THE ROGERS-RAMANUJAN IDENTITIES,
THE FINITE GENERAL LINEAR GROUPS,
AND THE HALL-LITTLEWOOD POLYNOMIALS

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Abstract. We connect Gordon’s generalization of the Rogers-Ramanujan identities with the Hall-Littlewood polynomials and with generating functions which arise in a probabilistic setting in the finite general linear groups. This yields a Rogers-Ramanujan type product formula for the \( n \to \infty \) probability that an element of \( GL(n, q) \) or \( Mat(n, q) \) is semisimple.

1. Background and notation

The Rogers-Ramanujan identities are among the most remarkable partition identities in number theory and combinatorics. This paper will be concerned with the following generalization of the Rogers-Ramanujan identities, due to Gordon. Let \( (x)_n \) denote \((1 - x)(1 - x^2) \cdots (1 - x^n)\).

**Theorem 1** ([A, page 111]). For \( 1 \leq i \leq k, k \geq 2 \), and complex \( x \) with \( |x| < 1 \),

\[
\sum_{n_1, \ldots, n_{k-1} \geq 0} \frac{x^{N_2 + \cdots + N_{k-1} + N_i + \cdots + N_{k-1}}}{(x)_{n_1} \cdots (x)_{n_{k-1}}} = \prod_{r=1}^{\infty} \frac{1}{1 - x^r}
\]

where \( N_j = n_j + \cdots + n_{k-1} \).

Gordon’s generalization of the Rogers-Ramanujan identities has been widely studied and appears in many places in mathematics and physics. Andrews [A] discusses combinatorial aspects of these identities. In an important series of papers, Lepowsky and Wilson [LW1], [LW2], [LW3] connect the Gordon identities with affine Lie algebras and structures that they called \( Z \)-algebras (later interpreted as parafermion algebras in conformal field theory). Meurman and Primc [MP] solve a problem left open in [LW3], proving the independence of a \( Z \)-algebra basis and obtaining a \( Z \)-algebra proof of the Gordon identities. Feigin and Frenkel [FF] interpret the Gordon identities as a character formula for the Virasoro algebra. Andrews, Baxter, and Forrester [ABF] and Warnaar [W] relate the Gordon identities with statistical mechanics. For some number theoretic connections see the conference proceedings [AABRR].

We use the following standard notation from the theory of partitions. Call \( \lambda = (\lambda_1, \lambda_2, \cdots) \) a partition of \( n = |\lambda| \) if \( \lambda_1 \geq \lambda_2 \geq \cdots \geq 0 \) where the \( \lambda_i \) are...
a sequence of positive integers stabilizing to 0 such that \( \sum \lambda_i = n \). The \( \lambda_i \) are referred to as the parts of \( \lambda \). Let \( m_i(\lambda) \) be the number of parts of \( \lambda \) of size \( i \), and set \( \lambda'_i = m_i(\lambda) + m_{i+1}(\lambda) + \cdots \). Define \( n(\lambda) \) by \( \sum_{i \geq 1} (i-1)\lambda_i \). Let \([u^n]f(u)\) denote the coefficient of \( u^n \) in a power series \( f(u) \).

2. Main Results

To begin we recall the Hall-Littlewood polynomials associated to a partition \( \lambda \) (page 208 of [Mac]). Let \( n \) be any integer such that \( n \geq \lambda_1 \). The permutation \( w \in S_n \) acts on the variables \( x_1, \cdots, x_n \) by sending \( x_i \) to \( x_{w(i)} \). Letting \( t \) be a complex number, the Hall-Littlewood polynomials are defined as

\[
P_\lambda(x_1, \cdots, x_n; t) = \prod_{i \geq 0} \prod_{r=1}^{m_i(\lambda)} \frac{1 - t^{1-r}}{1 - t} \sum_{w \in S_n} w(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i < j} x_i - tx_j).
\]

At first glance it is not obvious that these are polynomials, but the denominators cancel out after the symmetrization. The Hall-Littlewood polynomials interpolate between the Schur functions \( (t = 0) \) and the monomial symmetric functions \( (t = 1) \).

