

## ON CENTRAL LIMIT THEOREMS FOR SHRUNKEN RANDOM VARIABLES

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ABSTRACT. We discuss Central Limit Theorems and absence of limiting distributions for shrunken random variables.

### 1. INTRODUCTION

For  $r > 0$  let  $U_r$  be the shrinking operator:  $U_r(x) = \max(|x| - r, 0)\text{sgn}(x)$ , or, equivalently,

$$U_r(x) = \begin{cases} x + r & \text{for } -\infty < x < -r, \\ 0 & \text{for } -r \leq x \leq r, \\ x - r & \text{for } r < x < \infty. \end{cases}$$

This non-linear operator was studied by Jurek [2], [3] in relation to his work with  $s$ -self-decomposable distributions. Jurek showed that  $s$ -self-decomposable distributions were the limiting distributions of sums of the form

$$(1.1) \quad U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) + x_n$$

where  $X_1, X_2, \dots$  are independent random variables. Such distributions were also studied in a different context in [4] and [5].

If we let  $\mathcal{G}$  be the class of Gaussian distributions,  $\mathcal{S}$  the class of stable distributions,  $\mathcal{L}$  the class of self-decomposable distributions,  $\mathcal{U}$  the class of  $s$ -self-decomposable distributions, and  $\mathcal{ID}$  the class of infinitely divisible distributions, then we have the following hierarchy:  $\mathcal{G} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{U} \subset \mathcal{ID}$ . Thus the following question recently posed by Jurek naturally arises:

If  $X, X_1, X_2, \dots$  are independent and identically distributed (i.i.d.) random variables, what are the conditions on the distribution and the canonical form of  $r_n$  so that there exist  $x_n$  with (1.1) converging weakly to a standard normal distribution?

We have found that the condition on the distribution is that the tail must decay rapidly. We have:

**Theorem 1.** *Let  $X, X_1, X_2, \dots$  be independent and identically distributed (i.i.d.) random variables. Let*

$$(1.2) \quad G(x) = \int_0^\infty tP(|X| > t + x)dt.$$

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If

- (i)  $0 < G(x) < \infty$  for all  $x > 0$  and
- (ii)  $\lim_{x \rightarrow \infty} G(x + \epsilon)/G(x) = 0$  for all  $\epsilon > 0$

then for any  $r_n \rightarrow \infty$  satisfying

$$(1.3) \quad \lim_{n \rightarrow \infty} 2nG(r_n) = 1$$

and  $x_n = -nE[\max(|X| - r_n, 0)\text{sgn}(X)]$  we have

$$U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) + x_n \Rightarrow N(0, 1)$$

where  $\Rightarrow$  denotes weak convergence.

*Remark 1.* It is easy to see that  $G(x) < \infty$  for some  $x > 0$  if and only if  $EX^2 < \infty$ . When  $EX^2 < \infty$ ,  $G(x)$  is a non-increasing and continuous function. Therefore the solution  $r_n$  to (1.3) exists. Moreover,  $r_n$  must tend to  $\infty$  under condition (i).

*Remark 2.* For  $X$  with bounded support, one can use similar techniques as given below to show that the conclusion to Theorem 1 remains valid although one has to modify the choice of  $r_n$  so that  $n\text{Var}(\max(|X| - r_n, 0)\text{sgn}(X)) \rightarrow 1$ .

It is easy to check that (ii) is satisfied for the normal distribution. However, many other standard distributions, such as the exponential distribution, do not satisfy (ii). On the other hand, as a partial converse, our second theorem shows that one cannot have a central limit theorem type result for (1.1) for such distributions:

**Theorem 2.** *Let  $X, X_1, X_2, \dots$  be i.i.d. random variables. If there exists an  $\varepsilon_0 > 0$  such that*

$$(1.4) \quad \liminf_{x \rightarrow \infty} P(|X| > x + \varepsilon_0)/P(|X| > x) > 0,$$

then there do not exist  $r_n \rightarrow \infty$  and real numbers  $x_n$  such that

$$U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) + x_n \Rightarrow N(0, 1).$$

Thus, for the Weibull family, with density functions:

$$f(x; a) = ax^{a-1} \exp(x^{-a}), \quad x > 0, \quad a > 0,$$

we see that for  $a > 1$ , (1.1) converges weakly to a normal distribution with the proper choice of  $r_n$  given in Theorem 1, but no such Central Limit Theorem result holds for  $0 < a \leq 1$ .

