

**THE ROGERS-RAMANUJAN IDENTITIES,  
 THE FINITE GENERAL LINEAR GROUPS,  
 AND THE HALL-LITTLEWOOD POLYNOMIALS**

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ABSTRACT. We connect Gordon’s generalization of the Rogers-Ramanujan identities with the Hall-Littlewood polynomials and with generating functions which arise in a probabilistic setting in the finite general linear groups. This yields a Rogers-Ramanujan type product formula for the  $n \rightarrow \infty$  probability that an element of  $GL(n, q)$  or  $Mat(n, q)$  is semisimple.

1. BACKGROUND AND NOTATION

The Rogers-Ramanujan identities are among the most remarkable partition identities in number theory and combinatorics. This paper will be concerned with the following generalization of the Rogers-Ramanujan identities, due to Gordon. Let  $(x)_n$  denote  $(1-x)(1-x^2)\cdots(1-x^n)$ .

**Theorem 1** ([A, page 111]). *For  $1 \leq i \leq k, k \geq 2$ , and complex  $x$  with  $|x| < 1$ ,*

$$\sum_{n_1, \dots, n_{k-1} \geq 0} \frac{x^{N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}}}{(x)_{n_1} \cdots (x)_{n_{k-1}}} = \prod_{\substack{r=1 \\ r \neq 0, \pm i \pmod{2k+1}}}^{\infty} \frac{1}{1-x^r}$$

where  $N_j = n_j + \dots + n_{k-1}$ .

Gordon’s generalization of the Rogers-Ramanujan identities has been widely studied and appears in many places in mathematics and physics. Andrews [A] discusses combinatorial aspects of these identities. In an important series of papers, Lepowsky and Wilson [LW1], [LW2], [LW3] connect the Gordon identities with affine Lie algebras and structures that they called  $Z$ -algebras (later interpreted as parafermion algebras in conformal field theory). Meurman and Primc [MP] solve a problem left open in [LW3], proving the independence of a  $Z$ -algebra basis and obtaining a  $Z$ -algebra proof of the Gordon identities. Feigin and Frenkel [FF] interpret the Gordon identities as a character formula for the Virasoro algebra. Andrews, Baxter, and Forrester [ABF] and Warnaar [W] relate the Gordon identities with statistical mechanics. For some number theoretic connections see the conference proceedings [AABRR].

We use the following standard notation from the theory of partitions. Call  $\lambda = (\lambda_1, \lambda_2, \dots)$  a partition of  $n = |\lambda|$  if  $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$  where the  $\lambda_i$  are

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a sequence of positive integers stabilizing to 0 such that  $\sum_i \lambda_i = n$ . The  $\lambda_i$  are referred to as the parts of  $\lambda$ . Let  $m_i(\lambda)$  be the number of parts of  $\lambda$  of size  $i$ , and set  $\lambda'_i = m_i(\lambda) + m_{i+1}(\lambda) + \dots$ . Define  $n(\lambda)$  by  $\sum_{i \geq 1} (i-1)\lambda_i$ . Let  $[u^n]f(u)$  denote the coefficient of  $u^n$  in a power series  $f(u)$ .

## 2. MAIN RESULTS

To begin we recall the Hall-Littlewood polynomials associated to a partition  $\lambda$  (page 208 of [Mac]). Let  $n$  be any integer such that  $n \geq \lambda_1$ . The permutation  $w \in S_n$  acts on the variables  $x_1, \dots, x_n$  by sending  $x_i$  to  $x_{w(i)}$ . Letting  $t$  be a complex number, the Hall-Littlewood polynomials are defined as

$$P_\lambda(x_1, \dots, x_n; t) = \frac{1}{\prod_{i \geq 0} \prod_{r=1}^{m_i(\lambda)} \frac{1-t^r}{1-t}} \sum_{w \in S_n} w(x_1^{\lambda_1} \dots x_n^{\lambda_n} \prod_{i < j} \frac{x_i - tx_j}{x_i - x_j}).$$

At first glance it is not obvious that these are polynomials, but the denominators cancel out after the symmetrization. The Hall-Littlewood polynomials interpolate between the Schur functions ( $t = 0$ ) and the monomial symmetric functions ( $t = 1$ ).