**Theorem 2.** For \( q > 1 \) and an integer \( k \geq 2 \),

\[
\sum_{\lambda: \lambda_1 < k} P_\lambda \left( \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \frac{1}{q} \right) \frac{1}{q^{n(\lambda)}} = \prod_{r=1}^{\infty} \left( 1 - \frac{1}{q^r} \right), \quad r \neq 0, \pm k \mod 2k + 1
\]

\[
\sum_{\lambda: \lambda_1 < k} P_\lambda \left( \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \frac{1}{q} \right) \frac{1}{q^{n(\lambda)}} = \prod_{r=1}^{\infty} \left( 1 - \frac{1}{q^r} \right), \quad r \neq 0, \pm 1 \mod 2k + 1
\]

**Proof.** Macdonald’s principal specialization formula (page 337 of [Mac]) states that

\[
P_\lambda \left( \frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \frac{1}{q} \right) = \frac{1}{q^{n(\lambda) + n(\lambda')}} \prod_i \left( \frac{1}{q \lambda_i} \right)^{m_i(\lambda)}.
\]

Combining this with the elementary fact that \( n(\lambda) = \sum \lambda_i \) shows that

\[
P_\lambda \left( \frac{1}{q^2}, \frac{1}{q^3}, \frac{1}{q^4}, \cdots; \frac{1}{q} \right) = \frac{1}{q^{n(\lambda)}} \prod_{r=1}^{\infty} \left( \frac{1}{q^r} \right)^{m_r(\lambda)}.
\]

The first equality of the theorem follows by applying Theorem 1 with \( i = k, x = \frac{1}{q}, n_j = \lambda_j \) and \( N_j = \lambda'_j \). For the second equality, observe that for \( u \) complex,

\[
\frac{P_\lambda \left( \frac{u}{q}, \frac{u}{q^2}, \frac{u}{q^3}, \frac{u}{q^4}, \cdots; \frac{1}{q} \right)}{q^{n(\lambda)}} = \frac{u^{n(\lambda)}}{u^{n(\lambda)} \prod_{r=1}^{\infty} \left( \frac{1}{q^r} \right)^{m_r(\lambda)}}.
\]

Now set \( u = \frac{1}{q} \) and apply Theorem 1 with \( i = 1, x = \frac{1}{q}, n_j = \lambda_j \) and \( N_j = \lambda'_j \). \( \square \)

**Remark.** Stembridge [Ste] used properties of the Hall-Littlewood polynomials as a tool in giving proofs of the Rogers-Ramanujan identities. The statement of Theorem 2 gives a direct connection.
Recall that the conjugacy classes of \( GL(n, q) \) are parameterized by rational canonical form (Chapter 6 of Herstein [H]). This form corresponds to the following combinatorial data. To each monic non-constant irreducible polynomial \( \phi \) over a field of \( q \) elements, associate a partition \( \lambda_\phi \), of some non-negative integer \( |\lambda_\phi| \). Let \( \text{deg}(\phi) \) denote the degree of \( \phi \). This data represents a conjugacy class of \( GL(n, q) \) if and only if \( |\lambda_\phi| = 0 \) and \( \sum_{\phi} |\lambda_\phi| \cdot \text{deg}(\phi) = n \).

**Definition.** For \( \alpha \in GL(n, q) \) and \( \phi \) a monic, irreducible polynomial over \( F_q \), a field of \( q \) elements, define \( \lambda_\phi(\alpha) \) to be the partition corresponding to the polynomial \( \phi \) in the rational canonical form of \( \alpha \).

The following elementary lemmas will be of use in studying the partitions \( \lambda_\phi \).

