

## OSCILLATING GLOBAL CONTINUA OF POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS

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ABSTRACT. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 1$ , with  $C^2$  boundary  $\partial\Omega$ , and consider the semilinear elliptic boundary value problem

$$\begin{aligned} Lu &= \lambda au + g(\cdot, u)u, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{aligned}$$

where  $L$  is a uniformly elliptic operator on  $\bar{\Omega}$ ,  $a \in C^0(\bar{\Omega})$ ,  $a$  is strictly positive in  $\bar{\Omega}$ , and the function  $g : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable, with  $g(x, 0) = 0$ ,  $x \in \bar{\Omega}$ . A well known result of Rabinowitz shows that an unbounded continuum of positive solutions of this problem bifurcates from the principal eigenvalue  $\lambda_1$  of the linear problem. We show that under certain oscillation conditions on the nonlinearity  $g$ , this continuum oscillates about  $\lambda_1$ , in a certain sense, as it approaches infinity. Hence, in particular, the equation has infinitely many positive solutions for each  $\lambda$  in an open interval containing  $\lambda_1$ .

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 1$ , with  $C^2$  boundary  $\partial\Omega$ , and consider the semilinear elliptic boundary value problem

$$(1) \quad \begin{aligned} Lu := - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) + qu &= \lambda au + g(\cdot, u)u, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{aligned}$$

where  $a_{ij} \in C^1(\bar{\Omega})$ ,  $i, j = 1, \dots, n$ ,  $q \in C^0(\bar{\Omega})$ ,  $L$  is uniformly elliptic in  $\bar{\Omega}$ ,  $a \in C^0(\bar{\Omega})$ ,  $a$  is strictly positive in  $\bar{\Omega}$ . The function  $g : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable, with  $g(x, 0) = 0$ ,  $x \in \bar{\Omega}$ , and there are non-negative constants  $\gamma^{--}$ ,  $\gamma^{++}$  such that

$$(2) \quad -\gamma^{--} \leq \frac{g(x, \xi)}{a(x)} \leq \gamma^{++}, \quad (x, \xi) \in \bar{\Omega} \times \mathbb{R}.$$

For any integer  $r \geq 0$ ,  $C^r(\bar{\Omega})$  will denote the Banach space of real-valued, continuous functions on  $\bar{\Omega}$ , having continuous derivatives up to order  $r$  on  $\bar{\Omega}$ , and  $C^{r,\alpha}(\bar{\Omega})$ ,  $\alpha \in (0, 1)$ , will denote the set of functions in  $C^r(\bar{\Omega})$  whose derivatives of order  $r$  are Hölder continuous with exponent  $\alpha$ ; we let  $|\cdot|_r$ ,  $|\cdot|_{r,\alpha}$  denote the usual norms on these spaces. From now on we suppose that  $\alpha \in (0, 1)$  is fixed and let  $E$  be the subspace of  $C^{1,\alpha}(\bar{\Omega})$  consisting of those functions which are zero on the boundary

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$\partial\Omega$ . A *solution* of (1) is a pair  $(\lambda, u) \in \mathbb{R} \times C^2(\overline{\Omega})$  satisfying (1) (similarly for other equations below). Thus we may consider the structure of the set of solutions in the space  $\mathbb{R} \times E$ . A *positive* (*negative*) solution of (1) is a solution  $(\lambda, u)$  with  $u > 0$  ( $u < 0$ ) in  $\Omega$ . All the results below regarding positive solutions have analogues for negative solutions, but we will not discuss these further.

Let  $\lambda_1$  be the principal eigenvalue of the linear problem obtained from (1) by putting  $g \equiv 0$ , and let  $\phi$  be the corresponding eigenfunction with the properties

$$\phi(x) > 0, \quad x \in \Omega, \quad |\phi|_0 = 1.$$

Under the above hypotheses it is well known (see [2] and [7]) that a ‘simple bifurcation’ takes place at  $\lambda = \lambda_1$ , and a single, unbounded continuum  $\mathcal{C}_1^+ \subset \mathbb{R} \times E$  of positive solutions of (1) bifurcates from  $\lambda_1$  (that is, the closure of  $\mathcal{C}_1^+$  is  $\{(\lambda_1, 0)\} \cup \mathcal{C}_1^+$ ). We have the following preliminary result regarding  $\mathcal{C}_1^+$ . Let  $I_1 = [\lambda_1 - \gamma^{++}, \lambda_1 + \gamma^{--}]$ .

