THE ROGERS-RAMANUJAN IDENTITIES, THE FINITE GENERAL LINEAR GROUPS, AND THE HALL-LITTLEWOOD POLYNOMIALS

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Abstract. We connect Gordon’s generalization of the Rogers-Ramanujan identities with the Hall-Littlewood polynomials and with generating functions which arise in a probabilistic setting in the finite general linear groups. This yields a Rogers-Ramanujan type product formula for the probability that an element of $GL(n, q)$ or $Mat(n, q)$ is semisimple.

1. BACKGROUND AND NOTATION

The Rogers-Ramanujan identities are among the most remarkable partition identities in number theory and combinatorics. This paper will be concerned with the following generalization of the Rogers-Ramanujan identities, due to Gordon. Let $(x)_n$ denote $(1-x)(1-x^2)\cdots(1-x^n)$.

Theorem 1 (A, page 111). For $1 \leq i \leq k, k \geq 2$, and complex $x$ with $|x| < 1$,

$$\sum_{n_1, \ldots, n_{k-1} \geq 0} x^{N^2_1+\cdots+N^2_{k-1}+N_i+\cdots+N_{k-1}} = \prod_{r=1, r \neq 0, \pm i}^{\infty} \frac{1}{1-x^r}$$

where $N_j = n_j + \cdots + n_{k-1}$.

Gordon’s generalization of the Rogers-Ramanujan identities has been widely studied and appears in many places in mathematics and physics. Andrews [A] discusses combinatorial aspects of these identities. In an important series of papers, Lepowsky and Wilson [LW1], [LW2], [LW3] connect the Gordon identities with affine Lie algebras and structures that they called $Z$-algebras (later interpreted as parafermion algebras in conformal field theory). Meurman and Primc [MP] solve a problem left open in [LW3], proving the independence of a $Z$-algebra basis and obtaining a $Z$-algebra proof of the Gordon identities. Feigin and Frenkel [FF] interpret the Gordon identities as a character formula for the Virasoro algebra. Andrews, Baxter, and Forrester [ABF] and Warnaar [W] relate the Gordon identities with statistical mechanics. For some number theoretic connections see the conference proceedings [AABRR].

We use the following standard notation from the theory of partitions. Call $\lambda = (\lambda_1, \lambda_2, \cdots)$ a partition of $n = |\lambda|$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ where the $\lambda_i$ are...
a sequence of positive integers stabilizing to 0 such that $\sum \lambda_i = n$. The $\lambda_i$ are referred to as the parts of $\lambda$. Let $m_i(\lambda)$ be the number of parts of $\lambda$ of size $i$, and set $\lambda'_i = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$. Define $n(\lambda)$ by $\sum_{i \geq 1} (i-1) \lambda_i$. Let $[u^n]f(u)$ denote the coefficient of $u^n$ in a power series $f(u)$.

2. Main results

To begin we recall the Hall-Littlewood polynomials associated to a partition $\lambda$ (page 208 of [Mac]). Let $n$ be any integer such that $n \geq \lambda_1$. The permutation $w \in S_n$ acts on the variables $x_1, \cdots, x_n$ by sending $x_i$ to $x_{w(i)}$. Letting $t$ be a complex number, the Hall-Littlewood polynomials are defined as

$$P_\lambda(x_1, \cdots, x_n; t) = \frac{1}{\prod_{r=1}^{\infty} (1 - t^r)} \sum_{w \in S_n} w(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i < j} x_i - tx_i).$$

At first glance it is not obvious that these are polynomials, but the denominators cancel out after the symmetrization. The Hall-Littlewood polynomials interpolate between the Schur functions ($t = 0$) and the monomial symmetric functions ($t = 1$).

