \notin Y \) and let \( X = \{z\} \cup Y \). Declare the family \( \tau(y, Y) \) to be the local base in \( X \) at any \( y \in Y \). Let \( B = \{U_A = \{z\} \cup \bigcup \{Q_\beta : \beta \in \alpha \setminus A\} \} \) where \( A \) is a countable subset of \( E \) such that \( \tau \) is countable, we have a contradiction.

Suppose that \( X \) has a connected transversal Tychonoff topology \( \mu \). By Proposition 2.2 there exist \( U \in \tau(X) \) and \( V \in \mu \) such that \( \{z\} = U \cap V \). But the complement of \( U \) is countable, which implies that \( V \subset (X \setminus U) \cup \{z\} \) is countable. Since in a connected Tychonoff space no non-empty proper open set can be countable, we have a contradiction. \( \square \)

2.10. Lemma. Let \((X, \tau)\) be a space of cardinality \( \kappa \geq \omega \). Suppose that there exist families \( \{U_n : n \in \omega\} \) and \( \{V_n : n \in \omega\} \) of open subsets of \( X \) with the following properties:

1. \( \overline{U}_{n+1} \subset U_n \) and \( \overline{V}_{n+1} \subset V_n \) for any \( n \in \omega \);
2. \( \overline{U}_0 \cap \overline{V}_0 = \emptyset \);
3. \( |P| = \kappa \), where \( P = X \setminus (U_0 \cup V_0) \);
4. \( |U_n \setminus U_{n+1}| = |V_n \setminus V_{n+1}| = \kappa \) for all \( n \in \omega \).

Then there exists a \( \tau \)-transversal topology \( \mu \) on the set \( X \) such that \((X, \mu)\) is homeomorphic to \( E_\kappa \) from Corollary 2.7. In particular, \( \tau \) has a transversal dense-in-itself Tychonoff topology and if \( \kappa \geq \omega \), then \( \tau \) has a transversal connected Tychonoff topology.

Proof. Let \( F = \bigcap \{U_n : n \in \omega\} \) and \( G = \bigcap \{V_n : n \in \omega\} \) (the sets \( F \) and \( G \) can be empty). Take a point \( p' \in P \). Using evident decompositions of \( E_\kappa \) and \( X \) into countably many pieces of cardinality \( \kappa \), we conclude that there exists a bijection \( \varphi : E_\kappa \rightarrow X \) such that

(i) \( \varphi(p) = p' \);
(ii) \( \varphi(A_0^n) = F \cup (P \setminus \{p'\}) \);
(iii) \( \varphi(B_0^n) = G \cup (U_0 \setminus U_1) \);
(iv) \( \varphi(B_i^n) = U_i \setminus U_{i+1} \) for all \( i \geq 1 \);
(v) \( \varphi(A_i^{n+1}) = V_i \setminus V_{i+1} \) for every \( i \in \omega \).

Let \( \mu = \{\varphi(W) : W \text{ is open in } E_\kappa\} \). It is clear that \((X, \mu)\) is homeomorphic to \( E_\kappa \). We are going to prove that the topology \( \mu \) is \( \tau \)-transversal.

Given an \( x \in X \setminus (F \cup G) \), there exists an \( n \in \omega \) such that \( x \notin \overline{U}_n \cup \overline{V}_n \). Let \( U = X \setminus (\overline{U}_n \cup \overline{V}_n) \). Then \( x \in U \) and \( \varphi^{-1}(U) \) is contained in the closed and discrete subset

\[ A = \{p\} \cup A_0^0 \cup \ldots \cup A_\kappa^\kappa \cup B_0^0 \cup \ldots \cup B_\kappa^\kappa \]
of the space $E_\kappa$. Therefore, there is a $W \in \tau(E_\kappa)$ such that $\{\varphi^{-1}(x)\} = W \cap \varphi^{-1}(U)$. Then $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$.