**Theorem 1.** *Under the above hypotheses, any positive solution of (1) lies in the set  $I_1 \times E$ . In particular,*

$$\mathcal{C}_1^+ \subset I_1 \times E.$$

Furthermore, for any number  $\xi > 0$  there exists a solution  $(\lambda, u) \in \mathcal{C}_1^+$  with  $|u|_0 = \xi$ .

*Proof.* Suppose that  $(\lambda, u)$  is a positive solution of (1). Multiplying (1) by  $\phi$ , integrating by parts and using the definition of  $\phi$  yields

$$(3) \quad (\lambda_1 - \lambda) \int_{\Omega} au\phi = \int_{\Omega} g(\cdot, u)u\phi.$$

Hence,

$$(\lambda_1 - \lambda) \int_{\Omega} au\phi \leq \gamma^{++} \int_{\Omega} au\phi \implies \lambda \geq \lambda_1 - \gamma^{++}.$$

We obtain  $\lambda \leq \lambda_1 + \gamma^{--}$  similarly.

Now suppose that  $(\lambda_k, u_k)$ ,  $k = 1, 2, \dots$ , is a sequence of positive solutions of (1) which is bounded in  $I_1 \times C^0(\overline{\Omega})$ . Then the right hand side of (1) is bounded in  $L^\infty$ , so by elliptic regularity results the sequence  $u_k$  is bounded in  $W^{2,p}(\Omega)$  for all  $p > 1$ , and hence by the Sobolev embedding theorem this sequence is bounded in  $E$ . Thus any set of positive solutions of (1) which is bounded in  $I_1 \times C^0(\overline{\Omega})$  is bounded in  $I_1 \times E$ . But we know that the continuum  $\mathcal{C}_1^+$  is unbounded in  $I_1 \times E$ , so it must be unbounded in  $I_1 \times C^0(\overline{\Omega})$ . Also, the mapping  $(\lambda, u) \rightarrow |u|_0$  from  $\mathcal{C}_1^+ \subset I_1 \times E$  to  $\mathbb{R}$  is continuous so, since  $\mathcal{C}_1^+$  is connected and  $(\lambda_1, 0)$  belongs to the closure of  $\mathcal{C}_1^+$ , this image must be the interval  $(0, \infty)$ . Hence the second part of the theorem follows.  $\square$

Theorem 1 shows that  $\mathcal{C}_1^+$  goes to infinity in the cylinder  $I_1 \times E$ . Under a further oscillation type condition on  $g$  we will show that  $\mathcal{C}_1^+$  ‘oscillates’ in this cylinder, as it approaches infinity, over an open  $\lambda$  interval containing  $\lambda_1$ , and the amplitude of these oscillations is bounded away from zero (this will be made precise in Theorem 2 below). This type of behaviour is illustrated in Figure 1 in [4], or in the bifurcation diagram in [5] for the case where  $\mathcal{C}_1^+$  is a smooth curve (except that here the oscillations of  $\mathcal{C}_1^+$  are above  $\lambda_1$ ). Here, in general, the structure of  $\mathcal{C}_1^+$  could be more complicated.