**Lemma 1.** If the Taylor series of \( f(u) \) around \( 0 \) converges at \( u = 1 \), then

\[
\lim_{n \to \infty} [u^n] \frac{f(u)}{1-u} = f(1).
\]

**Proof.** Write the Taylor expansion \( f(u) = \sum_{n=0}^{\infty} a_n u^n \). Then observe that \( [u^n] \frac{f(u)}{1-u} = \sum_{i=0}^{n} a_i \).

**Lemma 2.** For \( t \geq 1 \), \( q \) a prime power, and \( u \) a formal variable,

\[
\prod_{\phi \text{ irred.}} (1 - \frac{u^{\text{deg}(\phi)}}{q^{r \cdot \text{deg}(\phi)}}) = 1 - \frac{u}{q^{t-1}}.
\]

**Proof.** Assume that \( t = 1 \), the general case following by replacing \( u \) with \( \frac{w}{q^{t-1}} \). Expanding \( \frac{1}{1-q^{t-1}u} \) as a geometric series, unique factorization in \( F_q[x] \) implies that the coefficient of \( u^d \) in the reciprocal of the left hand side is \( \frac{1}{q^d} \) times the number of monic polynomials of degree \( d \), hence 1. Comparing with the reciprocal of the right hand side proves the lemma.

**Lemma 3.** For \( t \geq 1 \), \( q \) a prime power, and \( u \) a formal variable,

\[
1 - u = \prod_{\phi \neq z} \prod_{r=1}^{\infty} (1 - \frac{u^{\text{deg}(\phi)}}{q^{r \cdot \text{deg}(\phi)}}).
\]

**Proof.** By Lemma 2,

\[
\prod_{r=1}^{\infty} (1 - \frac{u}{q^{r-1}}) = \prod_{\phi \text{ irred.}} \prod_{r=1}^{\infty} (1 - \frac{u^{\text{deg}(\phi)}}{q^{r \cdot \text{deg}(\phi)}}).
\]

The lemma follows by cancelling the terms corresponding to \( \phi = z \).

**Lemma 4.** For \( q \) a prime power and \( u \) complex with \( |u| \leq 1 \),

\[
\sum_{\lambda} q^{\sum_i (\lambda)_i^2} \prod_{i=1}^{\infty} \frac{1}{(\frac{1}{q})^{m_i(\lambda)}} = \prod_{r=1}^{\infty} \left( 1 - \frac{u}{q^r} \right).
\]

**Proof.** On page 225 of [Mac] it is proved that

\[
\prod_{r=1}^{\infty} (1 - x_r) \sum_{\lambda} t^{n(\lambda)} P_{\lambda}(x_1, x_2, x_3, \cdots ; t) = 1.
\]

Applying Macdonald’s principal specialization formula with \( t = \frac{1}{q}, x_i = \frac{w}{q^{r}} \) as in Theorem 2 proves the result.
Lemma 5 (Euler). For $|q| > 1$ and $u$ complex with $|u| \leq 1$,

1. $\prod_{r=1}^{\infty} \left(1 - \frac{u}{q^r}\right) = \sum_{n=0}^{\infty} \frac{(-u)^n}{(q^n - 1) \cdots (q-1)}$,
2. $\prod_{r=1}^{\infty} \left(1 - \frac{1}{q^r}\right) = \sum_{n=0}^{\infty} \frac{u^n q^{\binom{n}{2}}}{(q^n - 1) \cdots (q-1)}$.

Furthermore, these Taylor series converge at $u = 1$.

Theorem 3 relates the Gordon identities with probability in the finite general linear groups.