Theorem 2. For $q > 1$ and an integer $k \geq 2$,

$$\sum_{\lambda: \lambda_1 < k} \frac{P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} = \prod_{r \neq 0, \pm k \mod 2k+1} \left( \frac{1}{1 - \frac{1}{q^r}} \right),$$

$$\sum_{\lambda: \lambda_1 < k} \frac{P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} = \prod_{r \neq 0, \pm 1 \mod 2k+1} \left( \frac{1}{1 - \frac{1}{q^r}} \right).$$

Proof. Macdonald’s principal specialization formula (page 337 of [Mac]) states that

$$P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \cdots; \frac{1}{q}) = \frac{1}{q^{n(\lambda)}} \prod_i \frac{1}{(\frac{1}{q})^{m_i(\lambda)}}.$$

Combining this with the elementary fact that $n(\lambda) = \sum_i (\lambda'_i)$ shows that

$$\frac{P_\lambda(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} = \frac{1}{q^{\sum_i (\lambda'_i)}} \prod_i \left( \frac{1}{q} \right)^{m_i(\lambda)}.$$

The first equality of the theorem follows by applying Theorem 1 with $i = k, x = \frac{1}{q}, n_j = \lambda_j$ and $N_j = \lambda'_j$. For the second equality, observe that for $u$ complex,

$$\frac{P_\lambda(\frac{u}{q}, \frac{u}{q^2}, \frac{u}{q^3}, \cdots; \frac{1}{q})}{q^{n(\lambda)}} = \frac{u^{n(\lambda)}}{q^{\sum_i (\lambda'_i)}} \prod_i \left( \frac{1}{q} \right)^{m_i(\lambda)}.$$

Now set $u = \frac{1}{q}$ and apply Theorem 1 with $i = 1, x = \frac{1}{q}, n_j = \lambda_j$ and $N_j = \lambda'_j$. $\square$

Remark. Stembridge [Ste] used properties of the Hall-Littlewood polynomials as a tool in giving proofs of the Rogers-Ramanujan identities. The statement of Theorem 2 gives a direct connection.
Recall that the conjugacy classes of $GL(n, q)$ are parameterized by rational canonical form (Chapter 6 of Herstein [H]). This form corresponds to the following combinatorial data. To each monic non-constant irreducible polynomial $φ$ over a field of $q$ elements, associate a partition $λ_φ$ of some non-negative integer $|λ_φ|$. Let $deg(φ)$ denote the degree of $φ$. This data represents a conjugacy class of $GL(n, q)$ if and only if $|λ_2| = 0$ and $∑_{φ}|λ_φ| deg(φ) = n$.

**Definition.** For $α ∈ GL(n, q)$ and $φ$ a monic, irreducible polynomial over $F_q$, a field of $q$ elements, define $λ_φ(α)$ to be the partition corresponding to the polynomial $φ$ in the rational canonical form of $α$.

The following elementary lemmas will be of use in studying the partitions $λ_φ$.

**Lemma 1.** If the Taylor series of $f(u)$ around 0 converges at $u = 1$, then

$$\lim_{n→∞}[u^n]f(u) = f(1).$$

**Proof.** Write the Taylor expansion $f(u) = ∑_{n=0}^∞ a_n u^n$. Then observe that $[u^n]f(u) = ∑_{i=0}^n a_i$. □

**Lemma 2.** For $t ≥ 1$, a prime power, and $u$ a formal variable,

$$\prod_{φ \text{ irred.}} (1 - \frac{u^{deg(φ)}}{q^{r·deg(φ)}}) = 1 - \frac{u}{q^{t-1}}.$$

**Proof.** Assume that $t = 1$, the general case following by replacing $u$ with $\frac{u}{q^{t-1}}$. Expanding $\frac{1}{1 - \frac{u}{q^{t-1}}}$ as a geometric series, unique factorization in $F_q[x]$ implies that the coefficient of $u^d$ in the reciprocal of the left hand side is $\frac{1}{q^{t-1}}$ times the number of monic polynomials of degree $d$, hence 1. Comparing with the reciprocal of the right hand side proves the lemma. □

**Lemma 3.** For $t ≥ 1$, a prime power, and $u$ a formal variable,

$$1 - u = \prod_{φ \text{ irred.}} ∑_{r=1}^∞ (1 - \frac{u^{deg(φ)}}{q^{r·deg(φ)}}).$$

**Proof.** By Lemma 2,

$$\prod_{r=1}^∞ (1 - \frac{u}{q^{r-1}}) = \prod_{φ \text{ irred.}} ∏_{r=1}^∞ (1 - \frac{u^{deg(φ)}}{q^{r·deg(φ)}}).$$