We now briefly discuss earlier related results. The paper [10] considers the nonlinear Sturm-Liouville problem

$$u'' + \lambda r(x)u + p(x)g(u) = h(x), \quad x \in (0, 1),$$

with self-adjoint boundary conditions, and assumes that  $g(\xi)/\xi \rightarrow 0$ , as  $\xi \rightarrow \infty$ , and  $g$  satisfies an oscillation condition (e.g.,  $g$  is periodic). It is shown that there is a connected set of positive solutions  $\mathcal{C}_1^+$  which oscillates infinitely often above  $\lambda_1$  so, in particular, the equation has infinitely many positive solutions in  $\mathcal{C}_1^+$  with  $\lambda = \lambda_1$ . However, the amplitude of the oscillations tends to zero (in fact, the set  $\mathcal{C}_1^+$  bifurcates from  $(\lambda_1, \infty)$ ) so it does not oscillate over any open interval. An elliptic analogue, in a bounded domain in  $\mathbb{R}^n$ , of the above Sturm-Liouville problem is considered in [11] (the term  $u''$  is replaced with  $\Delta u$ ). By a slightly different technique it is again shown that the equation has infinitely many positive solutions in  $\mathcal{C}_1^+$  with  $\lambda = \lambda_1$ . Again it is assumed that  $g(\xi)/\xi \rightarrow 0$ , as  $\xi \rightarrow \infty$ , and satisfies an oscillation condition, and again the amplitude of the oscillations of  $\mathcal{C}_1^+$  may tend to zero. In [6] the equation

$$(4) \quad Lu + \lambda f(u) = 0$$

is considered. It is assumed that  $f$  is asymptotically linear and satisfies certain oscillation conditions, and it is shown that a solution continuum bifurcates from  $(\lambda_1, \infty)$  and this continuum oscillates infinitely often (here,  $\lambda_1$  need not be the principal eigenvalue, and the solutions need not be positive). Thus the equation has infinitely many solutions with  $\lambda = \lambda_1$  but, since the continuum bifurcates from  $(\lambda_1, \infty)$ , the amplitude of the oscillations again tends to zero and the continuum does not oscillate over any open  $\lambda$  interval.

In [5] the equation (4) is also considered. Here,  $f$  is not asymptotically linear, but it is strictly increasing and satisfies an oscillation condition (the condition we impose on  $g$ , described below, is based on this condition). In essence,  $f(\xi)/\xi$  oscillates and the amplitude of the oscillations is bounded away from zero, while the duration (or ‘period’) of the oscillations grows at a certain rate as  $\xi$  increases. In particular, this condition means that  $f(\xi)/\xi$  cannot be periodic. For each fixed  $\lambda$  in a certain interval, sequences of positive solutions are obtained by sub- and super-solution methods. However, these solutions may not lie on the bifurcating set  $\mathcal{C}_1^+$ , so this does not show that  $\mathcal{C}_1^+$  oscillates. It is then shown in [4] that for a class of planar domains  $\Omega \subset \mathbb{R}^2$  with certain symmetry properties the solutions found in [5] do lie on  $\mathcal{C}_1^+$ , and this set is a smooth curve parametrized by  $|u|_0$ . Illustrative bifurcation diagrams are given in [4] and [5].

The oscillation condition on  $g$  will now be described. We suppose that there are non-negative constants  $\gamma^-, \gamma^+$ , a constant  $\kappa \in (0, 1)$ , and an increasing sequence of positive numbers  $\xi_j, j = 1, 2, \dots$ , such that for each  $j, \xi_j < \kappa \xi_{j+1}$ , and for all  $x \in \bar{\Omega}$ ,

$$(5) \quad \begin{aligned} \gamma^+ &\leq g(x, \xi)/a(x), & \xi \in (\kappa \xi_{2j-1}, \xi_{2j-1}), \\ -\gamma^- &\geq g(x, \xi)/a(x), & \xi \in (\kappa \xi_{2j}, \xi_{2j}). \end{aligned}$$

With this condition we have the following result.

