INTEGRABILITY OF SUPERHARMONIC FUNCTIONS
IN A JOHN DOMAIN

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Abstract. The integrability of positive superharmonic functions on a bounded fat John domain is established. No exterior conditions are assumed. For a general bounded John domain the $L^p$-integrability is proved with the estimate of $p$ in terms of the John constant.

1. Introduction

Let $D$ be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. By $S^+(D)$ we denote the family of all positive superharmonic functions in $D$. Armitage [5], [6] proved that $S^+(D) \subset L^p(D)$ for $0 < p < n/(n-1)$, provided $D$ is smooth. This result was extended by Maeda-Suzuki [11] to a Lipschitz domain. They gave an estimate of $p$ in terms of Lipschitz constant. Their estimate has the correct asymptotic behavior: $p \to n/(n-1)$ as the Lipschitz constant tends to 0. As a result they showed that $S^+(D) \subset L^p(D)$ for $0 < p < n/(n-1)$, provided $D$ is a $C^1$ domain. Masumoto [12], [13] succeeded in obtaining the sharp value of $p$ for planar domains bounded by finitely many Jordan curves. For the higher dimensional case Aikawa [1] gave the sharp value of $p$ for Lipschitz domains with the aid of the coarea formula and the boundary Harnack principle.

On the other hand, Stegenga-Ullrich [16] treated very non-smooth domains, such as John domains and domains satisfying the quasihyperbolic boundary condition [7, 3.6], which are called “Hölder domains” by Smith-Stegenga [15]. Let $\delta_D(x) = \text{dist}(x, \partial D)$ and $x_0 \in D$. We say that $D$ is a John domain with John constant $c_J > 0$ if each $x \in D$ can be joined to $x_0$ by a rectifiable curve $\gamma$ such that

$$\delta_D(\xi) \geq c_J \ell(\gamma(x, \xi)) \quad \text{for all } \xi \in \gamma,$$

(1.1)

where $\gamma(x, \xi)$ is the subarc of $\gamma$ from $x$ to $\xi$ and $\ell(\gamma(x, \xi))$ is the length of $\gamma(x, \xi)$. A John domain may be visualized as a domain satisfying a twisted cone condition. The quasi-hyperbolic metric $k_D(x_1, x_2)$ is defined by

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{ds}{\delta_D(x)},$$
where the infimum is taken over all rectifiable arcs $\gamma$ joining $x_1$ to $x_2$ in $D$. We say that $D$ satisfies a quasi-hyperbolic boundary condition if there are positive constants $A_1$ and $A_2$ such that

$$k_D(x,x_0) \leq A_1 \log \left( \frac{1}{\delta_D(x)} \right) + A_2 \quad \text{for all } x \in D.$$  

Smith-Stegenga [15] called a domain satisfying the quasihyperbolic boundary condition a Hölder domain. It is easy to see that a John domain satisfies the quasihyperbolic boundary condition (see [7, Lemma 3.11]). Stegenga-Ullrich [16] proved that $S^+(D) \subset L^p(D)$ with small $p > 0$ for a domain satisfying the quasihyperbolic boundary condition. Lindqvist [10] extends the result to positive supersolutions of certain nonlinear elliptic equations, such as the $p$-Laplace equation. Gotoh [8] also studies $L^p$-integrability. Unfortunately, their $p > 0$ is very small and it does not seem that $p \geq 1$ is obtained by their methods.

The main aim of the present paper is to show that $S^+(D) \subset L^1(D)$ for a “fat” John domain.

**Theorem 1.** Let $D$ be a bounded John domain with John constant $c_J \geq 1 - 2^{-n-1}$. Then $S^+(D) \subset L^1(D)$.

The above bound $1 - 2^{-n-1}$ is not sharp. For more specific John domains we obtain the sharp bound. We say that $D$ satisfies the interior cone condition with aperture $\psi$, $0 < \psi < \pi/2$, if for each point $x \in D$ there is a truncated cone with vertex at $x$, aperture $\psi$ and a fixed radius lying in $D$. Obviously, a domain satisfying the interior cone condition with aperture $\psi$ is a John domain with John constant $\sin \psi$.

**Theorem 2.** Let $D$ be a bounded domain satisfying the interior cone condition with aperture $\psi$ with $\cos \psi > 1/\sqrt{n}$. Then $S^+(D) \subset L^1(D)$.