**Theorem 2.** For  $q > 1$  and an integer  $k \geq 2$ ,

$$\sum_{\lambda: \lambda_1 < k} \frac{P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \dots; \frac{1}{q})}{q^{n(\lambda)}} = \prod_{\substack{r=1 \\ r \neq 0, \pm k \pmod{2k+1}}}^{\infty} \left( \frac{1}{1 - \frac{1}{q^r}} \right),$$

$$\sum_{\lambda: \lambda_1 < k} \frac{P_\lambda(\frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \frac{1}{q^5}, \dots; \frac{1}{q})}{q^{n(\lambda)}} = \prod_{\substack{r=1 \\ r \neq 0, \pm 1 \pmod{2k+1}}}^{\infty} \left( \frac{1}{1 - \frac{1}{q^r}} \right).$$

*Proof.* Macdonald's principal specialization formula (page 337 of [Mac]) states that

$$P_\lambda\left(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \dots; \frac{1}{q}\right) = \frac{1}{q^{|\lambda|+n(\lambda)}} \prod_i \frac{1}{\left(\frac{1}{q}\right)_{m_i(\lambda)}}.$$

Combining this with the elementary fact that  $n(\lambda) = \sum_i \binom{\lambda'_i}{2}$  shows that

$$\frac{P_\lambda\left(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \dots; \frac{1}{q}\right)}{q^{n(\lambda)}} = \frac{1}{q^{\sum_i \binom{\lambda'_i}{2}} \prod_i \left(\frac{1}{q}\right)_{m_i(\lambda)}}.$$

The first equality of the theorem follows by applying Theorem 1 with  $i = k$ ,  $x = \frac{1}{q}$ ,  $n_j = \lambda_j$  and  $N_j = \lambda'_j$ . For the second equality, observe that for  $u$  complex,

$$\begin{aligned} \frac{P_\lambda\left(\frac{u}{q}, \frac{u}{q^2}, \frac{u}{q^3}, \frac{u}{q^4}, \dots; \frac{1}{q}\right)}{q^{n(\lambda)}} &= u^{|\lambda|} \frac{P_\lambda\left(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \dots; \frac{1}{q}\right)}{q^{n(\lambda)}} \\ &= \frac{u^{|\lambda|}}{q^{\sum_i \binom{\lambda'_i}{2}} \prod_i \left(\frac{1}{q}\right)_{m_i(\lambda)}}. \end{aligned}$$

Now set  $u = \frac{1}{q}$  and apply Theorem 1 with  $i = 1$ ,  $x = \frac{1}{q}$ ,  $n_j = \lambda_j$  and  $N_j = \lambda'_j$ .  $\square$

*Remark.* Stembridge [Ste] used properties of the Hall-Littlewood polynomials as a tool in giving proofs of the Rogers-Ramanujan identities. The statement of Theorem 2 gives a direct connection.

Recall that the conjugacy classes of  $GL(n, q)$  are parameterized by rational canonical form (Chapter 6 of Herstein [H]). This form corresponds to the following combinatorial data. To each monic non-constant irreducible polynomial  $\phi$  over a field of  $q$  elements, associate a partition  $\lambda_\phi$  of some non-negative integer  $|\lambda_\phi|$ . Let  $\deg(\phi)$  denote the degree of  $\phi$ . This data represents a conjugacy class of  $GL(n, q)$  if and only if  $|\lambda_z| = 0$  and  $\sum_\phi |\lambda_\phi| \deg(\phi) = n$ .

**Definition.** For  $\alpha \in GL(n, q)$  and  $\phi$  a monic, irreducible polynomial over  $F_q$ , a field of  $q$  elements, define  $\lambda_\phi(\alpha)$  to be the partition corresponding to the polynomial  $\phi$  in the rational canonical form of  $\alpha$ .

The following elementary lemmas will be of use in studying the partitions  $\lambda_\phi$ .