**Theorem 2.** *Choose a number  $\mu \in (0, 1)$ . If  $g$  satisfies (5), with the constant  $\kappa$  sufficiently small (depending on  $\mu$ ), then any positive solution  $(\lambda, u)$  of (1) with  $|u|_0 = \xi_{2j-1}, j \geq 1$  (respectively  $|u|_0 = \xi_{2j}$ ) has  $\lambda < \lambda_1 - \mu\gamma^+$  (respectively  $\lambda > \lambda_1 + \mu\gamma^-$ ).*

*Proof.* We begin by considering the problem

$$(6) \quad \begin{aligned} Lu + cu &= 0, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \end{aligned}$$

for a general function  $c \in C^0(\bar{\Omega})$  with  $|c|_0 \leq \gamma$ , where

$$\gamma = (1 + |a|_0)(|\lambda_1| + \max\{\gamma^{++}, \gamma^{--}\}).$$

Note that by Theorem 1 and the assumptions on  $g$ , equation (1) has the form of (6) with  $|c|_0 \leq \gamma$ . For any  $\delta > 0$ , let

$$\Omega(\delta) = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\},$$

and let  $|\cdot|_{0;\bar{\Omega}(\delta)}$  denote the sup norm on the closure,  $\bar{\Omega}(\delta)$ , of this set.

**Lemma 3.** *There exists  $\delta_0 > 0$  and a constant  $C_0$  (depending on  $\gamma$  but not on  $c$ ) such that if  $|c|_0 \leq \gamma$  and  $u > 0$  satisfies (6), then  $|u|_0 \leq C_0|u|_{0;\bar{\Omega}(\delta_0)}$ .*

*Remark.* The constants in the lemma and in the proof also depend on  $n$ ,  $\Omega$  and  $L$ , but we regard these as fixed so this dependence is unimportant here.

*Proof.* Suppose that the result is false. Then for each integer  $k > 0$  there exist functions  $c_k$  and  $u_k$  satisfying the conditions of the lemma but for which, if  $u_k(x_k) = |u_k|_0$ , we have

$$u_k(x_k) > r(k)|u_k|_{0;\bar{\Omega}(1/k)},$$

with  $r(k) \rightarrow \infty$  as  $k \rightarrow \infty$ . By considering a subsequence if necessary, we may suppose that  $x_k \rightarrow y \in \partial\Omega$  as  $k \rightarrow \infty$ . Now, by Theorem 9.26 of [3] we may choose a fixed  $R > 0$  and a constant  $C_1$  (which depends only on  $\gamma$  and  $R$ ) such that for sufficiently large  $k$  we have

$$u_k(x_k) \leq C_1 R^{-n} \int_{\Omega \cap B_{2R}(y)} u_k,$$

where  $B_{2R}(y)$  is the ball centred at  $y$ , of radius  $2R$ . Now let  $E_1 = \Omega(1/k) \cap B_{2R}(y)$ ,  $E_2 = (\Omega \setminus \Omega(1/k)) \cap B_{2R}(y)$ . Then there is a constant  $C_2 > 0$  such that

$$\int_{E_1} u_k \leq C_2 r(k)^{-1} R^n u_k(x_k), \quad \int_{E_2} u_k \leq C_2 k^{-1} R^{n-1} u_k(x_k)$$

(the second inequality holds, if  $R$  is chosen to be sufficiently small, since  $\partial\Omega$  is smooth). Hence

$$u_k(x_k) \leq C_1 C_2 (r(k)^{-1} + k^{-1} R^{-1}) u_k(x_k),$$

which for sufficiently large  $k$  is a contradiction. This proves the lemma.  $\square$

Now let  $\rho = \max_{\bar{\Omega}} a / \min_{\bar{\Omega}} a$ , and choose  $\epsilon > 0$  sufficiently small that

$$(7) \quad \gamma^+(1 - \rho\epsilon) - \gamma^-\rho\epsilon > \mu\gamma^+.$$

For any  $\delta > 0$ , let

$$K(\delta) = \{x \in \Omega : \phi(x) \geq \delta\},$$

and choose  $\delta_1 > 0$  sufficiently small that  $\Omega(\delta_0) \subset K(\delta_1)$ , and

$$(8) \quad |\Omega \setminus K(\delta_1)| \leq \epsilon |K(\delta_1)|,$$

where  $|\cdot|$  denotes the Lebesgue measure of a set here (this is possible since  $\phi > 0$  on  $\Omega$ ). We denote the set  $K(\delta_1)$  by  $K$ . By a slight extension of Corollary 9.25 in [3],