Our final result is that there is no non-degenerate limiting distribution for (1.1) for regular distributions:

**Theorem 3.** *Let  $X, X_1, X_2, \dots$  be i.i.d. random variables with density function  $f(x)$ . Assume that there exist  $p > 1$ ,  $c_1 \geq 0$ ,  $c_2 \geq 0$ ,  $c_1 + c_2 > 0$  and a slowly varying function  $l(x)$  (at  $\infty$ ) such that*

$$(1.5) \quad \lim_{x \rightarrow +\infty} \frac{f(x)}{(l(x)/x^p)} = c_1, \quad \lim_{x \rightarrow +\infty} \frac{f(-x)}{(l(x)/x^p)} = c_2.$$

Then for any  $r_n \rightarrow \infty$  and real numbers  $x_n$ ,  $U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) + x_n$  does not have a non-degenerate limiting distribution. If  $r_n$  satisfies

$$(1.6) \quad \frac{nl(r_n)}{r_n^{p-1}} \rightarrow 0,$$

then

$$U_{r_n}(X_1) + U_{r_n}(X_2) + \cdots + U_{r_n}(X_n) \rightarrow 0 \quad \text{in probability.}$$

## 2. PROOFS

Throughout our proofs we make extensive use of the following theorem in [1], Theorem 18, p. 95. The statement of the theorem is modified slightly here to match our problem:

**Theorem (A).** *Let  $\{Y_n, Y_{n,1}, Y_{n,2}, \dots, Y_{n,n}\}$  be a sequence of series of random variables i.i.d. within each series satisfying that for all  $\varepsilon > 0$ ,  $P(Y_n \geq \varepsilon) \rightarrow 0$ . There will exist a sequence of real constants  $x_n$  such that the distribution of the sums  $\sum_{k=1}^n Y_{nk} + x_n$  converges weakly to the standard normal distribution if and only if the following conditions are satisfied:*

$$(2.1) \quad nP(|Y_n| \geq \varepsilon) \rightarrow 0 \text{ for every } \varepsilon > 0$$

and

$$(2.2) \quad n \operatorname{Var}(Y_n I_{\{|Y_n| \leq 1\}}) \rightarrow 1.$$

If these conditions are satisfied, then we may write

$$x_n = -nE(Y_n I_{\{|Y_n| \leq 1\}}) + o(1).$$

*Proof of Theorem 1.* Let  $Y_{n,k} = U_{r_n}(X_k) = \max(|X_k| - r_n, 0)\operatorname{sgn}(X_k)$  and  $Y_n = U_{r_n}(X)$ . In order to show that we may choose our  $x_n$  as given in Theorem 1 without truncating the tail as in Theorem A, we will show

$$(2.3) \quad nE|Y_n|I_{\{|Y_n| > 1\}} \rightarrow 0.$$

In order to show that the variance term (2.2) tends to 1, we will show

$$(2.4) \quad nEY_n^2 I_{\{|Y_n| \leq 1\}} \rightarrow 1$$

and

$$(2.5) \quad n(E|Y_n|I_{\{|Y_n| \leq 1\}})^2 \rightarrow 0.$$

First, in order to show (2.1), observe that  $P(|Y_n| > \varepsilon) = P(|X| > r_n + \varepsilon)$ . Since  $P(|X| > r_n + \varepsilon) \leq P(|X| > t + r_n + \varepsilon/2)$  for all  $0 \leq t \leq \varepsilon/2$ , we see

$$\begin{aligned} P(|X| > r_n + \varepsilon) &\leq \frac{8}{\varepsilon^2} \int_0^{\varepsilon/2} tP(|X| > t + r_n + \varepsilon/2) dt \\ &\leq \frac{8}{\varepsilon^2} G(r_n + \varepsilon/2). \end{aligned}$$

Now (2.1) follows by our choice of  $r_n$  and condition (ii).