For a “slim” John domain we will show $S^+(D) \subset L^p(D)$ for some $0 < p < 1$ with the estimate of $p$. This will give a larger $p$ than that in Stegenga-Ullrich [16]. For details see Section 3.

In the previous paper [2], the above theorems are obtained with additional assumption: the capacity density condition (CDC). See [3] for more illustrations. For the 2-dimensional case CDC is equivalent to the uniform perfectness of the boundary; planar domains bounded by finitely many Jordan curves satisfy CDC. Thus all the known results for $S^+(D) \subset L^p(D)$ with $p \geq 1$ required CDC or some other stronger exterior condition. The above Theorems 1 and 2 first establish $S^+(D) \subset L^1(D)$ for a domain satisfying only an interior condition. Recently, Gustafsson, Sakai and Shapiro [9] considered the $L^1$-integrability in connection with quadrature domains. They showed that if $D$ is a quadrature domain and the Green functions do not decay so fast near the boundary, then $S^+(D) \subset L^1(D)$ ([9, Corollary 5.4]).

2. Proof of the theorems

For an open set $U$ we denote by $G_U$ the Green function for $U$. Throughout this section $D$ is a bounded John domain or a domain satisfying an interior condition. For simplicity we suppress the subscript $D$ and write $G$ for the Green function for $D$. Moreover, $x_0 \in D$ is a fixed point and let $g(x) = G(x,x_0)$. By the symbol $A$ we denote an absolute positive constant whose value is unimportant and may change.
from line to line. If necessary, we use $A_1, A_2, \ldots$, to specify them. We shall say that two positive functions $f_1$ and $f_2$ are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \geq 1$ such that $A^{-1} f_1 \leq f_2 \leq A f_1$. The constant $A$ will be called the constant of comparison.

The proof of the theorems uses the following lower estimate of the Green function. For $0 < c_J < 1$ we let

\[(2.1) \quad \alpha_J = \log \left[ \frac{1 - c_J}{(1 + c_J)^{n-1}} \right]/ \log(1 - c_J^2). \]

We observe that $\lim \alpha_J = 1$ as $c_J \to 1$. Let $c_n$ be the solution of the equation $(1 + t)^{n+1}(1 - t) = 1$ for $0 < t < 1$. Then $\alpha_J = 2$ for $c_J = c_n$ and $1 < \alpha_J < 2$ for $c_n < c_J < 1$. We see that $n/(n + 2) < c_n < 1 - 2^{-n-1}$.

**Lemma 1** (see [2, Lemma 12]).

(i) If $D$ is a John domain with John constant $c_J$, then $g(x) \geq A \delta_D(x)^{\alpha_J}$.

(ii) If $D$ satisfies the interior cone condition with aperture $\psi > 1/\sqrt{n}$, then there is $1 < \alpha(\psi) < 2$ such that $g(x) \geq A \delta_D(x)^{\alpha(\psi)}$.

Theorems 1 and 2 readily follow from Lemma 1 and the following.

**Theorem 3.** Let $D$ be a John domain and suppose $g(x) \geq A \delta_D(x)^{\alpha}$ for $\alpha > 0$. For $\varepsilon > 0$ let $V = (\min \{g, 1\})^{\varepsilon - 2/\alpha}$. Then

\[\int_D u(x)V(x)g(x)dx \leq Au(x_0) \quad \text{for any } u \in S^+(D),\]

where $A$ is independent of $u \in S^+(D)$. Moreover, if $0 < \alpha < 2$, then $S^+(D) \subset L^1(D)$.

We need one of the main results in [2]. Define the Green capacity $\text{Cap}_U(E)$ for $E \subset U$ by

\[\text{Cap}_U(E) = \sup \{\mu(E) : G_U \mu \leq 1 \text{ on } U, \mu \text{ is a Borel measure supported on } E\}.\]

By $B(x, r)$ we denote the open ball with center at $x$ and radius $r$.

**Lemma 2** ([2, Theorem 1]). Let $0 < \eta < 1$. Then for an open set $U$ with Green function $G_U$

\[\sup_{x \in U} \int_U G_U(x, y)dy \leq Aw_\eta(U)^2,\]

where

\[w_\eta(U) = \inf \left\{ \rho > 0 : \frac{\text{Cap}_{B(x, 2\rho)}(B(x, \rho) \setminus U)}{\text{Cap}_{B(x, 2\rho)}(B(x, \rho))} \geq \eta \quad \text{for all } x \in U \right\}.\]

The above quantity $w_\eta(U)$ is called the capacitary width of $U$. The definition of John domain readily implies the following.

