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OSCILLATING GLOBAL CONTINUA OF POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS

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ABSTRACT. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with C^2 boundary $\partial\Omega$, and consider the semilinear elliptic boundary value problem

$$\begin{aligned} Lu &= \lambda a u + g(\cdot, u)u, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

where L is a uniformly elliptic operator on $\overline{\Omega}$, $a \in C^0(\overline{\Omega})$, a is strictly positive in $\overline{\Omega}$, and the function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable, with $g(x, 0) = 0$, $x \in \overline{\Omega}$. A well known result of Rabinowitz shows that an unbounded continuum of positive solutions of this problem bifurcates from the principal eigenvalue λ_1 of the linear problem. We show that under certain oscillation conditions on the nonlinearity g, this continuum oscillates about λ_1, in a certain sense, as it approaches infinity. Hence, in particular, the equation has infinitely many positive solutions for each λ in an open interval containing λ_1.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with C^2 boundary $\partial\Omega$, and consider the semilinear elliptic boundary value problem

$$\begin{aligned} Lu := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + qu &= \lambda a u + g(\cdot, u)u, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $a_{ij} \in C^1(\overline{\Omega})$, $i, j = 1, \ldots, n$, $q \in C^0(\overline{\Omega})$, L is uniformly elliptic in $\overline{\Omega}$, $a \in C^0(\overline{\Omega})$, a is strictly positive in $\overline{\Omega}$. The function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable, with $g(x, 0) = 0$, $x \in \overline{\Omega}$, and there are non-negative constants γ^{--}, γ^{++} such that

$$-\gamma^{--} \leq \frac{g(x,\xi)}{a(x)} \leq \gamma^{++}, \quad (x, \xi) \in \overline{\Omega} \times \mathbb{R}. \tag{2}$$

For any integer $r \geq 0$, $C^r(\overline{\Omega})$ will denote the Banach space of real-valued, continuous functions on $\overline{\Omega}$, having continuous derivatives up to order r on $\overline{\Omega}$, and $C^{r,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$, will denote the set of functions in $C^r(\overline{\Omega})$ whose derivatives of order r are Hölder continuous with exponent α; we let $|\cdot|_r$, $|\cdot|_{r,\alpha}$ denote the usual norms on these spaces. From now on we suppose that $\alpha \in (0, 1)$ is fixed and let E be the subspace of $C^{1,\alpha}(\overline{\Omega})$ consisting of those functions which are zero on the boundary

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$\partial\Omega$. A *solution* of (1) is a pair $(\lambda, u) \in \mathbb{R} \times C^2(\overline{\Omega})$ satisfying (1) (similarly for other equations below). Thus we may consider the structure of the set of solutions in the space $\mathbb{R} \times E$. A *positive* (*negative*) solution of (1) is a solution (λ, u) with $u > 0$ ($u < 0$) in Ω. All the results below regarding positive solutions have analogues for negative solutions, but we will not discuss these further.

Let λ_1 be the principal eigenvalue of the linear problem obtained from (1) by putting $g \equiv 0$, and let ϕ be the corresponding eigenfunction with the properties

$$\phi(x) > 0, \quad x \in \Omega, \quad |\phi|_0 = 1.$$

Under the above hypotheses it is well known (see [2] and [7]) that a 'simple bifurcation' takes place at $\lambda = \lambda_1$, and a single, unbounded continuum $\mathcal{C}_1^+ \subset \mathbb{R} \times E$ of positive solutions of (1) bifurcates from λ_1 (that is, the closure of $\mathcal{C}_1^+$ is $\{(\lambda_1, 0)\} \cup \mathcal{C}_1^+$). We have the following preliminary result regarding $\mathcal{C}_1^+$. Let $I_1 = [\lambda_1 - \gamma^{++}, \lambda_1 + \gamma^{--}]$.

Theorem 1. *Under the above hypotheses, any positive solution of* (1) *lies in the set* $I_1 \times E$. *In particular,*

$$\mathcal{C}_1^+ \subset I_1 \times E.$$

Furthermore, for any number $\xi > 0$ *there exists a solution* $(\lambda, u) \in \mathcal{C}_1^+$ *with* $|u|_0 = \xi$.