To prove (2.3), we have

$$\begin{aligned} E|Y_n|I_{\{|Y_n| > 1\}} &= E(|X| - r_n)I_{\{|X| > r_n + 1\}} \\ &= P(|X| > r_n + 1) + \int_{1/2}^{\infty} P(|X| > r_n + 1/2 + t) dt \\ &\leq P(|X| > r_n + 1) + 2 \int_{1/2}^{\infty} tP(|X| > r_n + 1/2 + t) dt \\ &\leq P(|X| > r_n + 1) + 2G(r_n + 1/2). \end{aligned}$$

Note that  $P(|X| > r_n + 1) \leq 8G(r_n + 1/2)$  by the proof of (2.1) above with  $\varepsilon = 1$ . Thus,  $E|Y_n|I_{\{|Y_n| > 1\}} \leq 10G(r_n + 1/2)$  which yields (2.3) immediately by our choice of  $r_n$  and condition (ii).

To prove (2.4), let  $G_1(x) = \int_x^\infty P(|X| > t)dt$ . Note that

$$\begin{aligned}
EY_n^2 I_{\{|Y_n| \leq 1\}} &= E(|X| - r_n)^2 I_{\{r_n < |X| \leq r_n + 1\}} \\
&= \int_0^1 t^2 d(-P(|X| > r_n + t)) \\
&= -P(|X| > 1 + r_n) + 2 \int_0^1 tP(|X| > r_n + t)dt \\
&= -P(|X| > 1 + r_n) + 2 \int_0^1 td(-G_1(t + r_n)) \\
&= -P(|X| > 1 + r_n) - 2G_1(1 + r_n) \\
&\quad + 2 \int_0^1 \int_t^\infty P(|X| > s + r_n)dsdt \\
&= -P(|X| > 1 + r_n) - 2 \int_{1+r_n}^\infty P(|X| > t)dt \\
&\quad + 2 \int_0^1 \int_t^1 P(|X| > s + r_n)dsdt + 2 \int_1^\infty P(|X| > s + r_n)ds \\
&= -P(|X| > 1 + r_n) + 2 \int_0^1 sP(|X| > s + r_n)ds \\
&= -P(|X| > 1 + r_n) + 2G(r_n) - 2 \int_1^\infty sP(|X| > s + r_n)ds.
\end{aligned}$$

This proves (2.4) by (2.1), our choice of  $r_n$ , and by noting that

$$\int_1^\infty sP(|X| > s + r_n)ds \leq \int_{1/2}^\infty 2sP(|X| > s + r_n + 1/2)ds \leq 2G(r_n + 1/2)$$

so that  $n \int_1^\infty sP(|X| > s + r_n)ds \rightarrow 0$ .

Finally, we prove (2.5). It follows from the Hölder inequality that

$$\begin{aligned}
n (E|Y_n| I_{\{|Y_n| \leq 1\}})^2 &= n (E(|X| - r_n) I_{\{r_n < |X| \leq 1+r_n\}})^2 \\
&\leq n P(r_n < |X| \leq 1 + r_n) E(|X| - r_n)^2 I_{\{r_n < |X| \leq 1+r_n\}} \rightarrow 0
\end{aligned}$$

by (2.4) and the fact that  $P(|X| > r_n) \rightarrow 0$  since  $r_n \rightarrow \infty$ .

This completes the proof of Theorem 1.  $\square$

*Proof of Theorem 2.* Since  $r_n \rightarrow \infty$ , we have

$$\forall \varepsilon > 0, \quad P(\max(|X| - r_n, 0) > \varepsilon) \rightarrow 0.$$

Assume that (1.2) converges weakly to a standard normal distribution. Then, by Theorem A again, we have

$$(2.6) \quad \forall \varepsilon > 0, \quad nP(|X| \geq r_n + \varepsilon) = nP(\max(|X| - r_n, 0) > \varepsilon) \rightarrow 0$$

and

$$n \text{Var}(\max(|X| - r_n, 0) \text{sgn}(X) I_{\{|\max(|X| - r_n, 0) \text{sgn}(X)| < 1\}}) \rightarrow 1.$$

Thus, for sufficiently large  $n$

$$\begin{aligned}
1/2 &\leq n (E(|X| - r_n)^2 I_{\{r_n < |X| \leq 1+r_n\}}) \\
&\leq nP(|X| > r_n),
\end{aligned}$$

which together with (2.6) yields

$$\forall \varepsilon > 0, \quad \lim_{n \rightarrow \infty} P(|X| > r_n + \varepsilon) / P(|X| > r_n) = 0.$$

This contradicts the assumption (1.4) and completes the proof of Theorem 2.