there exists a constant  $C_3 > 0$  (depending on  $K$  and  $\gamma$ ) such that for any positive solution  $(\lambda, u)$  of (1),

$$(9) \quad \min_K u \geq C_3 \max_K u$$

(recall that (1) has the form of (6) with  $|c|_0 \leq \gamma$ ). Now suppose that (5) holds with  $\kappa \leq C_0^{-1}C_3$ , and suppose that  $(\lambda, u)$  is a positive solution of (1) with  $|u|_0 = \xi_{2j-1}$ , for some integer  $j \geq 1$ . Then by Lemma 3 and (9),

$$\min_K u \geq C_3|u|_{0;K} \geq C_3|u|_{0;\bar{\Omega}(\delta_0)} \geq C_0^{-1}C_3\xi_{2j-1} \geq \kappa\xi_{2j-1},$$

and hence

$$(10) \quad g(x, u(x))/a(x) \geq \gamma^+, \quad x \in K.$$

Now let

$$E(u) = \{x \in \Omega : u(x) < \min_K u\} \subset \Omega \setminus K.$$

Clearly, the inequality (10) also holds for  $x \in \Omega \setminus E(u)$ . On the other hand, for  $x \in E(u)$  we have  $u(x) \leq \min_K u$  and  $\phi(x) \leq \min_K \phi$ , hence, by (8),

$$(11) \quad \begin{aligned} \int_{E(u)} au\phi &\leq |E(u)| \sup_{E(u)}(au\phi) \leq \epsilon|K| \max_{\bar{\Omega}} a \min_K u \min_K \phi \\ &\leq \epsilon\rho \int_K au\phi \leq \epsilon\rho \int_{\Omega} au\phi. \end{aligned}$$

Hence from (3), (7), (10) and (11) we have

$$\begin{aligned} (\lambda_1 - \lambda) \int_{\Omega} au\phi &= \int_{\Omega} g(\cdot, u)u\phi \\ &\geq \int_{\Omega \setminus E(u)} g(\cdot, u)u\phi - \gamma^{--} \int_{E(u)} au\phi \\ &\geq \gamma^+ \int_{\Omega \setminus E(u)} au\phi - \gamma^{--}\rho\epsilon \int_{\Omega} au\phi \\ &\geq \gamma^+(1 - \rho\epsilon) \int_{\Omega} au\phi - \gamma^{--}\rho\epsilon \int_{\Omega} au\phi \\ &> \mu\gamma^+ \int_{\Omega} au\phi, \end{aligned}$$

and so  $\lambda < \lambda_1 - \mu\gamma^+$ . Similarly, if  $(\lambda, u)$  is a positive solution of (1) with  $|u|_0 = \xi_{2j}$ , we can show that  $\lambda > \lambda_1 + \mu\gamma^-$ . This completes the proof of Theorem 2.  $\square$

We now have the following simple corollary of Theorems 1 and 2.

**Corollary 4.** *Suppose that the hypotheses of Theorem 2 hold. Then the continuum  $\mathcal{C}_1^+$  lies in  $I_1 \times E$ , is unbounded and oscillates infinitely often over the interval  $J_\mu = [\lambda_1 - \mu\gamma^+, \lambda_1 + \mu\gamma^-]$  as it approaches infinity, in the sense that for each  $\lambda$  in  $J_\mu$  there are infinitely many solutions  $(\lambda, u) \in \mathcal{C}_1^+$ .*

*Proof.* By Theorem 1, for each integer  $j \geq 2$  we can choose a solution  $(\lambda_j, u_j) \in \mathcal{C}_1^+$  with  $|u_j|_0 = \xi_j$ . Since  $\mathcal{C}_1^+$  connects  $(\lambda_j, u_j)$  with  $(\lambda_1, 0)$  it follows from Theorem 2 that for each  $\lambda \in J_\mu$  there must be a solution  $(\lambda, u) \in \mathcal{C}_1^+$  with  $\xi_{j-1} \leq |u|_0 \leq \xi_j$ . Hence there must be infinitely many solutions  $(\lambda, u) \in \mathcal{C}_1^+$ .  $\square$