**Lemma 3.** Let $D$ be a John domain. Then $w_\eta(\{x \in D : \delta_D(x) \leq r\}) \leq Ar$.

The following estimate of the Green potential is called the basic estimate [4, Theorem 3].
Lemma 4. Let \( u \) be a positive continuous superharmonic function on \( D \). For an integer \( j \) we put \( D_j = \{ x \in D : 2^{j-1} < u(x) < 2^{j+2} \} \) and let \( G_j \) be the Green function for \( D_j \). If \( f \) is a nonnegative measurable function on \( D \), then

\[
\sup_{x \in D} \frac{1}{u(x)} \int_D G(x, y) f(y) dy \leq 4 \sum_{j=\infty}^{\infty} \sup_{x \in D_j} \frac{1}{u(x)} \int_{D_j} G_j(x, y) f(y) dy.
\]

**Proof of Theorem 3.** Apply Lemma 4 to \( u = g \) and \( f = Vg \) to obtain

\[
\sup_{x \in D} \frac{1}{g(x)} \int_D G(x, y) V(y) g(y) dy \leq 32 \sum_{j=\infty}^{\infty} \sup_{x \in D_j} \int_{D_j} G_j(x, y) V(y) dy,
\]

where \( D_j = \{ x \in D : 2^{j-1} < g(x) < 2^{j+2} \} \). Since \( g(x) \geq A \delta_D(x)^\alpha \), it follows that

\[
D_j \subset \{ x \in D : \delta_D(x) \leq A 2^{j/\alpha} \}
\]

and hence from Lemmas 2 and 3 that

\[
\sum_{j=\infty}^{0} \sup_{x \in D_j} \int_{D_j} G_j(x, y) V(y) dy \leq A \sum_{j=\infty}^{0} (2^j)^{\alpha - 2/\alpha} (2^{j/\alpha})^2 \leq A \sum_{j=\infty}^{0} 2^{\epsilon j} < \infty.
\]

On the other hand, if \( j \geq 1 \), then \( D_j \subset B(x_0, A 2^{j/(2-n)}) \) if \( n \geq 3 \) and \( D_j \subset B(x_0, \exp(-A 2^j)) \) if \( n = 2 \). Hence Lemma 2 implies

\[
\sum_{j=1}^{\infty} \sup_{x \in D_j} \int_{D_j} G_j(x, y) V(y) dy = \sum_{j=1}^{\infty} \sup_{x \in D_j} \int_{D_j} G_j(x, y) dy \leq \begin{cases} A \sum_{j=1}^{\infty} 2^{j/(2-n)} < \infty & \text{if } n \geq 3, \\ A \sum_{j=1}^{\infty} \exp(-A 2^j) < \infty & \text{if } n = 2. \end{cases}
\]

Thus

\[
\int_D G(x, y) V(y) g(y) dy \leq Ag(x) = AG(x, x_0).
\]

Integrate the above inequality with respect to \( d\mu(x) \) and use Fubini’s theorem. Then we have

\[
\int_D u(y) V(y) g(y) dy \leq Au(x_0)
\]

with \( u = G\mu \). Every \( u \in S^+(D) \) can be approximated from below by a Green potential, so that the monotone convergence theorem proves the first assertion.

Finally, suppose \( 0 < \alpha < 2 \). Let \( \epsilon = -1 + 2/\alpha > 0 \) and observe that \( Vg \geq 1 \) and \( \int_D u dx \leq Au(x_0) \) for \( u \in S^+(D) \). If \( u(x_0) < \infty \), then \( u \in L^1(D) \) obviously.