Proof. Suppose that (λ, u) is a positive solution of (1). Multiplying (1) by ϕ, integrating by parts and using the definition of ϕ yields

$$(\lambda_1 - \lambda) \int_\Omega au\phi = \int_\Omega g(\cdot, u)u\phi. \tag{3}$$

Hence,

$$(\lambda_1 - \lambda) \int_\Omega au\phi \le \gamma^{++} \int_\Omega au\phi \implies \lambda \ge \lambda_1 - \gamma^{++}.$$

We obtain $\lambda \le \lambda_1 + \gamma^{--}$ similarly.

Now suppose that (λ_k, u_k), $k = 1, 2, \ldots$, is a sequence of positive solutions of (1) which is bounded in $I_1 \times C^0(\overline{\Omega})$. Then the right hand side of (1) is bounded in L^∞, so by elliptic regularity results the sequence u_k is bounded in $W^{2,p}(\Omega)$ for all $p > 1$, and hence by the Sobolev embedding theorem this sequence is bounded in E. Thus any set of positive solutions of (1) which is bounded in $I_1 \times C^0(\overline{\Omega})$ is bounded in $I_1 \times E$. But we know that the continuum $\mathcal{C}_1^+$ is unbounded in $I_1 \times E$, so it must be unbounded in $I_1 \times C^0(\overline{\Omega})$. Also, the mapping $(\lambda, u) \to |u|_0$ from $\mathcal{C}_1^+ \subset I_1 \times E$ to $\mathbb{R}$ is continuous so, since $\mathcal{C}_1^+$ is connected and $(\lambda_1, 0)$ belongs to the closure of $\mathcal{C}_1^+$, this image must be the interval $(0, \infty)$. Hence the second part of the theorem follows. $\square$

Theorem 1 shows that $\mathcal{C}_1^+$ goes to infinity in the cylinder $I_1 \times E$. Under a further oscillation type condition on g we will show that $\mathcal{C}_1^+$ 'oscillates' in this cylinder, as it approaches infinity, over an open λ interval containing λ_1, and the amplitude of these oscillations is bounded away from zero (this will be made precise in Theorem 2 below). This type of behaviour is illustrated in Figure 1 in [4], or in the bifurcation diagram in [5] for the case where $\mathcal{C}_1^+$ is a smooth curve (except that here the oscillations of $\mathcal{C}_1^+$ are above λ_1). Here, in general, the structure of $\mathcal{C}_1^+$ could be more complicated.

We now briefly discuss earlier related results. The paper [10] considers the nonlinear Sturm-Liouville problem

$$u'' + \lambda r(x)u + p(x)g(u) = h(x), \quad x \in (0,1),$$

with self-adjoint boundary conditions, and assumes that $g(\xi)/\xi \to 0$, as $\xi \to \infty$, and g satisfies an oscillation condition (e.g., g is periodic). It is shown that there is a connected set of positive solutions $\mathcal{C}_1^+$ which oscillates infinitely often above λ_1 so, in particular, the equation has infinitely many positive solutions in $\mathcal{C}_1^+$ with $\lambda = \lambda_1$. However, the amplitude of the oscillations tends to zero (in fact, the set $\mathcal{C}_1^+$ bifurcates from '(λ_1, ∞)') so it does not oscillate over any open interval. An elliptic analogue, in a bounded domain in $\mathbb{R}^n$, of the above Sturm-Liouville problem is considered in [11] (the term u'' is replaced with Δu). By a slightly different technique it is again shown that the equation has infinitely many positive solutions in $\mathcal{C}_1^+$ with $\lambda = \lambda_1$. Again it is assumed that $g(\xi)/\xi \to 0$, as $\xi \to \infty$, and satisfies an oscillation condition, and again the amplitude of the oscillations of $\mathcal{C}_1^+$ may tend to zero. In [6] the equation

$$Lu + \lambda f(u) = 0 \tag{4}$$

is considered. It is assumed that f is asymptotically linear and satisfies certain oscillation conditions, and it is shown that a solution continuum bifurcates from (λ_1, ∞) and this continuum oscillates infinitely often (here, λ_1 need not be the principal eigenvalue, and the solutions need not be positive). Thus the equation has infinitely many solutions with $\lambda = \lambda_1$ but, since the continuum bifurcates from (λ_1, ∞), the amplitude of the oscillations again tends to zero and the continuum does not oscillate over any open λ interval.