The ‘simple bifurcation’ results in [2] show that near  $(\lambda_1, 0)$  in  $\mathbb{R} \times E$  the set  $\mathcal{C}_1^+$  is a single continuous curve, but the general topological results in [7] do not imply that the set  $\mathcal{C}_1^+$  is a curve globally. We now briefly discuss two simple situations in which it can be shown that the entire set  $\mathcal{C}_1^+$  is a single smooth, 1-dimensional curve: (i) generic equations of the form (1); (ii) sufficiently small functions  $g$ . In each of these cases the above results hold, so, in particular, the curve oscillates on either side of  $\lambda_1$ .

#### GENERIC EQUATIONS

Suppose that  $\partial\Omega$  is  $C^{3,\alpha}$ , the coefficients  $a_{ij}$ ,  $q$ ,  $a \in C^{3,\alpha}(\overline{\Omega})$ , and  $g : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  is  $C^{r,\alpha}$ , for some integer  $r \geq 3$ . Then Theorem 6 of [9] shows that for a generic set of coefficients  $\{a_{ij} : i, j = 1, \dots, n\}$  (in an appropriate open subset of the Cartesian product space  $C^{3,\alpha}(\overline{\Omega})^{n^2}$ ) the set  $\mathcal{C}_1^+$  consists of a single unbounded, 1-dimensional,  $C^r$  curve in  $I_1 \times E$ . We note that: (i) the precise meaning of the term ‘generic’, in this context, is given in [9]; (ii) the elliptic equation (1) above is in divergence form, whereas the equation (1.4) considered in [9] is not, so a slight adaptation of the proof in [9] is required here.

Probably this result also holds for generic domains  $\Omega$  in  $\mathbb{R}^n$ ,  $n \geq 2$ . If we choose a fixed  $g$  it can be shown that for generic  $\Omega$  the set  $\mathcal{C}_1^+$  is a smooth curve. Unfortunately, in the proof of the oscillation results the choice of  $g$  depends on  $\Omega$  (via the constant  $\kappa$ , which depends on  $\Omega$ ), which leads to a rather circular argument. Possibly a sufficiently small perturbation of the domain would work, but it is not clear how the constant  $\kappa$  depends on such a perturbation of  $\Omega$ . This difficulty does not arise when considering generic coefficients  $\{a_{ij}\}$  since  $\kappa$  depends only on bounds for the  $\{a_{ij}\}$ , so  $\kappa$  can be chosen independently of sufficiently small perturbations of the  $\{a_{ij}\}$ .

The results of [8] can also be used to show that for generic functions  $g$  the ‘vertical’ turning points on the curve  $\mathcal{C}_1^+$  are non-degenerate, i.e., quadratic — see [8] for more details.

#### SMALL $g$

Suppose that  $g : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  is  $C^r$ , for some integer  $r \geq 2$ . Let  $\lambda_2$  be the second eigenvalue of the linearization of (1) and let  $d_1 = \lambda_2 - \lambda_1 > 0$ . Define the numbers

$$\begin{aligned} \gamma_1 &= \max\{\gamma^{++}, \gamma^{--}\}, \\ \gamma_2 &= \sup\{|g_\xi(x, \xi)\xi + g(x, \xi)|/a(x) : (x, \xi) \in \overline{\Omega} \times \mathbb{R}\}, \end{aligned}$$

and suppose that

$$(12) \quad \gamma_1 < \frac{1}{2}d_1, \quad \gamma_2 < \frac{1}{2}d_1, \quad \frac{\gamma_1}{\frac{1}{2}d_1 - \gamma_1} + \frac{\gamma_2}{\frac{1}{2}d_1 - \gamma_2} < 1.$$

Also, let  $\langle \cdot, \cdot \rangle$  denote the standard inner product in  $L^2(\Omega)$ , and define the weighted Hilbert space  $L_a^2(\Omega)$  to be the space  $L^2(\Omega)$  endowed with the inner product  $\langle \cdot, \cdot \rangle_a$  defined by  $\langle u, v \rangle_a = \langle au, v \rangle$ , for  $u, v \in L^2(\Omega)$ ; let  $\|\cdot\|_a$  denote the corresponding norm. Then we have the following extension of Theorems 1.8 and 1.9 in [1].