If \( u(x_0) = \infty \), then replace \( u \) by its Poisson integral over a small ball with center at \( x_0 \). The replaced function belongs to \( S^+(D) \) and its value at \( x_0 \) is finite, so that it belongs to \( L^1(D) \) by the previous observation. This, together with the local integrability of \( u \), proves \( u \in L^1(D) \).

\[ \square \]

3. \( L^p \)-integrability

For a bounded John domain with John constant smaller than that in Theorem 1, we shall obtain \( L^p \)-integrability of positive superharmonic functions with \( 0 < p < 1 \). The exponent \( p \) will be estimated in terms of John constant. To this end we show the following lemma, which is inspired by [14, Theorem 4].
Lemma 5. Let \( D \) be a bounded John domain with John constant \( c_J \). Then there is a positive constant \( \tau_J \) depending only on \( c_J \) and the dimension \( n \) such that
\[
\int_D \delta_D(x)^{-\tau} \, dx < \infty \quad \text{for} \quad 0 < \tau < \tau_J.
\]
Here \( \tau_J \) can be estimated as \( \tau_J \geq \frac{\log(1 + (c_J/20)^n)}{\log 2} \).

Proof. Let \( \tilde{D}_j = \{ x \in D : 2^{-j-1} \leq \delta_D(x) < 2^{-j} \} \). Then \( \bigcup_{j=0}^{\infty} \tilde{D}_j \) is a disjoint decomposition of \( D \) with some \( j_0 \). Observe that \( \sum_{j=0}^{\infty} |\tilde{D}_j| = |D| < \infty \), where \( |\tilde{D}_j| \) denotes the volume of \( \tilde{D}_j \). Without loss of generality we may assume that \( j_0 = 0 \) and \( x_0 \in D_0 \). Suppose \( x \in \bigcup_{j=1}^{\infty} \tilde{D}_j \) with \( j \geq 1 \), i.e., \( \delta_D(x) < 2^{-j-1} \). By definition there is a rectifiable curve \( \gamma \) connecting \( x \) and \( x_0 \) with (1.1). We find a point \( \xi \in \gamma \) such that \( \delta_D(\xi) = 2^{-j} \). By (1.1)
\[
2^{-j} = \delta_D(\xi) \geq c_J \ell(\gamma(x, \xi)) \geq c_J |x - \xi|,
\]
so that \( |x - \xi| \leq c_J^{-1}2^{-j} \). Hence
\[
\bigcup_{i=j+1}^{\infty} \tilde{D}_i \subset \bigcup_{\delta_D(\xi) = 2^{-j}} C(\xi, c_J^{-1}2^{-j}),
\]
where \( C(\xi, c_J^{-1}2^{-j}) \) is the closed ball with center at \( \xi \) and radius \( c_J^{-1}2^{-j} \). Suppose for a moment \( \delta_D(\xi) = 2^{-j} \). Then, by definition, there is a point \( x_\xi \in \partial D \) such that \( |x_\xi - \xi| = 2^{-j} \). Let \( \xi' \) be the point on the line segment \( x_\xi \xi \) with \( |\xi - \xi'| = 2^{-j-1} \). Then an elementary geometrical observation shows that \( \delta_D(\xi') = \frac{1}{2}(2^{-j} + 2^{-j-1}) \) and \( B(\xi', 2^{-j-2}) \subset \tilde{D}_j \), so that
\[
|\tilde{D}_j \cap C(\xi, c_J^{-1}2^{-j})| \geq A_0(2^{-j-2})^n = \left( \frac{c_J}{20} \right)^n |C(\xi, 5c_J^{-1}2^{-j})|,
\]
where \( A_0 \) is the volume of a unit ball. By the covering lemma (see e.g. [17, Theorem 1.3.1]) we can find \( \xi_k \) such that \( \delta_D(\xi_k) = 2^{-j} \), \( \{C(\xi_k, c_J^{-1}2^{-j})\}_k \) is disjoint and
\[
\bigcup_{i=j+1}^{\infty} \tilde{D}_i \subset \bigcup_k C(\xi_k, 5c_J^{-1}2^{-j}).
\]
In view of (3.1) we have
\[
\sum_{i=j+1}^{\infty} |\tilde{D}_i| \leq \sum_k |C(\xi_k, 5c_J^{-1}2^{-j})| \leq \left( \frac{20}{c_J} \right)^n \sum_k |\tilde{D}_j \cap C(\xi_k, c_J^{-1}2^{-j})| \leq \left( \frac{20}{c_J} \right)^n |\tilde{D}_j|.
\]
Multiply the above inequalities by \( r^j \) and take the summation for \( j = 1, \ldots, N - 1 \), where \( r > 1 \) is a constant to be determined. Then
\[
\left( \frac{20}{c_J} \right)^n \sum_{j=1}^{N-1} r^j |\tilde{D}_j| \geq \sum_{j=1}^{N-1} \sum_{i=j+1}^{\infty} r^j |\tilde{D}_i| = \sum_{j=1}^{\min\{N, i\} - 1} \sum_{j=1}^{\infty} r^j |\tilde{D}_i| \\
\geq \sum_{i=2}^{N-1} \sum_{j=1}^{i-1} r^j |\tilde{D}_i| = \sum_{i=2}^{N-1} \frac{r^i - r}{r - 1} |\tilde{D}_i| \\
= \frac{1}{r - 1} \sum_{i=1}^{N-1} r^i |\tilde{D}_i| - \frac{r}{r - 1} \sum_{i=1}^{N-1} |\tilde{D}_i|,
\]
so that
\[ \frac{r}{r-1} \sum_{i=1}^{N-1} |\tilde{D}_i| \geq \left( \frac{1}{r-1} - \left( \frac{20}{c_J} \right)^n \right) \sum_{i=1}^{N-1} r^i |\tilde{D}_i|. \]