In [5] the equation (4) is also considered. Here, f is not asymptotically linear, but it is strictly increasing and satisfies an oscillation condition (the condition we impose on g, described below, is based on this condition). In essence, $f(\xi)/\xi$ oscillates and the amplitude of the oscillations is bounded away from zero, while the duration (or 'period') of the oscillations grows at a certain rate as ξ increases. In particular, this condition means that $f(\xi)/\xi$ cannot be periodic. For each fixed λ in a certain interval, sequences of positive solutions are obtained by sub- and super-solution methods. However, these solutions may not lie on the bifurcating set $\mathcal{C}_1^+$, so this does not show that $\mathcal{C}_1^+$ oscillates. It is then shown in [4] that for a class of planar domains $\Omega \subset \mathbb{R}^2$ with certain symmetry properties the solutions found in [5] do lie on $\mathcal{C}_1^+$, and this set is a smooth curve parametrized by $|u|_0$. Illustrative bifurcation diagrams are given in [4] and [5].

The oscillation condition on g will now be described. We suppose that there are non-negative constants γ^-, γ^+, a constant $\kappa \in (0,1)$, and an increasing sequence of positive numbers ξ_j, $j = 1, 2, \dots$, such that for each j, $\xi_j < \kappa \xi_{j+1}$, and for all $x \in \overline{\Omega}$,

$$\begin{aligned} \gamma^+ &\le g(x,\xi)/a(x), \quad \xi \in (\kappa\xi_{2j-1}, \xi_{2j-1}), \\ -\gamma^- &\ge g(x,\xi)/a(x), \quad \xi \in (\kappa\xi_{2j}, \xi_{2j}). \end{aligned} \tag{5}$$

With this condition we have the following result.

Theorem 2. *Choose a number $\mu \in (0,1)$. If g satisfies (5), with the constant κ sufficiently small (depending on μ), then any positive solution (λ, u) of (1) with $|u|_0 = \xi_{2j-1}$, $j \ge 1$ (respectively $|u|_0 = \xi_{2j}$) has $\lambda < \lambda_1 - \mu\gamma^+$ (respectively $\lambda > \lambda_1 + \mu\gamma^-$).*

Proof. We begin by considering the problem

$$\begin{aligned} Lu + cu &= 0, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{6}$$

for a general function $c \in C^0(\overline{\Omega})$ with $|c|_0 \le \gamma$, where

$$\gamma = (1 + |a|_0)(|\lambda_1| + \max\{\gamma^{++}, \gamma^{--}\}).$$

Note that by Theorem 1 and the assumptions on g, equation (1) has the form of (6) with $|c|_0 \le \gamma$. For any $\delta > 0$, let

$$\Omega(\delta) = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\},$$

and let $|\cdot|_{0;\overline{\Omega}(\delta)}$ denote the sup norm on the closure, $\overline{\Omega}(\delta)$, of this set.

Lemma 3. *There exists $\delta_0 > 0$ and a constant C_0 (depending on γ but not on c) such that if $|c|_0 \le \gamma$ and $u > 0$ satisfies* (6)*, then $|u|_0 \le C_0 |u|_{0;\overline{\Omega}(\delta_0)}$.*

Remark. The constants in the lemma and in the proof also depend on n, Ω and L, but we regard these as fixed so this dependence is unimportant here.