**Theorem 5.** *The set  $\mathcal{C}_1^+$  consists of a single unbounded, 1-dimensional,  $C^r$  curve in  $I_1 \times E$ , and this curve can be parametrized by  $\|u\|_a$ . Furthermore, the solutions on  $\mathcal{C}_1^+$  are the only positive solutions of (1).*

*Proof.* The results in Theorem 1.8 and Section 2 of [1] show that  $\mathcal{C}_1^+$  consists of a single unbounded, 1-dimensional,  $C^r$  curve in  $I_1 \times E$ . Note that to apply the results of [1] we divide (1) by  $a$ ; the operator  $a^{-1}L$  is self-adjoint in the space  $L_a^2(\Omega)$ , so the methods of [1] apply.

The parametrization result can now be obtained as follows. Let  $(\lambda^0, u^0)$  be an arbitrary point in  $\mathcal{C}_1^+$ . The proof of Theorem 1.8, using the implicit function theorem, shows that near  $(\lambda^0, u^0)$  the set  $\mathcal{C}_1^+$  has a  $C^r$  parametrization of the form  $s \rightarrow (\lambda(s), u(s)) : N^0 \rightarrow \mathbb{R} \times E$ , where  $N^0 \subset \mathbb{R}$  is a neighbourhood of a point  $s^0$ , with  $(\lambda(s^0), u(s^0)) = (\lambda^0, u^0)$ . Now define  $w(s) = \langle u(s), u(s) \rangle_a = \|u(s)\|_a^2$ . By the implicit function theorem, if  $w'(s^0) \neq 0$  then we can reparametrize the curve, near  $(\lambda^0, u^0)$ , with  $\|u(s)\|_a$  as the new parameter. Differentiating  $w(s)$  with respect to  $s$  yields

$$w'(s) = 2\langle u(s), u'(s) \rangle_a,$$

so  $w'(s^0) = 0$  if and only if either  $u'(s^0) = 0$ , or  $u(s^0)$  and  $u'(s^0) \neq 0$  are orthogonal. Now, using the notation in the proof of Theorem 1.8 in [1], we have

$$\begin{aligned} u(s^0) &\in N(\tilde{T} - \lambda^0 I - \tilde{A}(\lambda^0, u^0)), \\ u'(s^0) &\in N(\tilde{T} - \lambda^0 I - \widetilde{D_u F}(\lambda^0, u^0)), \end{aligned}$$

and hence  $u(s^0)$  and  $u'(s^0) \neq 0$  are orthogonal if and only if  $\|P_D - P_A\|_a = 1$ . However, it follows from (12), together with (1.19) in [1], that  $\|P_D - P_A\|_a < 1$ , so this alternative cannot hold. On the other hand, if  $u'(s^0) = 0$ , then  $N(DG)$  is spanned by  $\mu = \lambda'(s^0) \neq 0$ , which implies that  $Ju^0 = 0$ , and hence  $u^0 = 0$  (see the paragraph containing (1.18) in [1]), which also cannot hold on  $\mathcal{C}_1^+$ . Hence  $w'(s^0) \neq 0$ , which proves that the parametrization result holds near  $(\lambda^0, u^0)$ . Furthermore, since  $(\lambda^0, u^0) \in \mathcal{C}_1^+$  was arbitrary, the implicit function theorem argument allows us to extend the parametrization to hold globally.