*Proof of Theorem 3.* Assume that (1.1) has a non-degenerate limiting distribution for some  $r_n \rightarrow \infty$  and  $\{x_n\}$ . Let  $Y_n = U_{r_n}(X) = \max(|X| - r_n, 0)\text{sgn}(X)$ . Then

$$(2.7) \quad e^{itx_n} (Ee^{itY_n})^n \rightarrow g(t),$$

where  $g(t)$  is a non-degenerate characteristic function. Write

$$Ee^{itY_n} = E \cos(t \max(|X| - r_n, 0)) + iE \sin(t \max(|X| - r_n, 0)\text{sgn}(X)).$$

For  $t \neq 0$ , we have

$$\begin{aligned} & E \cos(t \max(|X| - r_n, 0)) \\ &= 1 - P(|X| > r_n) + \int_0^\infty \cos(x) d(-P(|X| > r_n + x/|t|)) \\ &= 1 - \int_0^\infty \sin(x) P(|X| > r_n + x/|t|) dx. \end{aligned}$$

Put

$$I_n(t) = \int_0^\infty \sin(x) P(|X| > r_n + x/|t|) dx.$$

Using the properties of regular distributions, one can check that  $I_n(t)$  is equal to

$$\begin{aligned} & \sum_{k=0}^{\infty} \left\{ \int_{2k\pi}^{(2k+1)\pi} \sin(x) P(|X| > r_n + x/|t|) dx + \int_{(2k+1)\pi}^{(2k+2)\pi} \sin(x) P(|X| > r_n + x/|t|) dx \right\} \\ &= \sum_{k=0}^{\infty} \int_0^\pi \sin(x) \left\{ P(|X| > r_n + (2k\pi + x)/|t|) \right. \\ & \quad \left. - P(|X| > r_n + ((2k+1)\pi + x)/|t|) \right\} dx \\ &= \int_0^\pi \sin(x) \left\{ \sum_{k=0}^{\infty} P((2k\pi + x)/|t| + r_n < |X| \leq r_n + ((2k+1)\pi + x)/|t|) \right\} dx \\ &\sim \int_0^\pi \sin(x) \left\{ \sum_{k=0}^{\infty} \frac{((c_1 + c_2)\pi/|t|)l(r_n + (2k\pi + x)/|t|)}{(r_n + (2k\pi + x)/|t|)^p} \right\} dx \quad [\text{by (1.5)}] \\ &\sim \left\{ \sum_{k=0}^{\infty} \frac{((c_1 + c_2)\pi/|t|)l(r_n + 2k\pi/|t|)}{(r_n + 2k\pi/|t|)^p} \right\} \int_0^\pi \sin(x) dx \\ &\sim 2(c_1 + c_2)(\pi/|t|) \int_0^\infty \frac{l(r_n + 2x\pi/|t|)}{(r_n + 2x\pi/|t|)^p} dx \\ &= (c_1 + c_2) \int_0^\infty \frac{l(r_n + x)}{(r_n + x)^p} dx \\ &\sim \frac{(c_1 + c_2)l(r_n)}{(p-1)r_n^{p-1}}. \end{aligned}$$

Therefore

$$(2.8) \quad E \cos(t \max(|X| - r_n, 0)) - 1 \sim -\frac{(c_1 + c_2)l(r_n)}{(p-1)r_n^{p-1}}.$$

To estimate  $E \sin(t \max(|X| - r_n, 0) \operatorname{sgn}(X))$ , write  $E \sin(t \max(|X| - r_n, 0) \operatorname{sgn}(X)) = E \sin(t(X - r_n))I_{\{X > r_n\}} - E \sin(t(-X - r_n))I_{\{-X > r_n\}}$ .