The lemma follows by cancelling the terms corresponding to $φ = z$. □

**Lemma 4.** For $q$ a prime power and $u$ complex with $|u| ≤ 1$,

$$∑_{λ} q^{∑_i(λ)_i} r^{|λ|} ∏_{i=1}^∞ (\frac{1}{q})^{|m_i(λ)|} = ∏_{r=1}^∞ (1 - \frac{u}{q^r}).$$

**Proof.** On page 225 of [Mac] it is proved that

$$\prod_{r=1}^∞ (1 - x_i) ∑_{λ} t^{n(λ)}P_λ(x_1, x_2, x_3, \cdots ; t) = 1.$$

Applying Macdonald’s principal specialization formula with $t = \frac{1}{q}$, $x_i = \frac{n}{q^r}$ as in Theorem 2 proves the result. □
Lemma 5 (Euler). For \(|q| > 1\) and \(u\) complex with \(|u| \leq 1\),

1. \(\prod_{r=1}^{\infty} (1 - \frac{u^r}{q^r}) = \sum_{n=0}^{\infty} \frac{(-u)^n}{(q^n-1)\cdots(q-1)}\),
2. \(\prod_{r=1}^{\infty} (\frac{1}{1 - \frac{u^r}{q^r}}) = \sum_{n=0}^{\infty} \frac{u^n q^{\binom{n}{2}}}{(q^n-1)\cdots(q-1)}\).

Furthermore, these Taylor series converge at \(u = 1\).

Theorem 3 relates the Gordon identities with probability in the finite general linear groups.

Theorem 3. Let \(\phi\) be a monic, irreducible polynomial over \(F_q\). Let \(k \geq 2\) be an integer. Then the \(n \to \infty\) limit of the chance that a uniformly chosen element \(\alpha\) of \(GL(n, q)\) has the largest part of the partition \(\lambda_{\phi}(\alpha)\) less than \(k\) is equal to

\[
\prod_{r=0, \pm (k \text{ mod } 2k+1)} (1 - \frac{1}{q^r \cdot \deg(\phi)}).
\]

Proof. Assume for simplicity that \(\phi = z - 1\). From the proof it will be clear that the general case follows. Strong [Sto] used Kung’s [Ku] formula for the sizes of the conjugacy classes of \(GL(n, q)\) to find a “cycle index” for the general linear groups. Using the notation

\[
d_i(\lambda) = 1m_1(\lambda) + 2m_2(\lambda) + \cdots + (i-1)m_{i-1}(\lambda) + i(m_i(\lambda) + m_{i+1}(\lambda) + \cdots + m_j(\lambda)),
\]

he obtained the equality

\[
1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{\alpha \in GL(n, q)_{\phi \text{ irr}}} \prod_{\phi \neq z} x_{\phi, \lambda_{\phi}(\alpha)} = \prod_{\phi \text{ irr.}} \left[ \sum_{\lambda} X_{\phi, \lambda} \frac{u^{\lambda \cdot \deg(\phi)}}{\prod_i \prod_{k=1}^{m_i(\lambda)} (q^{\deg(\phi) d_i - q^{\deg(\phi) (d_i - k)}) \right].
\]

Observe that

\[
\prod_{i} \prod_{k=1}^{m_i} (q^{d_i} - q^{d_i - k}) = q^{\sum_i m_i(\lambda) d_i(\lambda)} \prod_i \left(\frac{1}{q}\right)^{m_i(\lambda)}
\]

\[
= q^{\sum_i m_i(\lambda) [\sum_{h<i} h m_h(\lambda) + \sum_{i<k} h m_k(\lambda)]} \prod_i \left(\frac{1}{q}\right)^{m_i(\lambda)}
\]

\[
= q^{\sum_i [im_i(\lambda)^2 + 2m_i(\lambda) \sum_{h<i} h m_h(\lambda)]} \prod_i \left(\frac{1}{q}\right)^{m_i(\lambda)}
\]

\[
= q^{\sum_i [\sum_{h<i} m_h(\lambda)^2]} \prod_i \left(\frac{1}{q}\right)^{m_i(\lambda)}
\]

\[
= q^{\sum_i (\lambda)^2} \prod_i \left(\frac{1}{q}\right)^{m_i(\lambda)}.
\]
Combining this observation with Lemma 3 shows that

\[
1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{\alpha \in GL(n,q)} \prod_{\phi \neq \text{irred.}} x_{\phi,\lambda_\alpha(n)} = \frac{1}{1 - u} \prod_{\phi \neq \text{irred.}} \left[ \prod_{r=1}^{\infty} \left( 1 - \frac{u^{\text{deg} \phi}}{q^{\text{deg} \phi}} \right) \right] \sum_{\lambda} x_{\lambda, \lambda}(\alpha) \frac{u^{\text{deg} \lambda}}{q^{\text{deg} \lambda} \prod_{r=1}^{\infty} \left( \frac{1}{q^{\text{deg} \phi}} m_\lambda(\lambda) \right)}.
\]

