ON A THEOREM BY SERRE

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Abstract. We present a short proof of a theorem by Serre on the trace form of a finite separable field extension.

Let $M/K$ be a finite Galois extension in characteristic $\neq 2$, and assume that $M$ is the splitting field over $K$ of an irreducible polynomial $f(X) \in K[X]$ of degree $n$. We embed the Galois group $G = \text{Gal}(M/K)$ transitively into $S_n$ by considering the elements of $G$ as permutations of the roots of $f(X)$. From the ‘positive’ double cover

$$1 \to \mu_2 \to \tilde{S}_n^+ \to S_n \to 1$$

of $S_n$ (i.e., the double cover in which transpositions lift to elements of order 2) we then get an extension

$$(*) \quad 1 \to \mu_2 \to \tilde{G}^+ \to G \to 1$$

of $G$ with the cyclic group $\mu_2 = \{\pm 1\}$. Let $\gamma^+ \in H^2(G, \mu_2)$ be the characteristic class of $(*).$

We embed $S_n$ into the orthogonal group $O_n(\bar{K}_{\text{sep}})$ as permutations of the standard basis vectors $e_1, \ldots, e_n \in \bar{K}_{\text{sep}}^n$. ($\bar{K}_{\text{sep}}$ being the separable closure of $K$.) As the pre-image in the Clifford group $C_n^*(\bar{K}_{\text{sep}})$ of a transposition $(ij), \ i < j,$ we can then take the element $x_{ij} = (e_i - e_j)/\sqrt{2}$. The subgroup of $C_n^*(\bar{K}_{\text{sep}})$ generated by these is exactly the double cover $\tilde{S}_n^+$ of $S_n$, and we get a diagram

\[
\begin{array}{cccccc}
1 & \to & \mu_2 & \to & \tilde{G}^+ & \to & G & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \mu_2 & \to & \tilde{S}_n^+ & \to & S_n & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \bar{K}_{\text{sep}}^* & \to & C_n^*(\bar{K}_{\text{sep}}) & \to & O_n(\bar{K}_{\text{sep}}) & \to & 1,
\end{array}
\]

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where \( \text{Gal}(K) = \text{Gal}((K_{\text{sep}})/K) \) is the absolute Galois group of \( K \). The last row of this diagram is a short-exact sequence of \( G \)-groups, inducing a connecting map \( \delta : H^1(\text{Gal}(K), O_\text{Gal}(K_{\text{sep}})) \to H^2(\text{Gal}(K), K_{\text{sep}}^*) \), cf. [Sel], and we know from [Sp] that the image of the crossed homomorphism \( e : \text{Gal}(K) \to O_\text{Gal}(K_{\text{sep}}) \) in the last column of the diagram is the Hasse-Witt invariant of the quadratic form obtained from \( \langle 1, \ldots, 1 \rangle \) by Galois twist with \( e \). The \textit{Hasse-Witt invariant} of a regular quadratic form \( q \sim \langle a_1, \ldots, a_n \rangle \) is

\[
\text{lw}(q) = \prod_{i<j} (a_i, a_j) \in H^2(\text{Gal}(K), K_{\text{sep}}^*),
\]

where the elements \( (a_i, a_j) \in H^2(\text{Gal}(K), K_{\text{sep}}^*) \) are \textit{quaternion symbols}: For \( a, b \in K^* \) the quaternion symbol \( (a, b) \) is represented by the factor system \( (\sigma, \tau) \mapsto (-1)^{\chi_\sigma(\sigma)\chi_\tau(\tau)} \), where \( \chi_\sigma, \chi_\tau : \text{Gal}(K) \to \mathbb{F}_2 \) are the homomorphisms with kernels \( \text{Gal}(K(\sqrt{a})) \) and \( \text{Gal}(K(\sqrt{b})) \), resp.

Now, let \( L = K(\theta) \), where \( \theta \) is a root of \( f(X) \), and let \( \theta_1 = \theta, \theta_2, \ldots, \theta_n \in M \) be the conjugates. This numbering fixes our embedding of \( G \) into \( S_n \). The Galois twist corresponding to \( e \) above is obtained by restricting \( (1, \ldots, 1) \) from \( K_{\text{sep}}^* \) to the space of fixed points under the \( G \)-action \( \sigma X = e_\sigma(\sigma X) \). It is easy to see that the fixed points are exactly the points

\[
(g(\theta_1), \ldots, g(\theta_n)), \quad g(X) \in K[X],
\]

meaning that the twisted quadratic space is \( L \) equipped with the \textit{trace form} \( q_L : x \mapsto \text{Tr}_{L/K}(x^2) \).

We compute \( \delta(e) \) directly as follows: Let \( s_\sigma \in \tilde{G}^+ \) be a pre-image of \( \sigma \in G \). Then \( s_{\text{res} \sigma} \in C_n(K_{\text{sep}}) \) is a pre-image of \( e_\sigma \in O_\text{Gal}(K_{\text{sep}}) \) for \( \sigma \in \text{Gal}(K) \), and \( \delta(e) \) is given by the factor system

\[
(\sigma, \tau) \mapsto s_{\text{res} \sigma} s_{\text{res} \tau}^{-1} = (-1)^{\chi_\sigma(\sigma)\chi_\tau(\tau)} s_{\text{res} \sigma} s_{\text{res} \tau}^{-1} s_{\text{res} \sigma} s_{\text{res} \tau}^{-1}, \quad \sigma, \tau \in \text{Gal}(K),
\]

where \( d = \text{disc}(L/K) \) is the discriminant of \( L/K \), since \( \sigma \) operates on \( s_{\text{res} \tau} \) through the factor \( 1/\sqrt{2} \) contributed by each transposition. Here, \( (\sigma, \tau) \mapsto s_{\text{res} \sigma} s_{\text{res} \tau} s_{\text{res} \sigma}^{-1} \) is the inflation of \( \gamma^+ \) to \( H^2(\text{Gal}(K), K_{\text{sep}}^*) \), and \( (\sigma, \tau) \mapsto (-1)^{\chi_\sigma(\sigma)\chi_\tau(\tau)} \) is the quaternion symbol \( (2, d) \). Hence, we have

**Theorem** (Serre, [Se2]). With notation as above,

\[
\inf_{G \to \text{Gal}(K)}(\gamma^+) = \text{lw}(q_L) \cdot (2, \text{disc}(L/K)) \in H^2(\text{Gal}(K), K_{\text{sep}}^*).
\]

If we look instead at the ‘negative’ double cover

\[
1 \to \mu_2 \to \widetilde{S}_n^- \to S_n \to 1
\]

of \( S_n \), where transpositions lift to elements of order 4, we get an extension

\[
1 \to \mu_2 \to \widetilde{G}^- \to G \to 1
\]

of \( G \) with \( \mu_2 \). Let \( \gamma^- \in H^2(G, \mu_2) \) be the characteristic class of \((**)\). The elements \( \gamma_{ij} = (e_i - e_j)/\sqrt{-2} \in C_n(K_{\text{sep}}) \), \( i < j \), generate a copy of \( \widetilde{S}_n^- \) mapping onto \( S_n \subseteq O_\text{Gal}(K_{\text{sep}}) \), and we can repeat the entire argument above with \(-2\) instead of \( 2 \), getting

\[\footnote{The author would like to thank the referee for suggesting this modified argument.}\]
Theorem. With notation as above,
\[ \inf_G \text{Gal}(K)(\gamma^-) = \text{hw}(q_L) \cdot (-2, d_{L/K}) \in H^2(\text{Gal}(K), \bar{K}^*_{\text{sep}}). \]

References


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