**Lemma 1.** *If the Taylor series of  $f(u)$  around 0 converges at  $u = 1$ , then*

$$\lim_{n \rightarrow \infty} [u^n] \frac{f(u)}{1-u} = f(1).$$

*Proof.* Write the Taylor expansion  $f(u) = \sum_{n=0}^{\infty} a_n u^n$ . Then observe that  $[u^n] \frac{f(u)}{1-u} = \sum_{i=0}^n a_i$ .  $\square$

**Lemma 2.** *For  $t \geq 1$ ,  $q$  a prime power, and  $u$  a formal variable,*

$$\prod_{\phi \text{ irred.}} \left(1 - \frac{u^{\deg(\phi)}}{q^{t \cdot \deg(\phi)}}\right) = 1 - \frac{u}{q^{t-1}}.$$

*Proof.* Assume that  $t = 1$ , the general case following by replacing  $u$  with  $\frac{u}{q^{t-1}}$ . Expanding  $\frac{1}{1 - \frac{u^{\deg(\phi)}}{q^{\deg(\phi)}}}$  as a geometric series, unique factorization in  $F_q[x]$  implies that the coefficient of  $u^d$  in the reciprocal of the left hand side is  $\frac{1}{q^d}$  times the number of monic polynomials of degree  $d$ , hence 1. Comparing with the reciprocal of the right hand side proves the lemma.  $\square$

**Lemma 3.** *For  $t \geq 1$ ,  $q$  a prime power, and  $u$  a formal variable,*

$$1 - u = \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} \prod_{r=1}^{\infty} \left(1 - \frac{u^{\deg(\phi)}}{q^{r \cdot \deg(\phi)}}\right).$$

*Proof.* By Lemma 2,

$$\prod_{r=1}^{\infty} \left(1 - \frac{u}{q^{r-1}}\right) = \prod_{\phi \text{ irred.}} \prod_{r=1}^{\infty} \left(1 - \frac{u^{\deg(\phi)}}{q^{r \cdot \deg(\phi)}}\right).$$

The lemma follows by cancelling the terms corresponding to  $\phi = z$ .  $\square$

**Lemma 4.** *For  $q$  a prime power and  $u$  complex with  $|u| \leq 1$ ,*

$$\sum_{\lambda} \frac{u^{|\lambda|}}{q^{\sum_i (\lambda_i)^2} \prod_i \left(\frac{1}{q}\right)_{m_i(\lambda)}} = \prod_{r=1}^{\infty} \left(1 - \frac{u}{q^r}\right).$$

*Proof.* On page 225 of [Mac] it is proved that

$$\prod_{r=1}^{\infty} (1 - x_i) \sum_{\lambda} t^{n(\lambda)} P_{\lambda}(x_1, x_2, x_3, \dots; t) = 1.$$

Applying Macdonald's principal specialization formula with  $t = \frac{1}{q}$ ,  $x_i = \frac{u}{q^i}$  as in Theorem 2 proves the result.  $\square$

**Lemma 5** (Euler). *For  $|q| > 1$  and  $u$  complex with  $|u| \leq 1$ ,*

1.  $\prod_{r=1}^{\infty} (1 - \frac{u}{q^r}) = \sum_{n=0}^{\infty} \frac{(-u)^n}{(q^n-1)\cdots(q-1)},$
2.  $\prod_{r=1}^{\infty} (\frac{1}{1-\frac{u}{q^r}}) = \sum_{n=0}^{\infty} \frac{u^n q^{\binom{n}{2}}}{(q^n-1)\cdots(q-1)}.$

*Furthermore, these Taylor series converge at  $u = 1$ .*

Theorem 3 relates the Gordon identities with probability in the finite general linear groups.

**Theorem 3.** *Let  $\phi$  be a monic, irreducible polynomial over  $F_q$ . Let  $k \geq 2$  be an integer. Then the  $n \rightarrow \infty$  limit of the chance that a uniformly chosen element  $\alpha$  of  $GL(n, q)$  has the largest part of the partition  $\lambda_\phi(\alpha)$  less than  $k$  is equal to*

$$\prod_{\substack{r=1 \\ r=0, \pm k \pmod{2k+1}}}^{\infty} (1 - \frac{1}{q^{r \cdot \deg(\phi)}}).$$