Theorem 3. Let $\phi$ be a monic, irreducible polynomial over $F_q$. Let $k \geq 2$ be an integer. Then the $n \to \infty$ limit of the chance that a uniformly chosen element $\alpha$ of $GL(n, q)$ has the largest part of the partition $\lambda_\phi(\alpha)$ less than $k$ is equal to

$$\prod_{r=0, \pm \frac{k}{2k+1}}^{\infty} \left(1 - \frac{1}{q^{r \cdot \deg(\phi)}}\right).$$

Proof. Assume for simplicity that $\phi = z - 1$. From the proof it will be clear that the general case follows. Strong [Sto] used Kung’s [Ku] formula for the sizes of the conjugacy classes of $GL(n, q)$ to find a “cycle index” for the general linear groups. Using the notation

$$d_i(\lambda) = 1m_1(\lambda) + 2m_2(\lambda) + \cdots + (i-1)m_{i-1}(\lambda) + i(m_i(\lambda) + m_{i+1}(\lambda) + \cdots + m_j(\lambda)),$$

he obtained the equality

$$1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{\alpha \in GL(n, q)} \prod_{\phi \neq z} x_{\phi, \lambda_\phi(\alpha)} = \prod_{\phi \neq z} \left[ \sum_{\lambda} x_{\phi, \lambda} \prod_{i} \prod_{k=1}^{m_i(\lambda)} \frac{u^{\lambda|\deg(\phi)}}{(q^{\deg(\phi)d_i - q^{\deg(\phi)}(d_i - k)})} \right].$$

Observe that

$$\prod_{i} \prod_{k=1}^{m_i} (q^{d_i} - q^{d_i - k}) = q^{\sum_i m_i(\lambda)d_i(\lambda)} \prod_{i} \left(\frac{1}{q}\right)^{m_i(\lambda)}$$

$$= q^{\sum_i m_i(\lambda)(\sum_{h<i} m_h(\lambda)) + \sum_i m_i(\lambda)} \prod_{i} \left(\frac{1}{q}\right)^{m_i(\lambda)}$$

$$= q^{\sum_i \sum_{h<i} m_i(\lambda)^2} \sum_{h<i} m_h(\lambda) \prod_{i} \left(\frac{1}{q}\right)^{m_i(\lambda)}$$

$$= q^{\sum_i \sum_{h<i} m_h(\lambda)^2} \prod_{i} \left(\frac{1}{q}\right)^{m_i(\lambda)}$$

$$= q^{\sum_i (\lambda)^2} \prod_{i} \left(\frac{1}{q}\right)^{m_i(\lambda)}.$$

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Combining this observation with Lemma 3 shows that
\[
1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{x \in GL(n,q)} \prod_{\phi \not\equiv \psi} x_{\phi,\lambda_n(\alpha)} = \frac{1}{1-u} \prod_{\phi \not\equiv \psi} \left[ \prod_{r=1}^{\infty} (1 - \frac{u^{\deg(\phi)}}{q^r}) \right] \left[ \sum_{\lambda} x_{\phi,\lambda} u^{\deg(\phi)} \prod_{k=1}^{\infty} \left( \frac{1}{q^r} \right)^{m_\lambda(\lambda)} \right].
\]

Setting \(x_{\phi,\lambda} = 0\) if the largest part of \(\lambda\) is greater than or equal to \(k\), and all \(x_{\phi,\lambda} = 1\) otherwise shows by Lemma 4 that the sought probability is
\[
\lim_{n \to \infty} \frac{[u^n]}{1-u} \left( \prod_{i=1}^{\infty} (1 - \frac{u}{q^i}) \right) \left( \sum_{\lambda: \lambda_i < k} \frac{u^{\deg(\lambda)}}{q^{\sum_i (\lambda_i)^2} \prod_{k=1}^{\infty} \left( \frac{1}{q^r} \right)^{m_\lambda(\lambda)}} \right).
\]

By Lemmas 4 and 5, Lemma 1 applies and
\[
\lim_{n \to \infty} \frac{[u^n]}{1-u} \left( \prod_{i=1}^{\infty} (1 - \frac{u}{q^i}) \right) \left( \sum_{\lambda: \lambda_i < k} \frac{u^{\deg(\lambda)}}{q^{\sum_i (\lambda_i)^2} \prod_{k=1}^{\infty} \left( \frac{1}{q^r} \right)^{m_\lambda(\lambda)}} \right) = \left( \prod_{i=1}^{\infty} (1 - \frac{1}{q^i}) \right) \left( \sum_{\lambda: \lambda_i < k} \frac{u^{\deg(\lambda)}}{q^{\sum_i (\lambda_i)^2} \prod_{k=1}^{\infty} \left( \frac{1}{q^r} \right)^{m_\lambda(\lambda)}} \right).
\]

The theorem follows from the Gordon identities with \(i = k, n_j = m_j(\lambda)\), and \(N_j = \lambda_j^n\).