Now if $x \in F$, then let $U = U_1$. Observe that $\varphi^{-1}(U) = (B_\kappa \setminus B_\kappa^0) \cup \varphi^{-1}(F) \subset B_\kappa \cup A_\kappa^0$, and $\varphi^{-1}(x) \in \varphi^{-1}(F) \subset A_\kappa^0$. Thus $A_\kappa \setminus \{p\}$ is an open neighbourhood of $\varphi^{-1}(x)$, whose intersection with $\varphi^{-1}(U)$ is contained in $A_\kappa^0$. Therefore, $\varphi^{-1}(x)$ is isolated in $\varphi^{-1}(U)$. Take any $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $V \cap U = \{x\}$. We have established that for any $x \in X$ property (2) of Proposition 2.2 is fulfilled and hence $\mu$ is $\tau$-transversal.

\[ \Box \]

2.11. Lemma. Suppose that in a space $(X, \tau)$ of cardinality $\kappa$ there exist $a, b \in X$ and $U_0, U_1, V_0, V_1 \in \tau$ such that
\begin{enumerate}[\itshape (1)]
\item $a \in U_1 \subset U_0 \subset U_0$;
\item $b \in V_1 \subset V_0 \subset V_0$;
\item $|U_1| = |V_1| = \kappa$;
\item $U_0 \cap V_0 = \emptyset$ and $X \setminus (U_0 \cup V_0) \neq \emptyset$;
\item for any $x \in (U_0 \setminus \{a\}) \cup (V_0 \setminus \{b\})$ there exists $U \in \tau(x, X)$ such that $|U| < \kappa$.
\end{enumerate}
Then there exists a $\tau$-transversal topology $\mu$ on the set $X$ such that $(X, \mu)$ is homeomorphic to the space $E_\kappa$ from Corollary 2.7. In particular, $\tau$ has a transversal dense-in-itself Tychonoff topology and if $\kappa \geq \mathfrak{c}$, then $\tau$ has a transversal connected Tychonoff topology.

Proof. Pick any $p' \in X \setminus (U_0 \cup V_0)$ and denote by $P$ the set $X \setminus (U_0 \cup V_1 \cup \{p'\})$. Using evident decompositions of $X$ and $E_\kappa$ into finitely many pieces we can construct a bijection $\varphi : E_\kappa \to X$ such that
\begin{enumerate}[\itshape (i)]
\item $\varphi(p) = p'$;
\item $\varphi(A_\kappa^0) \supset P \cup \{b\}$ and $\varphi(A_\kappa \setminus \{p\}) = (U_1 \setminus \{a\}) \cup P \cup \{b\}$;
\item $\varphi(B_\kappa^0) \supset U_0 \setminus U_1$ and $\varphi(B_\kappa \setminus \{p\}) = (V_1 \setminus \{b\}) \cup (U_0 \setminus U_1) \cup \{a\}$.
\end{enumerate}
Let $\mu = \{\varphi(W) : W \text{ is open in } E_\kappa\}$. It is clear that $(X, \mu)$ is homeomorphic to $E_\kappa$. We are going to prove that the topology $\mu$ is $\tau$-transversal.

Given an $x \in X \setminus (U_0 \cup V_0)$ let $U = X \setminus (U_1 \cup V_1)$. The set $U$ is a $\tau$-open neighbourhood of the point $x$ and $\varphi^{-1}(U) \subset \varphi^{-1}(P \cup \{p'\} \cup (U_0 \setminus U_1)) \subset A_\kappa^0 \cup \{p\} \cup B_\kappa^0$. Since the set $A_\kappa^0 \cup \{p\} \cup B_\kappa^0$ is closed and discrete in $E_\kappa$, there is a $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$.

Suppose that $x \in (U_0 \setminus \{a\}) \cup (V_0 \setminus \{b\})$. Apply (5) to find a $U \in \tau$ such that $x \in U$ and $|U| < \kappa$. Then $|\varphi^{-1}(U)| < \kappa$ and therefore $\varphi^{-1}(U)$ is closed and discrete in $E_\kappa$ by condition (1) of Proposition 2.7. Pick a $W \in \tau(E_\kappa)$ such that $W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\}$. Then $V = \varphi(W) \in \mu$ and $U \cap V = \{x\}$.

If $x = a$, let $U = U_1$. Then $\varphi^{-1}(U) \cap (B_\kappa \setminus \{p\}) = \{\varphi^{-1}(a)\}$. Thus for $W = B_\kappa \setminus \{p\}$ we have $V = \varphi(W) \in \mu$ and $U \cap V = \{a\}$.

If $x = b$, let $U = V_1$. Then $\varphi^{-1}(U) \cap (A_\kappa \setminus \{p\}) = \{\varphi^{-1}(b)\}$. Thus for $W = A_\kappa \setminus \{p\}$ we have $V = \varphi(W) \in \mu$ and $U \cap V = \{b\}$.