Proof. Suppose that the result is false. Then for each integer $k > 0$ there exist functions c_k and u_k satisfying the conditions of the lemma but for which, if $u_k(x_k) = |u_k|_0$, we have

$$u_k(x_k) > r(k)|u_k|_{0;\overline{\Omega}(1/k)},$$

with $r(k) \to \infty$ as $k \to \infty$. By considering a subsequence if necessary, we may suppose that $x_k \to y \in \partial\Omega$ as $k \to \infty$. Now, by Theorem 9.26 of [3] we may choose a fixed $R > 0$ and a constant C_1 (which depends only on γ and R) such that for sufficiently large k we have

$$u_k(x_k) \le C_1 R^{-n} \int_{\Omega \cap B_{2R}(y)} u_k,$$

where $B_{2R}(y)$ is the ball centred at y, of radius $2R$. Now let $E_1 = \Omega(1/k) \cap B_{2R}(y)$, $E_2 = (\Omega \setminus \Omega(1/k)) \cap B_{2R}(y)$. Then there is a constant $C_2 > 0$ such that

$$\int_{E_1} u_k \le C_2 r(k)^{-1} R^n u_k(x_k), \quad \int_{E_2} u_k \le C_2 k^{-1} R^{n-1} u_k(x_k)$$

(the second inequality holds, if R is chosen to be sufficiently small, since $\partial\Omega$ is smooth). Hence

$$u_k(x_k) \le C_1 C_2 (r(k)^{-1} + k^{-1} R^{-1}) u_k(x_k),$$

which for sufficiently large k is a contradiction. This proves the lemma. □

Now let $\rho = \max_{\overline{\Omega}} a / \min_{\overline{\Omega}} a$, and choose $\epsilon > 0$ sufficiently small that

$$\gamma^{+}(1 - \rho\epsilon) - \gamma^{--}\rho\epsilon > \mu\gamma^{+}. \tag{7}$$

For any $\delta > 0$, let

$$K(\delta) = \{x \in \Omega : \phi(x) \ge \delta\},$$

and choose $\delta_1 > 0$ sufficiently small that $\Omega(\delta_0) \subset K(\delta_1)$, and

$$|\Omega \setminus K(\delta_1)| \le \epsilon |K(\delta_1)|, \tag{8}$$

where $|\cdot|$ denotes the Lebesgue measure of a set here (this is possible since $\phi > 0$ on Ω). We denote the set $K(\delta_1)$ by K. By a slight extension of Corollary 9.25 in [3], there exists a constant $C_3 > 0$ (depending on K and γ) such that for any positive solution (λ, u) of (1),

$$\min_K u \geq C_3 \max_K u \tag{9}$$

(recall that (1) has the form of (6) with $|c|_0 \leq \gamma$). Now suppose that (5) holds with $\kappa \leq C_0^{-1} C_3$, and suppose that (λ, u) is a positive solution of (1) with $|u|_0 = \xi_{2j-1}$, for some integer $j \geq 1$. Then by Lemma 3 and (9),

$$\min_K u \geq C_3 |u|_{0;K} \geq C_3 |u|_{0;\overline{\Omega}(\delta_0)} \geq C_0^{-1} C_3 \xi_{2j-1} \geq \kappa \xi_{2j-1},$$

and hence

$$g(x, u(x))/a(x) \geq \gamma^+, \quad x \in K. \tag{10}$$

Now let

$$E(u) = \{x \in \Omega : u(x) < \min_K u\} \subset \Omega \setminus K.$$

Clearly, the inequality (10) also holds for $x \in \Omega \setminus E(u)$. On the other hand, for $x \in E(u)$ we have $u(x) \leq \min_K u$ and $\phi(x) \leq \min_K \phi$, hence, by (8),

$$\begin{aligned}\int_{E(u)} au\phi &\leq |E(u)| \sup_{E(u)} (au\phi) \leq \epsilon |K| \max_{\overline{\Omega}} a \min_K u \min_K \phi \\ &\leq \epsilon\rho \int_K au\phi \leq \epsilon\rho \int_\Omega au\phi.\end{aligned} \tag{11}$$

Hence from (3), (7), (10) and (11) we have

$$\begin{aligned}(\lambda_1 - \lambda) \int_\Omega au\phi &= \int_\Omega g(\cdot, u) u\phi \\ &\geq \int_{\Omega \setminus E(u)} g(\cdot, u) u\phi - \gamma^{--} \int_{E(u)} au\phi \\ &\geq \gamma^+ \int_{\Omega \setminus E(u)} au\phi - \gamma^{--} \rho\epsilon \int_\Omega au\phi \\ &\geq \gamma^+ (1 - \rho\epsilon) \int_\Omega au\phi - \gamma^{--} \rho\epsilon \int_\Omega au\phi \\ &> \mu\gamma^+ \int_\Omega au\phi,\end{aligned}$$

and so $\lambda < \lambda_1 - \mu\gamma^+$. Similarly, if (λ, u) is a positive solution of (1) with $|u|_0 = \xi_{2j}$, we can show that $\lambda > \lambda_1 + \mu\gamma^-$. This completes the proof of Theorem 2. □

We now have the following simple corollary of Theorems 1 and 2.