Let \(Mat(n,q)\) denote the set of \(n \times n\) matrices with entries in the finite field \(F_q\). Recall that \(\alpha \in Mat(n,q)\) is said to be semisimple if it is diagonalizable over the algebraic closure \(\overline{F_q}\).

**Theorem 4.** The \(n \to \infty\) limiting probability that an element of \(GL(n,q)\) is semisimple is
\[
\prod_{r=1}^{\infty} \frac{1}{\left( 1 - \frac{1}{q^r} \right)}.
\]

**Proof.** From the theory of Jordan canonical forms, an element \(\alpha \in GL(n,q)\) is semisimple if and only if the largest part of \(\lambda_n(\alpha)\) is less than two for all \(\phi\). Thus by the cycle index for \(GL(n,q)\) (see the proof of Theorem 3) and Lemma 3, the sought probability is
\[
\lim_{n \to \infty} \frac{[u^n]}{1-u} \prod_{\phi \not\equiv \psi} \left( \prod_{r=1}^{\infty} (1 - \frac{u^{\deg(\phi)}}{q^r}) \right) \left( \sum_{\lambda: \lambda_i < 2} \frac{u^{\deg(\phi)}}{q^{\sum_i (\lambda_i)^2} \prod_{k=1}^{\infty} \left( \frac{1}{q^r} \right)^{m_\lambda(\lambda)}} \right).
\]
By Theorem 1 with $i = 1, k = 2, x = \frac{1}{q^{\deg(\phi)}}, n_j = \lambda_j, N_j = \lambda'_j$ and Lemma 2, the desired probability becomes

\[
\prod_{\phi \text{ irred.}} (1 - \frac{1}{q^{r \deg(\phi)}})
= \prod_{r \equiv 1 \pmod{5}} (1 - \frac{1}{q^r}) \prod_{r \equiv 0, \pm 2 \pmod{5}} \prod_{r \equiv 1} (1 - \frac{1}{q^{r \deg(\phi)}})
= \prod_{r \equiv 0, \pm 2 \pmod{5}} (1 - \frac{1}{q^r}) (1 - \frac{1}{q^r}).
\]

\[\square\]

**Theorem 5.** The $n \to \infty$ limiting probability that an element of $\text{Mat}(n, q)$ is semisimple is

\[
\prod_{r \equiv 0, \pm 2 \pmod{5}} (1 - \frac{1}{q^{r-1}}).
\]

**Proof.** The orbits of $GL(n, q)$ on $\text{Mat}(n, q)$ under conjugation are also parameterized by the data $\lambda_\phi$. However the polynomial $z$ may appear with non-zero multiplicity, so the restriction that $|\lambda_z| = 0$ does not apply. Strong [Sto] obtained the generating function equality

\[
1 + \sum_{n=1}^{\infty} \frac{u^n}{[GL(n, q)]} \sum_{\alpha \in \text{Mat}(n, q)} \prod_{\phi \text{ irred.}} X_{\phi, \lambda_\phi}(\alpha)
= \prod_{\phi \text{ irred.}} \left[ \sum_{\lambda} X_{\phi, \lambda} \frac{u^{\lambda|\deg(\phi)}}{\prod_i \Pi_{k=1}^{m_i(\lambda)} (q^{\deg(\phi)}d_i - q^{\deg(\phi)}d_i - k)} \right],
\]

where $d_i(\lambda)$ is defined by

\[
d_i(\lambda) = m_1(\lambda) + 2m_2(\lambda) + \cdots + (i - 1)m_{i-1}(\lambda) + i(m_i(\lambda) + m_{i+1}(\lambda) + \cdots + m_j(\lambda)).
\]