*Proof.* Assume for simplicity that  $\phi = z - 1$ . From the proof it will be clear that the general case follows. Stong [Sto] used Kung's [Ku] formula for the sizes of the conjugacy classes of  $GL(n, q)$  to find a "cycle index" for the general linear groups. Using the notation

$$d_i(\lambda) = 1m_1(\lambda) + 2m_2(\lambda) + \cdots + (i-1)m_{i-1}(\lambda) + i(m_i(\lambda) + m_{i+1}(\lambda) + \cdots + m_j(\lambda)),$$

he obtained the equality

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{\alpha \in GL(n, q)} \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} x_{\phi, \lambda_\phi(\alpha)} \\ = \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} \left[ \sum_{\lambda} x_{\phi, \lambda} \frac{u^{|\lambda| \deg(\phi)}}{\prod_i \prod_{k=1}^{m_i(\lambda)} (q^{\deg(\phi) d_i} - q^{\deg(\phi)(d_i-k)})} \right]. \end{aligned}$$

Observe that

$$\begin{aligned} \prod_i \prod_{k=1}^{m_i} (q^{d_i} - q^{(d_i-k)}) &= q^{\sum_i m_i(\lambda) d_i(\lambda)} \prod_i \left(\frac{1}{q}\right)_{m_i(\lambda)} \\ &= q^{\sum_i m_i(\lambda) [(\sum_{h<i} h m_h(\lambda)) + i m_i(\lambda) + \sum_{i<k} i m_k(\lambda)]} \prod_i \left(\frac{1}{q}\right)_{m_i(\lambda)} \\ &= q^{\sum_i [i m_i(\lambda)^2 + 2 m_i(\lambda) \sum_{h<i} h m_h(\lambda)]} \prod_i \left(\frac{1}{q}\right)_{m_i(\lambda)} \\ &= q^{\sum_i [\sum_{h<i} m_h(\lambda)]^2} \prod_i \left(\frac{1}{q}\right)_{m_i(\lambda)} \\ &= q^{\sum_i (\lambda'_i)^2} \prod_i \left(\frac{1}{q}\right)_{m_i(\lambda)}. \end{aligned}$$

Combining this observation with Lemma 3 shows that

$$\begin{aligned} & 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{\alpha \in GL(n, q)} \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} x_{\phi, \lambda_{\phi}(\alpha)} \\ &= \frac{1}{1-u} \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} \left[ \prod_{r=1}^{\infty} \left( 1 - \frac{u^{\deg(\phi)}}{q^{r \cdot \deg(\phi)}} \right) \right] \left[ \sum_{\lambda} x_{\phi, \lambda} \frac{u^{|\lambda| \deg(\phi)}}{q^{\deg(\phi) \sum_i (\lambda'_i)^2} \prod_i \left( \frac{1}{q^{\deg(\phi)}} \right)_{m_i(\lambda)}} \right]. \end{aligned}$$

Setting  $x_{z-1, \lambda} = 0$  if the largest part of  $\lambda$  is greater than or equal to  $k$ , and all  $x_{\phi, \lambda} = 1$  otherwise shows by Lemma 4 that the sought probability is

$$\lim_{n \rightarrow \infty} [u^n] \frac{1}{1-u} \left( \prod_{r=1}^{\infty} \left( 1 - \frac{u}{q^r} \right) \right) \left( \sum_{\lambda: \lambda_1 < k} \frac{u^{|\lambda|}}{q^{\sum_i (\lambda'_i)^2} \prod_i \left( \frac{1}{q} \right)_{m_i(\lambda)}} \right).$$

By Lemmas 4 and 5, Lemma 1 applies and

$$\begin{aligned} & \lim_{n \rightarrow \infty} [u^n] \frac{1}{1-u} \left( \prod_{r=1}^{\infty} \left( 1 - \frac{u}{q^r} \right) \right) \left( \sum_{\lambda: \lambda_1 < k} \frac{u^{|\lambda|}}{q^{\sum_i (\lambda'_i)^2} \prod_i \left( \frac{1}{q} \right)_{m_i(\lambda)}} \right) \\ &= \left( \prod_{r=1}^{\infty} \left( 1 - \frac{1}{q^r} \right) \right) \left( \sum_{\lambda: \lambda_1 < k} \frac{1}{q^{\sum_i (\lambda'_i)^2} \prod_i \left( \frac{1}{q} \right)_{m_i(\lambda)}} \right). \end{aligned}$$

The theorem follows from the Gordon identities with  $i = k$ ,  $x = \frac{1}{q}$ ,  $n_j = m_j(\lambda)$ , and  $N_j = \lambda'_j$ .  $\square$

Let  $Mat(n, q)$  denote the set of  $n \times n$  matrices with entries in the finite field  $F_q$ . Recall that  $\alpha \in Mat(n, q)$  is said to be semisimple if it is diagonalizable over the algebraic closure  $\bar{F}_q$ .