Setting \( x_{z-1,\lambda} = 0 \) if the largest part of \( \lambda \) is greater than or equal to \( k \), and all \( x_{\phi,\lambda} = 1 \) otherwise shows by Lemma 4 that the sought probability is

\[
\lim_{n \to \infty} \left[ u^n \right] \frac{1}{1 - u} \left( \prod_{r=1}^{\infty} \left( 1 - \frac{u}{q^r} \right) \right) \left( \sum_{\lambda : \lambda_1 < k} \frac{u^{\text{deg} \lambda}}{q^{\text{deg} \lambda} \prod_{r=1}^{\infty} \left( \frac{1}{q^r} m_\lambda(\lambda) \right)} \right).
\]

By Lemmas 4 and 5, Lemma 1 applies and

\[
\lim_{n \to \infty} \left[ u^n \right] \frac{1}{1 - u} \left( \prod_{r=1}^{\infty} \left( 1 - \frac{u}{q^r} \right) \right) \left( \sum_{\lambda : \lambda_1 < k} \frac{1}{q^{\text{deg} \lambda} \prod_{r=1}^{\infty} \left( \frac{1}{q^r} m_\lambda(\lambda) \right)} \right).
\]

The theorem follows from the Gordon identities with \( i = k, x = \frac{1}{q}, n_j = m_j(\lambda) \), and \( N_j = \lambda'_j \).

Let \( \text{Mat}(n,q) \) denote the set of \( n \times n \) matrices with entries in the finite field \( F_q \). Recall that \( \alpha \in \text{Mat}(n,q) \) is said to be semisimple if it is diagonalizable over the algebraic closure \( F_q \).

**Theorem 4.** The \( n \to \infty \) limiting probability that an element of \( GL(n,q) \) is semisimple is

\[
\prod_{r=1}^{\infty} \frac{1 - \frac{1}{q^{r-1}}}{1 - \frac{1}{q^{r}}}. 
\]

**Proof.** From the theory of Jordan canonical forms, an element \( \alpha \) of \( GL(n,q) \) is semisimple if and only if the largest part of \( \lambda_\phi(\alpha) \) is less than two for all \( \phi \). Thus by the cycle index for \( GL(n,q) \) (see the proof of Theorem 3) and Lemma 3, the sought probability is

\[
\lim_{n \to \infty} [u^n] \prod_{\phi \neq \text{irred.}} x_{\phi, \lambda}(\alpha) \frac{u^{\text{deg} \lambda}}{q^{\text{deg} \lambda} \prod_{r=1}^{\infty} \left( \frac{1}{q^{\text{deg} \phi}} m_\lambda(\lambda) \right)} = \lim_{n \to \infty} [u^n] \frac{1}{1 - u} \prod_{\phi \neq \text{irred.}} \left( \prod_{r=1}^{\infty} \left( 1 - \frac{u^{\text{deg} \phi}}{q^{\text{deg} \phi}} \right) \right) \left( \sum_{\lambda : \lambda_1 < 2} \frac{u^{\text{deg} \lambda}}{q^{\text{deg} \lambda} \prod_{r=1}^{\infty} \left( \frac{1}{q^{\text{deg} \phi}} m_\lambda(\lambda) \right)} \right).
\]
By Theorem 1 with $i = 1, k = 2, x = \frac{1}{q^{\text{deg}(\phi)}}, n_j = \lambda_j, N_j = \lambda_j'$ and Lemma 2, the desired probability becomes

$$
\prod_{\phi \text{ irred.} \atop \phi \equiv \lambda \pmod{5}} \prod_{r=1}^{\infty} \left(1 - \frac{1}{q^r \cdot \text{deg}(\phi)}\right) = \prod_{r=1}^{\infty} \left(1 - \frac{1}{q^r}\right) \prod_{\phi \text{ irred.} \atop \phi \equiv \lambda \pmod{5}} \left(1 - \frac{1}{q^r - 1}\right).
$$

\[\square\]