Manipulations identical to those in Theorem 3 show that

\[
1 + \sum_{n=1}^{\infty} \frac{u^n}{[GL(n, q)]} \sum_{\alpha \in \text{Mat}(n, q)} \prod_{\phi \text{ irred.}} X_{\phi, \lambda_\phi}(\alpha)
= \frac{1}{1 - u} \prod_{\phi \text{ irred.}} \left[ \sum_{r=1}^{\infty} \frac{u^{r\deg(\phi)}}{q^{r\deg(\phi)}} \right] \left[ \sum_{\lambda} X_{\phi, \lambda} \frac{u^{\lambda|\deg(\phi)}q^{\deg(\phi)}\prod_i (\frac{1}{q^{\deg(\phi)}m_i(\lambda)})^2}{\prod_i (\frac{1}{q^{\deg(\phi)}}m_i(\lambda))} \right].
\]
Arguing as in Theorem 4 shows that the $n \to \infty$ limit of the chance that an element of $\text{Mat}(n,q)$ is semisimple is

$$
\lim_{n \to \infty} \frac{|GL(n,q)|}{q^{n^2}} [u^n] \prod_{\phi \ irreduc.} \sum_{\lambda: \lambda_1 \leq 2} q^{\deg(\phi) \sum_i (\lambda_i)^2} \prod_i \left( \frac{1}{q^{\deg(\phi)}} \right) m_i(\lambda)
$$

\[
= \lim_{n \to \infty} \frac{|GL(n,q)|}{q^{n^2}} [u^n] \frac{1}{1-u} \prod_{r=1}^{\infty} \left( 1 - \frac{1}{q^{r \deg(\phi)}} \right) \prod_{\lambda: \lambda_1 \leq 2} q^{\deg(\phi) \sum_i (\lambda_i)^2} \prod_i \left( \frac{1}{q^{\deg(\phi)}} \right) m_i(\lambda) \]

\[
= \prod_{\phi \ irreduc.} \prod_{r=1}^{\infty} \left( 1 - \frac{1}{q^{r \deg(\phi)}} \right) \left[ \sum_{\lambda: \lambda_1 \leq 2} q^{\deg(\phi) \sum_i (\lambda_i)^2} \frac{1}{\prod_i \left( \frac{1}{q^{\deg(\phi)}} \right) m_i(\lambda)} \right] = \prod_{r=0, \pm 2 \text{mod } 5} \left( 1 - \frac{1}{q^{r \deg(\phi)}} \right). \tag*{\surd}
\]

Theorems 2 and 3 suggest a connection between the Hall-Littlewood polynomials and the finite general linear groups. Theorem 6 makes this connection precise.

**Theorem 6.** For $q$ a prime power and $u$ a formal variable,

$$
(1-u) \left[ 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \prod_{\phi \ irreduc.} X_{\phi,\lambda_\phi(\alpha)} \right]
$$

\[
= \prod_{\phi \ irreduc.} \prod_{r=1}^{\infty} \left( 1 - \frac{u^{\deg(\phi)}}{q^{r \deg(\phi)}} \right) \left[ \sum_{\lambda} X_{\phi,\lambda} \frac{u^{\deg(\phi)} \sum_i (\lambda_i)^2 \prod_i \left( \frac{1}{q^{\deg(\phi)}} \right) m_i(\lambda)}{q^{\deg(\phi)}} \right].
\]

**Proof.** The proof of Theorem 3 contained the equality

$$
(1-u) \left[ 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \prod_{\phi \ irreduc.} X_{\phi,\lambda_\phi(\alpha)} \right]
$$

\[
= \prod_{\phi \ irreduc.} \prod_{r=1}^{\infty} \left( 1 - \frac{u^{\deg(\phi)}}{q^{r \deg(\phi)}} \right) \left[ \sum_{\lambda} X_{\phi,\lambda} \frac{u^{\deg(\phi)} \sum_i (\lambda_i)^2 \prod_i \left( \frac{1}{q^{\deg(\phi)}} \right) m_i(\lambda)}{q^{\deg(\phi)}} \right].
\]