To prove the final result, suppose that there exists a component  $\mathcal{E} \subset \mathbb{R} \times E$  of positive solutions of (1) which does not intersect  $\mathcal{C}_1^+$ . By Theorem 1,  $\mathcal{E} \subset I_1 \times E$ . Let  $\beta = \inf\{\|u\|_a : (\lambda, u) \in \mathcal{E}\}$ , and let  $(\lambda_n, u_n) \in \mathcal{E}$ ,  $n = 1, 2, \dots$ , be a sequence with  $\|u_n\|_a \rightarrow \beta$ . It now follows, from (1), (2) and standard a priori estimates for elliptic operators, that the  $W^{2,2}(\Omega)$  norm  $\|u_n\|_{2,2}$  is bounded. By standard bootstrapping arguments we can also show that the sequence  $(\lambda_n, u_n)$  (or a subsequence) converges in  $I_1 \times E$  to some point  $(\lambda_\infty, u_\infty)$  as  $n \rightarrow \infty$ . Now suppose that  $\beta = 0$ . Then the bootstrapping argument shows that  $u_n \rightarrow 0$  in  $E$ , and hence we must have  $\lambda_\infty = \lambda_1$ . But this cannot happen since the theory of simple bifurcation at  $(\lambda_1, 0)$  ensures that  $\mathcal{C}_1^+$  contains all the positive solutions of (1) in a neighbourhood of  $(\lambda_1, 0)$ , which would imply that  $\mathcal{C}_1^+ \cap \mathcal{E} \neq \emptyset$ , contrary to our assumption. Thus we must have  $\|u_\infty\|_a = \beta > 0$ , i.e.,  $u_\infty \neq 0$ . But now the above parametrization argument applies to the set  $\mathcal{E}$  near  $(\lambda_\infty, u_\infty)$  and shows that, locally,  $\mathcal{E}$  consists of a smooth curve parametrized by  $\|u\|_a$ . Thus there must be elements  $(\lambda, u) \in \mathcal{E}$  arbitrarily close to  $(\lambda_\infty, u_\infty)$  with  $\|u\|_a < \beta$ . But this contradicts the definition of  $\beta$ , so we conclude that  $\mathcal{E}$  cannot exist. This completes the proof of the theorem.  $\square$

Finally, we note that under appropriate hypotheses the above results for (1) could be extended to allow  $g$  to depend on  $\lambda$  and on  $\nabla u$  — we will not consider this further. However, we briefly consider the case of a slight generalization of the equation (4) considered in [4] and [5]. We allow the function  $f$  to depend on  $x$ ,

and write it in the form  $f(x, \xi) = -a(x)\xi - g(x, \xi)\xi$ , where  $a(x) = -f_\xi(x, 0)$  and  $g(x, 0) \equiv 0$ , and hence the equation we consider is

$$(13) \quad Lu = \lambda(a + g(\cdot, u))u.$$

We assume that  $L$ ,  $a$  and  $g$  satisfy the same hypotheses as before, together with the conditions: (i)  $\gamma^{--} < 1$ ; (ii)  $\lambda_1 > 0$ . Using similar arguments to those above we obtain the following result.

**Theorem 6.** *For some  $\mu \in (0, 1)$ , let*

$$I_1 = \left[ \frac{\lambda_1}{1 + \gamma^{++}}, \frac{\lambda_1}{1 - \gamma^{--}} \right], \quad J_\mu = \left[ \frac{\lambda_1}{1 + \mu\gamma^+}, \frac{\lambda_1}{1 - \mu\gamma^-} \right].$$

*Under the above hypotheses, any positive solution of (13) lies in the set  $I_1 \times E$ . The continuum  $\mathcal{C}_1^+$  lies in  $I_1 \times E$ , is unbounded and, if the constant  $\kappa$  in (5) is sufficiently small, then  $\mathcal{C}_1^+$  oscillates infinitely often over the interval  $J_\mu$ .*

The above genericity results also hold in this case and, with some minor changes, the small  $g$  results hold.

The above assumptions on the size of the oscillations of  $g$  are weaker than the conditions imposed in [5] (see (3.15) in [5]) and are rather more explicit (apart from the unknown value of  $\kappa$ , which also appears in [5]). The figure on p. 1228 of [5] illustrates the results in this case (in the case of our parametrization result, the  $y$ -axis would represent  $\|u\|_a$  rather than  $|u|_0$ ). Our oscillation interval  $J_\mu$  is expressed rather more explicitly than in [5].

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