**Theorem 4.** *The  $n \rightarrow \infty$  limiting probability that an element of  $GL(n, q)$  is semisimple is*

$$\prod_{r=0, \pm 2 \pmod{5}}^{\infty} \frac{\left( 1 - \frac{1}{q^{r-1}} \right)}{\left( 1 - \frac{1}{q^r} \right)}.$$

*Proof.* From the theory of Jordan canonical forms, an element  $\alpha$  of  $GL(n, q)$  is semisimple if and only if the largest part of  $\lambda_{\phi}(\alpha)$  is less than two for all  $\phi$ . Thus by the cycle index for  $GL(n, q)$  (see the proof of Theorem 3) and Lemma 3, the sought probability is

$$\begin{aligned} & \lim_{n \rightarrow \infty} [u^n] \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} \sum_{\lambda: \lambda_1 < 2} \frac{u^{\deg(\phi)|\lambda|}}{q^{\deg(\phi) \sum_i (\lambda'_i)^2} \prod_i \left( \frac{1}{q^{\deg(\phi)}} \right)_{m_i(\lambda)}} \\ &= \lim_{n \rightarrow \infty} [u^n] \frac{1}{1-u} \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} \left( \prod_{r=1}^{\infty} \left( 1 - \frac{u^{\deg(\phi)}}{q^{r \cdot \deg(\phi)}} \right) \right) \\ & \quad \cdot \left( \sum_{\lambda: \lambda_1 < 2} \frac{u^{\deg(\phi)|\lambda|}}{q^{\deg(\phi) \sum_i (\lambda'_i)^2} \prod_i \left( \frac{1}{q^{\deg(\phi)}} \right)_{m_i(\lambda)}} \right). \end{aligned}$$

By Theorem 1 with  $i = 1, k = 2, x = \frac{1}{q^{\deg(\phi)}}, n_j = \lambda_j, N_j = \lambda'_j$  and Lemma 2, the desired probability becomes

$$\begin{aligned}
& \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} \prod_{r=0, \pm 2(\bmod 5)}^{\infty} \left(1 - \frac{1}{q^{r \cdot \deg(\phi)}}\right) \\
&= \prod_{r=0, \pm 2(\bmod 5)}^{\infty} \left(\frac{1}{1 - \frac{1}{q^r}}\right) \prod_{\phi \text{ irred.}} \prod_{r=0, \pm 2(\bmod 5)}^{\infty} \left(1 - \frac{1}{q^{r \cdot \deg(\phi)}}\right) \\
&= \prod_{r=0, \pm 2(\bmod 5)}^{\infty} \frac{\left(1 - \frac{1}{q^{r-1}}\right)}{\left(1 - \frac{1}{q^r}\right)}.
\end{aligned}$$

□

**Theorem 5.** *The  $n \rightarrow \infty$  limiting probability that an element of  $Mat(n, q)$  is semisimple is*

$$\prod_{r=0, \pm 2(\bmod 5)}^{\infty} \left(1 - \frac{1}{q^{r-1}}\right).$$

*Proof.* The orbits of  $GL(n, q)$  on  $Mat(n, q)$  under conjugation are also parameterized by the data  $\lambda_\phi$ . However the polynomial  $z$  may appear with non-zero multiplicity, so the restriction that  $|\lambda_z| = 0$  does not apply. Stong [Sto] obtained the generating function equality

$$\begin{aligned}
& 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{\alpha \in Mat(n, q)} \prod_{\phi \text{ irred.}} x_{\phi, \lambda_\phi(\alpha)} \\
&= \prod_{\phi \text{ irred.}} \left[ \sum_{\lambda} x_{\phi, \lambda} \frac{u^{|\lambda| \deg(\phi)}}{\prod_i \prod_{k=1}^{m_i(\lambda)} (q^{\deg(\phi) d_i} - q^{\deg(\phi)(d_i - k)})} \right],
\end{aligned}$$

where  $d_i(\lambda)$  is defined by

$$\begin{aligned}
d_i(\lambda) &= 1m_1(\lambda) + 2m_2(\lambda) + \cdots + (i-1)m_{i-1}(\lambda) \\
&\quad + i(m_i(\lambda) + m_{i+1}(\lambda) + \cdots + m_j(\lambda)).
\end{aligned}$$