Condition (2) of Proposition 2.2 having been checked for every $x \in X$ we conclude that $\tau$ and $\mu$ are transversal.

\[ \Box \]
2.12. Theorem. Given a cardinal number \( \lambda \geq \omega \) suppose that \((X, \tau)\) is a regular space such that \(|U| \geq \lambda \) for any \( U \in \tau^* \). Then \( \tau \) has a dense-in-itself transversal Tychonoff topology \( \mu \). Moreover, if \( \lambda \geq \omega \), then \( \mu \) can be chosen to be Tychonoff and connected.

Proof. Let

\[
M = \{ x \in X : |U| = |X| = \kappa \text{ for any } U \in \tau(x) \}.
\]

There are three cases to consider.

Case 1. The set \( M \) has at least two cluster points, say \( a \) and \( b \). It is clear that \( M \) has to be infinite and \( a, b \in M \).

Pick two distinct points \( x_0, y_0 \in M \setminus \{ a, b \} \). There exist \( U_0 \in \tau(a) \), \( V_0 \in \tau(b) \) such that \( U_0 \cap V_0 = \emptyset \) and \( \{ x_0, y_0 \} \subset X \setminus (U_0 \cup V_0) \). In particular \(|P| = \kappa \), where \( P = X \setminus (U_0 \cup V_0) \).

Suppose that we have constructed open sets \( U_i \in \tau(a) \), \( V_i \in \tau(b) \) and points \( x_i, y_i \in M \) for all \( i \leq n \) in such a way that

(i) \( U_{i+1} \subset U_i \) and \( V_{i+1} \subset V_i \) for all \( i < n \);

(ii) \( x_{i+1} \in U_i \setminus V_{i+1} \) and \( y_{i+1} \in V_i \setminus U_{i+1} \) for all \( i < n \).

Since \( a \) and \( b \) are cluster points of \( \mu \), there exist \( x_{n+1} \in (U_n \setminus \{ a \}) \cap M \) and \( y_{n+1} \in (V_n \setminus \{ b \}) \cap M \). Since the space \( X \) is regular, we can find \( U_{n+1} \in \tau(a) \) and \( V_{n+1} \in \tau(b) \) such that \( U_{n+1} \subset (U_n \setminus \{ x_{n+1} \}) \) and \( V_{n+1} \subset (V_n \setminus \{ y_{n+1} \}) \). It is evident that the properties (i) and (ii) are fulfilled for all \( i \leq n \).

Observe that the families \( U = \{ U_i : i \in \omega \} \) and \( V = \{ V_i : i \in \omega \} \) satisfy the conditions (1)-(3) of Lemma 2.10. The condition (4) is also fulfilled because each of the sets \( U_n \setminus U_{n+1} \) and \( V_n \setminus V_{n+1} \) is open and meets \( M \) for any \( n \in \omega \). Therefore, we can apply Lemma 2.10 and conclude that \((X, \tau)\) has a dense-in-itself transversal topology \( \mu \) which will be connected if \( \kappa \geq \omega \).

Case 2. The set \( M \) has at least two isolated points, say \( a \) and \( b \).

Pick any \( p' \in X \setminus \{ a, b \} \). Since \( X \) is Hausdorff, there exist \( U_0 \in \tau(a) \), \( V_0 \in \tau(b) \) such that \( U_0 \cap V_0 = \emptyset \), \( p' \in X \setminus (U_0 \cup V_0) \) and \( U_0 \cap M = \{ a \} \), \( V_0 \cap M = \{ b \} \).

Applying the regularity of \( X \) find \( U_1 \in \tau(a) \) and \( V_1 \in \tau(b) \) such that \( U_1 \subset U_0 \) and \( V_1 \subset V_0 \). It is clear that the conditions (1), (2) and (4) of Lemma 2.11 are satisfied for \( a, b, U_0, U_1, V_0, V_1 \). The condition (3) is fulfilled because \( a \in M \) and \( b \in M \). The condition (5) holds due to the fact that there are no points of \( M \) in \( U_0 \cup V_0 \) distinct from \( a \) and \( b \).

Thus we can apply Lemma 2.11 and conclude that \((X, \tau)\) has a dense-in-itself transversal Tychonoff topology \( \mu \) which will be connected if \( \kappa \geq \omega \).

Case 3. The set \( M \) has at most one point.