Corollary 4. *Suppose that the hypotheses of Theorem 2 hold. Then the continuum $\mathcal{C}_1^+$ lies in $I_1 \times E$, is unbounded and oscillates infinitely often over the interval $J_\mu = [\lambda_1 - \mu\gamma^+, \lambda_1 + \mu\gamma^-]$ as it approaches infinity, in the sense that for each λ in J_μ there are infinitely many solutions $(\lambda, u) \in \mathcal{C}_1^+$.*

Proof. By Theorem 1, for each integer $j \geq 2$ we can choose a solution $(\lambda_j, u_j) \in \mathcal{C}_1^+$ with $|u_j|_0 = \xi_j$. Since $\mathcal{C}_1^+$ connects (λ_j, u_j) with $(\lambda_1, 0)$ it follows from Theorem 2 that for each $\lambda \in J_\mu$ there must be a solution $(\lambda, u) \in \mathcal{C}_1^+$ with $\xi_{j-1} \leq |u|_0 \leq \xi_j$. Hence there must be infinitely many solutions $(\lambda, u) \in \mathcal{C}_1^+$. □

The 'simple bifurcation' results in [2] show that near $(\lambda_1, 0)$ in $\mathbb{R} \times E$ the set $\mathcal{C}_1^+$ is a single continuous curve, but the general topological results in [7] do not imply that the set $\mathcal{C}_1^+$ is a curve globally. We now briefly discuss two simple situations in which it can be shown that the entire set $\mathcal{C}_1^+$ is a single smooth, 1-dimensional curve: (i) generic equations of the form (1); (ii) sufficiently small functions g. In each of these cases the above results hold, so, in particular, the curve oscillates on either side of λ_1.

Generic equations

Suppose that $\partial\Omega$ is $C^{3,\alpha}$, the coefficients a_{ij}, q, $a \in C^{3,\alpha}(\overline{\Omega})$, and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is $C^{r,\alpha}$, for some integer $r \geq 3$. Then Theorem 6 of [9] shows that for a generic set of coefficients $\{a_{ij} : i,\, j = 1, \dots, n\}$ (in an appropriate open subset of the Cartesian product space $C^{3,\alpha}(\overline{\Omega})^{n^2}$) the set $\mathcal{C}_1^+$ consists of a single unbounded, 1-dimensional, C^r curve in $I_1 \times E$. We note that: (i) the precise meaning of the term 'generic', in this context, is given in [9]; (ii) the elliptic equation (1) above is in divergence form, whereas the equation (1.4) considered in [9] is not, so a slight adaptation of the proof in [9] is required here.

Probably this result also holds for generic domains Ω in $\mathbb{R}^n$, $n \geq 2$. If we choose a fixed g it can be shown that for generic Ω the set $\mathcal{C}_1^+$ is a smooth curve. Unfortunately, in the proof of the oscillation results the choice of g depends on Ω (via the constant κ, which depends on Ω), which leads to a rather circular argument. Possibly a sufficiently small perturbation of the domain would work, but it is not clear how the constant κ depends on such a perturbation of Ω. This difficulty does not arise when considering generic coefficients $\{a_{ij}\}$ since κ depends only on bounds for the $\{a_{ij}\}$, so κ can be chosen independently of sufficiently small perturbations of the $\{a_{ij}\}$.

The results of [8] can also be used to show that for generic functions g the 'vertical' turning points on the curve $\mathcal{C}_1^+$ are non-degenerate, i.e., quadratic — see [8] for more details.