Manipulations identical to those in Theorem 3 show that

$$\begin{aligned}
& 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{\alpha \in Mat(n, q)} \prod_{\phi \text{ irred.}} x_{\phi, \lambda_\phi(\alpha)} \\
&= \frac{1}{1-u} \prod_{\phi \text{ irred.}} \left[ \prod_{r=1}^{\infty} \left(1 - \frac{u^{\deg(\phi)}}{q^{r \cdot \deg(\phi)}}\right) \right] \left[ \sum_{\lambda} x_{\phi, \lambda} \frac{u^{|\lambda| \deg(\phi)}}{q^{\deg(\phi) \sum_i (\lambda'_i)^2} \prod_i \left(\frac{1}{q^{\deg(\phi)}}\right)^{m_i(\lambda)}} \right].
\end{aligned}$$

Arguing as in Theorem 4 shows that the  $n \rightarrow \infty$  limit of the chance that an element of  $Mat(n, q)$  is semisimple is

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{|GL(n, q)|}{q^{n^2}} [u^n] \prod_{\phi \text{ irred.}} \sum_{\lambda: \lambda_1 < 2} \frac{u^{deg(\phi)|\lambda|}}{q^{deg(\phi) \sum_i (\lambda'_i)^2} \prod_i \left(\frac{1}{q^{deg(\phi)}}\right)_{m_i(\lambda)}} \\
&= \lim_{n \rightarrow \infty} \frac{|GL(n, q)|}{q^{n^2}} [u^n] \frac{1}{1-u} \frac{1}{\prod_{r=1}^{\infty} \left(1 - \frac{u}{q^r}\right)} \prod_{\phi \text{ irred.}} \\
&\quad \left[ \prod_{r=1}^{\infty} \left(1 - \frac{u^{deg(\phi)}}{q^{r \cdot deg(\phi)}}\right) \sum_{\lambda: \lambda_1 < 2} \frac{u^{deg(\phi)|\lambda|}}{q^{deg(\phi) \sum_i (\lambda'_i)^2} \prod_i \left(\frac{1}{q^{deg(\phi)}}\right)_{m_i(\lambda)}} \right] \\
&= \prod_{\phi \text{ irred.}} \left[ \prod_{r=1}^{\infty} \left(1 - \frac{1}{q^{r \cdot deg(\phi)}}\right) \right] \left[ \sum_{\lambda: \lambda_1 < 2} \frac{1}{q^{deg(\phi) \sum_i (\lambda'_i)^2} \prod_i \left(\frac{1}{q^{deg(\phi)}}\right)_{m_i(\lambda)}} \right] \\
&= \prod_{\phi \text{ irred.}} \prod_{\substack{r=1 \\ r=0, \pm 2 \pmod{5}}}^{\infty} \left(1 - \frac{1}{q^{r \cdot deg(\phi)}}\right) = \prod_{\substack{r=1 \\ r=0, \pm 2 \pmod{5}}}^{\infty} \left(1 - \frac{1}{q^{r-1}}\right). \quad \square
\end{aligned}$$

Theorems 2 and 3 suggest a connection between the Hall-Littlewood polynomials and the finite general linear groups. Theorem 6 makes this connection precise.

**Theorem 6.** For  $q$  a prime power and  $u$  a formal variable,

$$\begin{aligned}
& (1-u) \left[ 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{\alpha \in GL(n, q)} \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} x_{\phi, \lambda_{\phi}(\alpha)} \right] \\
&= \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} \prod_{r=1}^{\infty} \left(1 - \frac{u^{deg(\phi)}}{q^{r \cdot deg(\phi)}}\right) \left[ \frac{P_{\lambda} \left( \left(\frac{u}{q}\right)^{deg(\phi)}, \left(\frac{u}{q}\right)^{2deg(\phi)}, \left(\frac{u}{q}\right)^{3deg(\phi)}, \dots; \left(\frac{1}{q}\right)^{deg(\phi)} \right)}{q^{n(\lambda) \cdot deg(\phi)}} \right].
\end{aligned}$$