Following the proof of (2.8), we have

$$\begin{aligned} E \sin(t(|X| - r_n)I_{\{X > r_n\}}) &= \int_0^\infty \sin(tx) d(-P(X > r_n + x)) \\ &= t \int_0^\infty \cos(tx) P(X > r_n + x) dx \\ &= (t/|t|) \int_0^\infty \cos(x) P(X > r_n + x/|t|) dx \\ &= \operatorname{sgn}(t) \int_0^{\pi/2} \cos(x) P(X > r_n + x/|t|) dx \\ &\quad + \operatorname{sgn}(t) \sum_{k=0}^\infty \left\{ \int_{2k\pi + \pi/2}^{(2k+1)\pi + \pi/2} \cos(x) P(X > r_n + x/|t|) dx \right. \\ &\quad \left. + \int_{(2k+1)\pi + \pi/2}^{(2k+2)\pi + \pi/2} \cos(x) P(X > r_n + x/|t|) dx \right\} \\ &= \operatorname{sgn}(t) \int_0^{\pi/2} \cos(x) P(X > r_n + x/|t|) dx \\ &\quad + \operatorname{sgn}(t) \sum_{k=0}^\infty \left\{ \int_0^\pi \sin(x) \left\{ -P(X > r_n + (x + 2k\pi + \pi/2)/|t|) \right. \right. \\ &\quad \left. \left. + P(X > r_n + (x + (2k+1)\pi + \pi/2)/|t|) \right\} dx \right\} \\ &\sim \operatorname{sgn}(t) \frac{c_1 l(r_n)}{r_n^p} \int_0^{\pi/2} \cos(x) dx \\ &\quad - (c_1 \pi / |t|) \operatorname{sgn}(t) \sum_{k=0}^\infty \frac{l(r_n + 2k\pi/|t|)}{(r_n + 2k\pi/|t|)^p} \int_0^\pi \sin(x) dx \\ &\sim -\frac{c_1 \operatorname{sgn}(t) l(r_n)}{(p-1)r_n^{p-1}}. \end{aligned}$$

Similarly,

$$E \sin(t(-X - r_n)I_{\{-X > r_n\}}) \sim -\frac{c_2 \operatorname{sgn}(t) l(r_n)}{(p-1)r_n^{p-1}}.$$

Hence

$$(2.9) \quad E \sin(t \max(|X| - r_n, 0) \operatorname{sgn}(X)) \sim (c_2 - c_1) \operatorname{sgn}(t) \frac{l(r_n)}{(p-1)r_n^{p-1}}.$$

From (2.8) and (2.9) it follows that

$$(2.10) \quad (E e^{itY_n})^n = \left( 1 - \frac{l(r_n)}{(p-1)r_n^{p-1}} (c_1 + c_2 + i \operatorname{sgn}(t)(c_1 - c_2) + o(1)) \right)^n.$$

Consider four different cases:

*Case 1.* When  $\liminf_{n \rightarrow \infty} \frac{nl(r_n)}{r_n^{p-1}} < \limsup_{n \rightarrow \infty} \frac{nl(r_n)}{r_n^{p-1}}$ , (2.7) contradicts (2.10).

*Case 2.* When  $\lim_{n \rightarrow \infty} \frac{nl(r_n)}{r_n^{p-1}} = \infty$ , (2.10) implies that for every  $t \neq 0$ ,

$$(Ee^{itY_n})^n \rightarrow 0,$$

which is also in contradiction with (2.7).

*Case 3.* When  $\lim_{n \rightarrow \infty} \frac{nl(r_n)}{r_n^{p-1}} = a$ , where  $0 < a < \infty$ , we obtain from (2.10) that

$$\forall t \neq 0, (Ee^{itY_n})^n \rightarrow \exp\left(-\frac{a}{p-1}(c_1 + c_2 + i\operatorname{sgn}(t)(c_1 - c_2))\right).$$

Since  $(Ee^{i0Y_n})^n \equiv 1$ , the limiting function of  $(Ee^{itY_n})^n$  is not continuous, which contradicts the assumption that  $g(t)$  is a characteristic function.

*Case 4.* When  $\lim_{n \rightarrow \infty} \frac{nl(r_n)}{r_n^{p-1}} = 0$ , then we have

$$(Ee^{itY_n})^n \rightarrow 1$$

and hence the limiting distribution is degenerate.

This completes the proof of the theorem.

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#### REFERENCES

- [1] V. V. Petrov, *Sums of Independent Random Variables*, Springer-Verlag, New York, 1975. MR **52**:9335
- [2] Z. Jurek, Limit distributions for sums of shrunken random variables, *Dissertationes Math.* **185** (1981) PWN Warszawa. MR **82i**:60018
- [3] Z. Jurek, Relations between the s-selfdecomposable and selfdecomposable measures, *Ann. Probab.* **13** (1985), 592–608. MR **87a**:60016
- [4] P. Medgyessy, On a new class of unimodal infinitely divisible distribution functions and related topics, *Studia Sci. Math. Hungar.* **2** (1967), 441–446. MR **36**:5979
- [5] T. A. O'Connor, Infinitely divisible distributions with unimodal Levy spectral functions, *Ann. Probab.* **7** (1979), 494–499. MR **80b**:60035

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