Small g

Suppose that $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is C^r, for some integer $r \geq 2$. Let λ_2 be the second eigenvalue of the linearization of (1) and let $d_1 = \lambda_2 - \lambda_1 > 0$. Define the numbers

$$\gamma_1 = \max\{\gamma^{++},\, \gamma^{--}\},$$
$$\gamma_2 = \sup\{|g_\xi(x,\xi)\xi + g(x,\xi)|/a(x) : (x,\xi) \in \overline{\Omega} \times \mathbb{R}\},$$

and suppose that

$$\gamma_1 < \tfrac{1}{2}d_1,\ \gamma_2 < \tfrac{1}{2}d_1, \quad \frac{\gamma_1}{\frac{1}{2}d_1 - \gamma_1} + \frac{\gamma_2}{\frac{1}{2}d_1 - \gamma_2} < 1. \tag{12}$$

Also, let $\langle \cdot, \cdot \rangle$ denote the standard inner product in $L^2(\Omega)$, and define the weighted Hilbert space $L^2_a(\Omega)$ to be the space $L^2(\Omega)$ endowed with the inner product $\langle \cdot, \cdot \rangle_a$ defined by $\langle u, v \rangle_a = \langle au, v \rangle$, for u, $v \in L^2(\Omega)$; let $\| \cdot \|_a$ denote the corresponding norm. Then we have the following extension of Theorems 1.8 and 1.9 in [1].

Theorem 5. *The set $\mathcal{C}_1^+$ consists of a single unbounded, 1-dimensional, C^r curve in $I_1 \times E$, and this curve can be parametrized by $\|u\|_a$. Furthermore, the solutions on $\mathcal{C}_1^+$ are the only positive solutions of* (1).

Proof. The results in Theorem 1.8 and Section 2 of [1] show that $\mathcal{C}_1^+$ consists of a single unbounded, 1-dimensional, C^r curve in $I_1 \times E$. Note that to apply the results of [1] we divide (1) by a; the operator $a^{-1}L$ is self-adjoint in the space $L_a^2(\Omega)$, so the methods of [1] apply.

The parametrization result can now be obtained as follows. Let (λ^0, u^0) be an arbitrary point in $\mathcal{C}_1^+$. The proof of Theorem 1.8, using the implicit function theorem, shows that near (λ^0, u^0) the set $\mathcal{C}_1^+$ has a C^r parametrization of the form $s \to (\lambda(s), u(s)) : N^0 \to \mathbb{R} \times E$, where $N^0 \subset \mathbb{R}$ is a neighbourhood of a point s^0, with $(\lambda(s^0), u(s^0)) = (\lambda^0, u^0)$. Now define $w(s) = \langle u(s), u(s)\rangle_a = \|u(s)\|_a^2$. By the implicit function theorem, if $w'(s^0) \neq 0$ then we can reparametrize the curve, near (λ^0, u^0), with $\|u(s)\|_a$ as the new parameter. Differentiating $w(s)$ with respect to s yields

$$w'(s) = 2\langle u(s), u'(s)\rangle_a,$$

so $w'(s^0) = 0$ if and only if either $u'(s^0) = 0$, or $u(s^0)$ and $u'(s^0) \neq 0$ are orthogonal. Now, using the notation in the proof of Theorem 1.8 in [1], we have

$$u(s^0) \in N(\widetilde{T} - \lambda^0 I - \widetilde{A}(\lambda^0, u^0)),$$
$$u'(s^0) \in N(\widetilde{T} - \lambda^0 I - \widetilde{D_u F}(\lambda^0, u^0)),$$

and hence $u(s^0)$ and $u'(s^0) \neq 0$ are orthogonal if and only if $\|P_D - P_A\|_a = 1$. However, it follows from (12), together with (1.19) in [1], that $\|P_D - P_A\|_a < 1$, so this alternative cannot hold. On the other hand, if $u'(s^0) = 0$, then $N(DG)$ is spanned by $\mu = \lambda'(s^0) \neq 0$, which implies that $Ju^0 = 0$, and hence $u^0 = 0$ (see the paragraph containing (1.18) in [1]), which also cannot hold on $\mathcal{C}_1^+$. Hence $w'(s^0) \neq 0$, which proves that the parametrization result holds near (λ^0, u^0). Furthermore, since $(\lambda^0, u^0) \in \mathcal{C}_1^+$ was arbitrary, the implicit function theorem argument allows us to extend the parametrization to hold globally.