**Theorem 5.** The $n \to \infty$ limiting probability that an element of $\text{Mat}(n, q)$ is semisimple is

$$
\prod_{r=1}^{\infty} \left(1 - \frac{1}{q^r - 1}\right).
$$

**Proof.** The orbits of $GL(n, q)$ on $\text{Mat}(n, q)$ under conjugation are also parameterized by the data $\lambda_\phi$. However the polynomial $z$ may appear with non-zero multiplicity, so the restriction that $|\lambda_z| = 0$ does not apply. Stong [Sto] obtained the generating function equality

$$
1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{\alpha \in \text{Mat}(n, q) \atop \phi \text{ irred.}} \prod_{\phi \text{ irred.}} X_{\phi, \lambda_\phi}(\alpha)
= \prod_{\phi \text{ irred.}} \left[ \sum_{\lambda} \frac{X_{\phi, \lambda} \cdot u^{\text{deg}(\phi)} \prod_{i=1}^{n_i(\lambda)} (q^{\text{deg}(\phi)} - q^{\text{deg}(\phi)}(d_i - k))}{\prod_{i=1}^{n_i(\lambda)} (q^{\text{deg}(\phi)} - q^{\text{deg}(\phi)}(d_i - k))} \right],
$$

where $d_i(\lambda)$ is defined by

$$
d_i(\lambda) = 1m_1(\lambda) + 2m_2(\lambda) + \cdots + (i - 1)m_{i-1}(\lambda) + i(m_i(\lambda) + m_{i+1}(\lambda) + \cdots + m_j(\lambda)).
$$

Manipulations identical to those in Theorem 3 show that

$$
1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{\alpha \in \text{Mat}(n, q) \atop \phi \text{ irred.}} \prod_{\phi \text{ irred.}} X_{\phi, \lambda_\phi}(\alpha)
= \frac{1}{1 - u} \prod_{\phi \text{ irred.}} \left[ \sum_{\lambda} \frac{X_{\phi, \lambda} \cdot u^{\text{deg}(\phi)} \prod_{i=1}^{n_i(\lambda)} (q^{\text{deg}(\phi)} - q^{\text{deg}(\phi)}(d_i - k))}{\prod_{i=1}^{n_i(\lambda)} (q^{\text{deg}(\phi)} - q^{\text{deg}(\phi)}(d_i - k))} \right].
$$
Arguing as in Theorem 4 shows that the \( n \to \infty \) limit of the chance that an element of \( \text{Mat}(n, q) \) is semisimple is

\[
\lim_{n \to \infty} \frac{|GL(n, q)|}{q^{n^2}} [u^n] \prod_{\phi \text{ irred.}} \sum_{\lambda : 1 \leq |\lambda|} u^{\deg(\phi)|\lambda|} q^{\deg(\phi) \sum_i (\lambda_i)^2} \prod_i (\frac{1}{q^{\deg(\phi)}}) m_i(\lambda)
\]

\[
= \lim_{n \to \infty} \frac{|GL(n, q)|}{q^{n^2}} [u^n] \frac{1}{1-u} \prod_{r=1}^{\infty} \left( 1 - \frac{1}{q^{r \deg(\phi)}} \right) \prod_{\phi \text{ irred.}} \sum_{\lambda : 1 \leq |\lambda|} q^{\deg(\phi) |\lambda|} \prod_i (\frac{1}{q^{\deg(\phi)}}) m_i(\lambda)
\]

\[
= \prod_{\phi \text{ irred.}} \prod_{r=1}^{\infty} (1 - \frac{1}{q^{r \deg(\phi)}}) \sum_{\lambda : 1 \leq |\lambda|} q^{\deg(\phi) \sum_i (\lambda_i)^2} \prod_i (\frac{1}{q^{\deg(\phi)}}) m_i(\lambda)
\]

Theorems 2 and 3 suggest a connection between the Hall-Littlewood polynomials and the finite general linear groups. Theorem 6 makes this connection precise.