If \( M = \emptyset \), then let \( \varphi : E_\kappa \to X \) be any bijection. The topology \( \mu = \{ \varphi(W) : W \in \tau(E_\kappa) \} \) is as promised and to prove it we must only establish \( \tau \)-transversality of \( \mu \). Let \( x \in X \). As \( M = \emptyset \), there is a \( U \in \tau(x) \) with \(|U| < \kappa \). Therefore, the set \( \varphi^{-1}(U) \) is closed and discrete in \( E_\kappa \) by (1) of Corollary 2.7. Take any \( W \in \tau(E_\kappa) \) such that \( W \cap \varphi^{-1}(U) = \{ \varphi^{-1}(x) \} \). Then \( V = \varphi(W) \in \mu \) and \( U \cap V = \{ x \} \) so that \( \mu \) is \( \tau \)-transversal.

Assume that \( M = \{ a \} \). Let \( \nu = \min\{|Z| : Z \in \tau^*\} \). Since the space \( X \) is regular, there exists an \( H \in \tau^* \) such that \(|H| = \nu \) and \( a \notin \overline{H} \). Since \(|Z| = \nu \geq \lambda \) for any \( Z \in \tau^*(H) \), the conclusion of Case 1 is applicable to the space \( H \). This makes it possible to find a bijection \( \xi : E_\nu \to H \) such that the topology \( \{ \xi(W) : W \in \tau(E_\nu) \} \) is \( \tau(H) \)-transversal.
Since \( \nu < \kappa \) we have \(|X \setminus \{a\} \cup H| = \kappa\). Let \( \psi : E_\kappa \to X \setminus \{H \cup \{a\}\} \) be any bijection. The spaces \( E_\nu \) and \( E_\kappa \) are not compact by Corollary 2.7, so it is possible to choose points \( z \in \beta E_\nu \setminus E_\nu \) and \( t \in \beta E_\kappa \setminus E_\kappa \). Take any \( q \in H \) and in the space \( E^* = (E_\kappa \cup \{t\}) \oplus (E_\nu \cup \{z\}) \) identify the points \( q' = \xi^{-1}(q) \) and \( t \). Denote the resulting quotient space by \( E \), the point \( \{q, t\} \) by \( w \), and let \( f : E^* \to E \) be the relevant quotient map. It is clear that \( E \) is a dense-in-itself Tychonoff space which is connected if \( \lambda \geq c \). Identifying \( E_\kappa \cup (E_\nu \setminus \{q'\}) \) with \( f(E_\kappa \cup (E_\nu \setminus \{q'\})) \) we have

\[
E = E_\kappa \cup (E_\nu \setminus \{q'\}) \cup \{w\} \cup \{z\}.
\]

Now let

\[
\varphi(y) = \begin{cases} 
\xi(y), & \text{if } y \in E_\nu \setminus \{q'\}; \\
\psi(y), & \text{if } y \in E_\kappa; \\
a, & \text{if } y = z; \\
q, & \text{if } y = w.
\end{cases}
\]

Then \( \varphi : E \to X \) is a bijection. To conclude our proof it suffices to establish that \( \mu = \{\varphi(W) : W \in \tau(E)\} \) is a \( \tau \)-transversal topology.

Take any \( x \in X \setminus \{a\} \). If \( x \in H \), then there is a \( U \in \tau(H) \) and \( W' \in \tau(E_\nu) \) such that \( W' \cap \xi^{-1}(U) = \{\xi^{-1}(x)\} \). If \( x \neq q \), then \( \varphi^{-1}(x) = \xi^{-1}(x) \). Otherwise \( \varphi^{-1}(x) = w \). But in both cases \( \varphi^{-1}(U) \subset \xi^{-1}(U) \cup \{w\} \) and the set \( W = f(W') \cup E_\kappa \) is open in \( E \). It is immediate that \( U \cap V = \{x\} \), where \( V = \varphi(W) \in \mu \). This shows that the condition (2) of Proposition 2.2 holds for \( x \).

Assume that \( x \in X \setminus \{H \cup \{a\}\} \). There exists a \( U \in \tau(x) \) with \(|U| < \kappa\). Therefore, \( \psi^{-1}(U \setminus H) \) is closed and discrete in \( E_\kappa \). Take any \( W \in \tau(E_\kappa) \) such that \( W \cap \psi^{-1}(U \setminus H) = \{\psi^{-1}(x)\} \). Then \( W \in \tau(E) \) and \( W \cap \varphi^{-1}(U) = \{\varphi^{-1}(x)\} \). Therefore, \( V = \varphi(W) \in \mu \) and \( U \cap V = \{x\} \) which proves the property (2) of Proposition 2.2 for \( x \).