The theorem now follows from the use of Macdonald’s principal specialization formula (as in Theorem 2) to conclude that

$$
P_{\chi} \left( \frac{u}{q}, \frac{u}{q^3}, \frac{u}{q^5}, \cdots, \frac{1}{q} \right) = \frac{u^{\deg(\phi)} \sum_i (\lambda_i)^2 \prod_i \left( \frac{1}{q^{\deg(\phi)}} \right) m_i(\lambda)}{q^{\deg(\phi)}} \frac{u^{\deg(\phi)} \sum_i (\lambda_i)^2 \prod_i \left( \frac{1}{q^{\deg(\phi)}} \right) m_i(\lambda)}{q^{\deg(\phi)}} = \frac{u^{\deg(\phi)}}{q^{\deg(\phi)}} \prod_i \left( \frac{1}{q^{\deg(\phi)}} \right) m_i(\lambda). \tag*{\surd}
\]

**Remark.** Although Theorem 6 followed easily from techniques in Theorems 2 and 3, its formulation admits an attractive probabilistic interpretation. Fix $u$ such that $0 < u < 1$ and fix an irreducible monic polynomial $\phi \neq z$. Then pick a natural number with probability of getting $n$ equal to $(1-u)u^n$. Finally, choose $\alpha$ uniformly at random in $GL(n,q)$ and let $\lambda_{\phi}(\alpha)$ be the random partition so defined. This procedure induces a probability measure on the set of all partitions of all integers.
Theorem 6 implies that measures obtained for different $\phi$ are independent, and that the mass the measure for a given $\phi$ assigns to a partition $\lambda$ is equal to

$$\prod_{r=1}^{\infty} \left( 1 - \frac{u^{\deg(\phi)}}{q^{r \cdot \deg(\phi)}} \right) \left[ \frac{P_{\lambda} \left( \left( \frac{u}{q} \right)^{\deg(\phi)}, \left( \frac{u^2}{q} \right)^{2 \deg(\phi)}, \ldots ; \left( \frac{u^r}{q} \right)^{r \deg(\phi)} \right)}{q^{n(\lambda) \cdot \deg(\phi)}} \right].$$

This connection with symmetric functions leads to a probabilistic algorithm for growing the random partitions $\lambda_\phi$. This viewpoint is used in [Fu1] to give probabilistic proofs of some group theoretic results of Steinberg, Rudvalis/Shinoda, and Lusztig. Cycle indices for the finite unitary, symplectic, and orthogonal groups appear in [Fu2].

3. Conclusion

This paper has offered connections between Gordon’s generalization of the Rogers-Ramanujan identities, the Hall-Littlewood polynomials, and generating functions which arise in the study of the finite general linear groups. The following questions seem natural.

- Are there analogs of the Gordon identities for the finite unitary, symplectic, and orthogonal groups? The Gordon identities do play a role in Misra’s work on affine symplectic Lie algebras [Mi1] and in Mandia’s work [Man] on the affine Lie algebras $B_l^{(1)}, F_4^{(1)}$, and $G_2^{(1)}$. Misra has also used $Z$-algebra methods to relate the Gordon identities to $A_n$.

- Ian Macdonald has suggested that the results of this paper should carry over to the affine finite general linear groups. The Hall-Littlewood polynomials have affine analogs [EK].

- It is known (e.g. Kac [Ka]) that the product side of the Rogers-Ramanujan identities has interesting modular properties. Can this phenomenon be understood group theoretically in terms of the conjugacy classes of the general linear groups; i.e. is there “general linear group moonshine”? In this regard note that Jing [J1, J2] has connected the Hall-Littlewood polynomials with vertex operators.

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