To prove the final result, suppose that there exists a component $\mathcal{E} \subset \mathbb{R} \times E$ of positive solutions of (1) which does not intersect $\mathcal{C}_1^+$. By Theorem 1, $\mathcal{E} \subset I_1 \times E$. Let $\beta = \inf\{\|u\|_a : (\lambda, u) \in \mathcal{E}\}$, and let $(\lambda_n, u_n) \in \mathcal{E}$, $n = 1, 2, \dots$, be a sequence with $\|u_n\|_a \to \beta$. It now follows, from (1), (2) and standard a priori estimates for elliptic operators, that the $W^{2,2}(\Omega)$ norm $\|u_n\|_{2,2}$ is bounded. By standard bootstrapping arguments we can also show that the sequence (λ_n, u_n) (or a subsequence) converges in $I_1 \times E$ to some point $(\lambda_\infty, u_\infty)$ as $n \to \infty$. Now suppose that $\beta = 0$. Then the bootstrapping argument shows that $u_n \to 0$ in E, and hence we must have $\lambda_\infty = \lambda_1$. But this cannot happen since the theory of simple bifurcation at $(\lambda_1, 0)$ ensures that $\mathcal{C}_1^+$ contains all the positive solutions of (1) in a neighbourhood of $(\lambda_1, 0)$, which would imply that $\mathcal{C}_1^+ \cap \mathcal{E} \neq \emptyset$, contrary to our assumption. Thus we must have $\|u_\infty\|_a = \beta > 0$, i.e., $u_\infty \neq 0$. But now the above parametrization argument applies to the set $\mathcal{E}$ near $(\lambda_\infty, u_\infty)$ and shows that, locally, $\mathcal{E}$ consists of a smooth curve parametrized by $\|u\|_a$. Thus there must be elements $(\lambda, u) \in \mathcal{E}$ arbitrarily close to $(\lambda_\infty, u_\infty)$ with $\|u\|_a < \beta$. But this contradicts the definition of β, so we conclude that $\mathcal{E}$ cannot exist. This completes the proof of the theorem. □

Finally, we note that under appropriate hypotheses the above results for (1) could be extended to allow g to depend on λ and on ∇u — we will not consider this further. However, we briefly consider the case of a slight generalization of the equation (4) considered in [4] and [5]. We allow the function f to depend on x, and write it in the form $f(x, \xi) = -a(x)\xi - g(x, \xi)\xi$, where $a(x) = -f_\xi(x, 0)$ and $g(x, 0) \equiv 0$, and hence the equation we consider is

$$Lu = \lambda(a + g(\cdot, u))u. \tag{13}$$

We assume that L, a and g satisfy the same hypotheses as before, together with the conditions: (i) $\gamma^{--} < 1$; (ii) $\lambda_1 > 0$. Using similar arguments to those above we obtain the following result.

Theorem 6. *For some $\mu \in (0, 1)$, let*

$$I_1 = \left[\frac{\lambda_1}{1 + \gamma^{++}}, \frac{\lambda_1}{1 - \gamma^{--}}\right], \quad J_\mu = \left[\frac{\lambda_1}{1 + \mu\gamma^{+}}, \frac{\lambda_1}{1 - \mu\gamma^{-}}\right].$$

Under the above hypotheses, any positive solution of (13) *lies in the set $I_1 \times E$. The continuum $\mathcal{C}_1^+$ lies in $I_1 \times E$, is unbounded and, if the constant κ in* (5) *is sufficiently small, then $\mathcal{C}_1^+$ oscillates infinitely often over the interval J_μ.*

The above genericity results also hold in this case and, with some minor changes, the small g results hold.

The above assumptions on the size of the oscillations of g are weaker than the conditions imposed in [5] (see (3.15) in [5]) and are rather more explicit (apart from the unknown value of κ, which also appears in [5]). The figure on p. 1228 of [5] illustrates the results in this case (in the case of our parametrization result, the y-axis would represent $\|u\|_a$ rather than $|u|_0$). Our oscillation interval J_μ is expressed rather more explicitly than in [5].

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