**Theorem 6.** For \( q \) a prime power and \( u \) a formal variable,

\[
(1-u) \left[ 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \prod_{\phi \text{ irred.}} \prod_{\lambda} X_{\phi, \lambda}(\alpha) \right]
\]

\[
= \prod_{\phi \text{ irred.}} \prod_{r=1}^{\infty} (1 - \frac{1}{q^{r \deg(\phi)}}) \left[ P_n \left( \frac{u}{q} \deg(\phi), \frac{u}{q} \deg(\phi)^2, \frac{u}{q} \deg(\phi)^3, \ldots ; \frac{1}{q} \deg(\phi) \right) \right].
\]

**Proof.** The proof of Theorem 3 contained the equality

\[
(1-u) \left[ 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \prod_{\phi \text{ irred.}} \prod_{\lambda} X_{\phi, \lambda}(\alpha) \right]
\]

\[
= \prod_{\phi \text{ irred.}} \prod_{r=1}^{\infty} (1 - \frac{1}{q^{r \deg(\phi)}}) \left[ \sum_{\lambda} X_{\phi, \lambda} q^{\deg(\phi) \sum_i (\lambda_i)^2} \prod_i (\frac{1}{q^{\deg(\phi)}}) m_i(\lambda) \right].
\]

The theorem now follows from the use of Macdonald’s principal specialization formula (as in Theorem 2) to conclude that

\[
\frac{P_n(u, u, u, u, \ldots ; \frac{1}{q})}{q^{n(\lambda)}} = \frac{u^{\lambda}}{q^{\sum_i (\lambda_i)^2} \prod_i (\frac{1}{q}) m_i(\lambda)}.
\]

**Remark.** Although Theorem 6 followed easily from techniques in Theorems 2 and 3, its formulation admits an attractive probabilistic interpretation. Fix \( u \) such that \( 0 < u < 1 \) and fix an irreducible monic polynomial \( \phi \neq z \). Then pick a natural number with probability of getting \( n \) equal to \((1-u)u^n\). Finally, choose \( \alpha \) uniformly at random in \( GL(n, q) \) and let \( \lambda_\phi(\alpha) \) be the random partition so defined. This procedure induces a probability measure on the set of all partitions of all integers.
Theorem 6 implies that measures obtained for different $\phi$ are independent, and that the mass the measure for a given $\phi$ assigns to a partition $\lambda$ is equal to

$$\prod_{r=1}^{\infty} \left(1 - \frac{u^{\deg(\phi)}}{q^{r^{\deg(\phi)}}}\right) \frac{P_{\lambda}\left(\left(\frac{u}{q}\right)^{\deg(\phi)}, \left(\frac{u}{q}\right)^{2\deg(\phi)}, \ldots ; \frac{1}{q} \deg(\phi)\right)}{q^{n(\lambda)\deg(\phi)}}.$$

This connection with symmetric functions leads to a probabilistic algorithm for growing the random partitions $\lambda_\phi$. This viewpoint is used in [Fu1] to give probabilistic proofs of some group theoretic results of Steinberg, Rudvalis/Shinoda, and Lusztig. Cycle indices for the finite unitary, symplectic, and orthogonal groups appear in [Fu2].

3. Conclusion

This paper has offered connections between Gordon’s generalization of the Rogers-Ramanujan identities, the Hall-Littlewood polynomials, and generating functions which arise in the study of the finite general linear groups. The following questions seem natural.

- Are there analogs of the Gordon identities for the finite unitary, symplectic, and orthogonal groups? The Gordon identities do play a role in Misra’s work on affine symplectic Lie algebras [Mi1] and in Mandia’s work [Man] on the affine Lie algebras $B(l)^{(1)}_1$, $F(1)_4$, and $G(1)_2$. Misra has also used $Z$-algebra methods to relate the Gordon identities to $A_n$.
- Ian Macdonald has suggested that the results of this paper should carry over to the affine finite general linear groups. The Hall-Littlewood polynomials have affine analogs [EK].
- It is known (e.g. Kac [Ka]) that the product side of the Rogers-Ramanujan identities has interesting modular properties. Can this phenomenon be understood group theoretically in terms of the conjugacy classes of the general linear groups; i.e. is there “general linear group moonshine”? In this regard note that Jing [J1, J2] has connected the Hall-Littlewood polynomials with vertex operators.

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THE ROGERS-RAMANUJAN IDENTITIES


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