Finally, if \( x = a \), then \( W = \{z\} \cup (E_\nu \setminus \{q'\}) \) is an open neighbourhood of \( z \) in \( E \). If \( U = X \setminus \Xi \), then \( \varphi^{-1}(U) \cap W = \{z\} = \{\varphi^{-1}(a)\} \). Therefore, \( U \in \tau \), \( V = \varphi(W) \in \mu \) and \( U \cap V = \{a\} \) which shows that the condition (2) of Proposition 2.2 is fulfilled for \( x = a \). Since this condition holds for every \( x \in X \) we conclude that \( \mu \) is a \( \tau \)-transversal topology.

2.13. Corollary. Let \( X \) be a regular space without isolated points. Then \( X \) has a transversal dense-in-itself Tychonoff topology.

2.14. Corollary. Let \( X \) be a connected Tychonoff space. Then \( X \) has a connected Tychonoff transversal topology.

Proof. Any open subset of a Tychonoff connected space has cardinality \( \geq c \). Now apply Theorem 2.12.

3. Complementary topologies

Recall that topologies \( \tau \) and \( \mu \) on the same set \( X \) are called \( T_1 \)-complementary [Wa2], if their join is discrete and \( \tau \cap \mu = C\mathcal{F}(X) \).

3.1. Definition. A closed subset \( F \) of a space \( X \) is called well-placed if \( F \) and \( X \setminus F \) are infinite.
3.2. Proposition. A space $(X, \tau)$ has a well-placed subset if and only if
\[ \tau_{|X\setminus A} = \{ U \cap (X\setminus A) : U \in \tau \} \neq CF(X\setminus A) \]
for each finite $A \subset X$.

Proof. If there is a finite $A \subset X$ such that the topology of $X\setminus A$ is cofinite, then
for any infinite closed $F \subset X$ the set $F \cap (X\setminus A)$ is infinite and closed in $X\setminus A$. Therefore
$(X\setminus A)\setminus F$ is finite whence $X\setminus F$ is finite.

Suppose that there is no finite $A \subset X$ with $\tau_{|X\setminus A} = CF(X\setminus A)$. If $D$ is an infinite
discrete subspace of $X$, then splitting it into two disjoint infinite parts $D_0, D_1$ we see that $D_0$ is closed, infinite and does not intersect $D_1$ which implies $F = D_0$ is well-placed.

Thus, if the set $A$ of isolated points of $X$ is infinite, our proof is complete. If not, then $\tau_{|X\setminus A}$ is not cofinite and hence $X\setminus A$ has a proper infinite closed subset $F$. If $B = (X\setminus A)\setminus F$ is finite, then every point of $B$ is isolated in $X\setminus A$ and hence in $X$ which is a contradiction because $B \cap A = \emptyset$. Thus $F$ is well-placed. \qed

3.3. Lemma. Let $\tau$ and $\mu$ be transversal topologies on a set $X$. Suppose that $\mu_{|A} = CF(A)$ for some $A \subset X$. Then $A$ is a discrete subspace of $(X, \tau)$.

Proof. By Proposition 2.2 there exist $U \in \tau$ and $V \in \mu$ such that $U \cap V = \{ a \}$ for every $a \in A$. But $F = A\setminus V$ is finite, so that $U \cap A \subset \{ a \} \cup F$ is also finite. Hence $a$ is a $\tau$-isolated point of $A$. \qed

3.4. Corollary. Let $(X, \tau)$ be a space in which every discrete subset is closed. If $\mu$ is a $T_1$-complementary topology for $\tau$, then for every well-placed set $F$ of $(X, \mu)$ there exists a well-placed set $G$ of $(X, \mu)$ such that $G \subset F$ and $G \neq F$.

Proof. Indeed, if $\mu_{|F} = CF(F)$, then by Lemma 3.3 the set $F$ is $\tau$-discrete and hence closed in $(X, \tau)$, which is a contradiction with $\tau \cap \mu = CF(X)$. Thus there exists an infinite closed proper subset $G$ of $F$. It is clear that $G$ is as required. \qed

3.5. Lemma. Let $(X, \tau)$ be a Hausdorff space in which every discrete subset is closed. Suppose that $\mu$ is a $T_1$-complementary topology for $\tau$ and $\mathcal{F} = \{ F_n : n \in \omega \}$ is a family of well-placed subsets of $(X, \mu)$ with $F_{n+1} \subset F_n$ for each $n \in \omega$. Then $F = \bigcap \mathcal{F}$ is well-placed in $(X, \mu)$.