*Proof.* The proof of Theorem 3 contained the equality

$$\begin{aligned}
& (1-u) \left[ 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{\alpha \in GL(n, q)} \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} x_{\phi, \lambda_{\phi}(\alpha)} \right] \\
&= \prod_{\substack{\phi \neq z \\ \phi \text{ irred.}}} \left[ \prod_{r=1}^{\infty} \left(1 - \frac{u^{deg(\phi)}}{q^{r \cdot deg(\phi)}}\right) \right] \left[ \sum_{\lambda} x_{\phi, \lambda} \frac{u^{|\lambda| deg(\phi)}}{q^{deg(\phi) \sum_i (\lambda'_i)^2} \prod_i \left(\frac{1}{q^{deg(\phi)}}\right)_{m_i(\lambda)}} \right].
\end{aligned}$$

The theorem now follows from the use of Macdonald's principal specialization formula (as in Theorem 2) to conclude that

$$\frac{P_{\lambda} \left( \frac{u}{q}, \frac{u}{q^2}, \frac{u}{q^3}, \frac{u}{q^4}, \dots; \frac{1}{q} \right)}{q^{n(\lambda)}} = \frac{u^{|\lambda|}}{q^{\sum_i (\lambda'_i)^2} \prod_i \left(\frac{1}{q}\right)_{m_i(\lambda)}}. \quad \square$$

*Remark.* Although Theorem 6 followed easily from techniques in Theorems 2 and 3, its formulation admits an attractive probabilistic interpretation. Fix  $u$  such that  $0 < u < 1$  and fix an irreducible monic polynomial  $\phi \neq z$ . Then pick a natural number with probability of getting  $n$  equal to  $(1-u)u^n$ . Finally, choose  $\alpha$  uniformly at random in  $GL(n, q)$  and let  $\lambda_{\phi}(\alpha)$  be the random partition so defined. This procedure induces a probability measure on the set of all partitions of all integers.

Theorem 6 implies that measures obtained for different  $\phi$  are independent, and that the mass the measure for a given  $\phi$  assigns to a partition  $\lambda$  is equal to

$$\prod_{r=1}^{\infty} \left(1 - \frac{u^{\deg(\phi)}}{q^{r \cdot \deg(\phi)}}\right) \left[ \frac{P_{\lambda} \left( \left(\frac{u}{q}\right)^{\deg(\phi)}, \left(\frac{u}{q}\right)^{2\deg(\phi)}, \left(\frac{u}{q}\right)^{3\deg(\phi)}, \dots; \left(\frac{1}{q}\right)^{\deg(\phi)} \right)}{q^{n(\lambda) \cdot \deg(\phi)}} \right].$$

This connection with symmetric functions leads to a probabilistic algorithm for growing the random partitions  $\lambda_{\phi}$ . This viewpoint is used in [Fu1] to give probabilistic proofs of some group theoretic results of Steinberg, Rudvalis/Shinoda, and Lusztig. Cycle indices for the finite unitary, symplectic, and orthogonal groups appear in [Fu2].

### 3. CONCLUSION

This paper has offered connections between Gordon's generalization of the Rogers-Ramanujan identities, the Hall-Littlewood polynomials, and generating functions which arise in the study of the finite general linear groups. The following questions seem natural.

- Are there analogs of the Gordon identities for the finite unitary, symplectic, and orthogonal groups? The Gordon identities do play a role in Misra's work on affine symplectic Lie algebras [Mi1] and in Mandia's work [Man] on the affine Lie algebras  $B_l^{(1)}$ ,  $F_4^{(1)}$ , and  $G_2^{(1)}$ . Misra has also used  $Z$ -algebra methods to relate the Gordon identities to  $\widetilde{A}_n$ .
- Ian Macdonald has suggested that the results of this paper should carry over to the affine finite general linear groups. The Hall-Littlewood polynomials have affine analogs [EK].
- It is known (e.g. Kac [Ka]) that the product side of the Rogers-Ramanujan identities has interesting modular properties. Can this phenomenon be understood group theoretically in terms of the conjugacy classes of the general linear groups; i.e. is there "general linear group moonshine"? In this regard note that Jing [J1, J2] has connected the Hall-Littlewood polynomials with vertex operators.

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