Letting \( N \to \infty \), we obtain
\[ \infty > \frac{r}{r-1} |D| = \frac{r}{r-1} \sum_{i=1}^{\infty} |\tilde{D}_i| \geq \left( \frac{1}{r-1} - \left( \frac{20}{c_J} \right)^n \right) \sum_{i=1}^{\infty} r^i |\tilde{D}_i|. \]

Let \( 1 < r < 1 + (c_J/20)^n \). Then
\[ \left( \frac{1}{r-1} - \left( \frac{20}{c_J} \right)^n \right) > 0 \]
and the above inequality implies
\[ \sum_{i=1}^{\infty} r^i |\tilde{D}_i| < \infty. \]

We observe that
\[ r^i \approx \delta_D(x)^{-\log r / \log 2} \quad \text{for} \ x \in \tilde{D}_i, \]
whence
\[ \int_D \delta_D(x)^{-\log r / \log 2} dx < \infty. \]
This proves the lemma. \( \square \)

**Theorem 4.** Let \( D \) be a bounded John domain with John constant \( c_J \). Suppose \( g(x) \geq A\delta_D(x)^\alpha \) for \( x \in D \) with \( \alpha \geq 2 \) and
\[ \int_D \delta_D(x)^{-\tau} dx < \infty \] with \( \tau > 0 \). Then \( S^+(D) \subset L^p(D) \) for \( 0 < p < \tau/(\alpha - 2 + \tau) \).

**Remark.** Let \( \alpha_J \) be as in (2.1) and let \( \tau_J \) be as in Lemma 5. If \( \alpha_J \geq 2 \), then Lemmas 1 and 5 show that \( S^+(D) \subset L^p(D) \) with \( 0 < p < p_J = \tau_J/(\alpha_J - 2 + \tau_J) \).

Observe that \( p_J \approx c_J^{n+1} \) as \( c_J \to 0 \); \( p_J \to 1 \) as \( c_J \to c_n \).

**Proof.** Let \( 0 < p < \tau/(\alpha - 2 + \tau) \). Put
\[ \varepsilon = \frac{1}{\alpha} \left( \frac{(1-p)\tau}{p} - \alpha + 2 \right). \]
Then \( \varepsilon > 0 \) and \( \alpha(\varepsilon - 2/\alpha + 1)p/(1-p) = \tau \). Let \( D' = \{ x \in D : g(x) \leq 1 \} \). Take \( u \in S^+(D) \). Then Hölder’s inequality and Theorem 3 yield
\[ \int_{D'} u^p dx \leq \left( \int_{D'} u g^{\varepsilon-2/\alpha+1} dx \right)^p \left( \int_{D'} g^{-(\varepsilon-2/\alpha+1)p/(1-p)} dx \right)^{1-p} \]
\[ \leq Au(x_0)^p \left( \int_{D'} \delta_D^{-\tau} dx \right)^{1-p} \leq Au(x_0)^p, \]
where (3.2) is used in the last inequality. By the same reasoning as in the proof of Theorem 3, we have \( \int_{D'} u^p dx < \infty \). This, together with the local integrability of a superharmonic function, proves the theorem. \( \square \)
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