Proof. It suffices to prove that $F$ is infinite. Suppose not. Since every $F_n$ is infinite, we can assume that $F_n \setminus F_{n+1} \neq \emptyset$ for each $n \in \omega$. Let $x_n \in F_n \setminus F_{n+1}$. It is straightforward that $F \cup A$ is $\mu$-closed for every $A \subset Y = \{ x_n : n \in \omega \}$. Any infinite subspace of a Hausdorff space has an infinite discrete subspace, so there is an infinite $A \subset Y$ such that $A$ is $\tau$-discrete and hence closed in $(X, \tau)$. Then $F \cup A$ is also $\tau$-closed and hence it is well-placed in $(X, \mu)$ and $(X, \tau)$, which is a contradiction with $\tau \cap \mu = CF(X)$. \qed

3.6. Theorem. Let $(X, \tau)$ be a dense-in-itself Hausdorff countable space in which every discrete subset is closed. Then $\tau$ does not have a $T_1$-complement.

Proof. Assume that $\mu$ is a $T_1$-complementary topology for $\tau$. If $A$ is a finite subset of $X$ and $\mu_{|X\setminus A} = CF(A)$, then by Lemma 3.3 the set $X\setminus A$ is closed and discrete in $(X, \tau)$. Then $X = (X\setminus A)\cup A$ is discrete, which is a contradiction. Consequently, we can apply Proposition 3.2 to conclude that $(X, \mu)$ has a well-placed subset $F$. Let $F_0 = F$. 

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Suppose that for some $\alpha < \omega_1$ we have constructed $\mu$-well-placed subsets 
{$F_\beta : \beta < \alpha$} such that $F_\beta$ is a proper subset of $F_\delta$ if $\beta < \delta < \alpha$. If $\alpha$ is a limit ordinal, let $F_\alpha = \bigcap \{F_\beta : \beta < \alpha\}$. Lemma 3.5 makes it possible to assert that $F_\alpha$ is $\mu$-well-placed. If $\alpha = \beta + 1$ use Corollary 3.4 to find a $\mu$-well-placed $G$ which is a proper subset of $F_\beta$. Putting $F_\alpha = G$ we finish our transfinite construction.

As a result we get a family $\mathcal{F} = \{F_\alpha : \alpha < \omega_1\}$ of subsets of a countable set $X$ such that $F_\beta \subset F_\alpha$ and $F_\beta \neq F_\alpha$ if $\alpha < \beta$. It is evident that such an $\mathcal{F}$ cannot exist so the theorem is proved.

3.7. Corollary. If $(X, \tau)$ is a submaximal Hausdorff countable space, then $\tau$ has no $T_1$-complement.

Proof. It is well-known (see e.g. [ArCo]) that in a submaximal Hausdorff space any discrete subspace is closed.

Corollary 3.7 gives a negative answer to Question 1 from [An].

3.8. Corollary. There exists a Tychonoff countable dense-in-itself space $(X, \tau)$ which has no $T_1$-complement.

Proof. Van Douwen constructed in [vD] an example of a Tychonoff maximal (and hence submaximal) countable space $(X, \tau)$. Now apply 3.6 to see that $(X, \tau)$ is as promised.

Corollary 3.8 gives a negative answer to Problem 162 (Problem 94 in its internal enumeration) of [Wa1] as well as to Problem 6.6 of [Wa2].

3.9. Question. Let $X$ be a Hausdorff dense-in-itself space. Does $\tau(X)$ have a transversal dense-in-itself Hausdorff (or Tychonoff) topology?

3.10. Question. Let $X$ be a Hausdorff connected space. Does $\tau(X)$ have a transversal connected Hausdorff (or Tychonoff) topology? What is the answer if $X$ is regular?

References

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(M. G. Tkachenko, V. V. Tkachuk and R. G. Wilson) Departamento de Matematicas, Universidad Autónoma Metropolitana, Av. Michoacan y La Purísima, Iztapalapa, A.P. 55-532, C.P. 09340, México D.F.
E-mail address: mich@xanum.uam.mx

E-mail address: vova@xanum.uam.mx

Current address, R. G. Wilson: Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad Universitaria, México 20, D.F.
E-mail address: rgw@xanum.uam.mx

(I. V. Yaschenko) Moscow Center for Continuous Mathematical Education, B. Vlas’evskij, 11, 121002, Moscow, Russia
E-mail address: ivan@mccme.ru