PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 2, Pages 407–411
S 0002-9939(99)05020-0
Article electronically published on July 6, 1999

ASYMPTOTIC BEHAVIOUR OF CASTELNUOVO-MUMFORD REGULARITY

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(Communicated by Wolmer V. Vasconcelos)

Abstract. Let S be a polynomial ring over a field. For a graded S-module generated in degree at most P, the Castelnuovo-Mumford regularity of each of (i) its n^{th} symmetric power, (ii) its n^{th} torsion-free symmetric power and (iii) the integral closure of its n^{th} torsion-free symmetric power is bounded above by a linear function in n with leading coefficient at most P. For a graded ideal I of S, the regularity of I^n is given by a linear function of n for all sufficiently large n. The leading coefficient of this function is identified.

Let $S = k[x_1, \cdots, x_d]$ be a polynomial ring over a field k with its usual grading, i.e., each x_i has degree 1, and let $\mathfrak{m}$ denote the maximal graded ideal of S. Let N be a finitely generated non-zero graded S-module. The Castelnuovo-Mumford regularity of N, denoted $\operatorname{reg}(N)$, is defined to be the least integer m so that, for every j, the j^{th} syzygy of N is generated in degrees $\leq m + j$. By Hilbert's syzygy theorem, N has a graded free resolution over S of the form

$$0 \to F_k \to \cdots \to F_1 \to F_0 \to N \to 0$$

where $F_i = \bigoplus_{j=1}^{t_i} S(-a_{ij})$ for some integers a_{ij} — which we will refer to as the twists of F_i. Then, $\operatorname{reg}(N) \leq \max_{i,j}\{a_{ij} - i\}$ with equality holding if the resolution is minimal. For other equivalent definitions and properties of this invariant, see [Snb].

For a graded ideal I in S, the behaviour of the regularity of I^n as a function of n has been of some interest. If I defines a smooth complex projective variety, it is shown in [BrtEinLzr, Proposition 1] using the Kawamata-Viehweg vanishing theorem that $\operatorname{reg}(I^n) \leq Pn + Q$ where P is the maximal degree of a minimal generator of I and Q is a constant expressed in terms of the degrees of generators of I. In [GrmGmgPtt, Theorem 1.1] and in [Chn, Theorem 1] it is shown that if $\dim(R/I) \leq 1$, then $\operatorname{reg}(I^n) \leq n.\operatorname{reg}(I)$ for all $n \in \mathbb{N}$. In [Chn, Conjecture 1], this is conjectured to be true for an arbitrary graded ideal. Supporting this conjecture is the result of [Swn, Theorem 3.6] that $\operatorname{reg}(I^n) \leq Pn$ for some constant P and for all $n \in \mathbb{N}$. The method of proof makes it difficult to explicitly identify such a constant. For monomial ideals, such a P is explicitly calculated in [SmtSwn, Theorem 3.1] and improved upon in [HoaTrn, Corollary 3.2].

We show that with S and N as above, the regularity of $\operatorname{Sym}_n(N)$ — and related modules — is bounded above by a linear function of n with leading coefficient at

Received by the editors October 28, 1997 and, in revised form, April 15, 1998.
1991 *Mathematics Subject Classification.* Primary 13D02; Secondary 13D40.

most the maximal degree of a minimal generator of N. For a graded ideal I of S, we get the sharper result that $\mathrm{reg}(I^n)$ is actually given by a linear function of n for all sufficiently large n. The leading coefficient of this function is identified as a certain invariant $\rho(I)$ of I. This verifies Chandler's conjecture in the case $\mathrm{reg}(I) > \rho(I)$ and for all n sufficiently large.

One of the main ingredients of the proof is an analysis of a bigrading ($= \mathbb{N}^2$ grading) on the Rees ring, $S[It]$, of a graded ideal $I \subseteq S$. This is defined by decreeing an element of $S[It]$ to be homogeneous of bidegree (p, n) if it is of the form ft^n where f is a homogeneous element of degree p in S. For a recent application of this bigrading, see [CncHrzTrnVll].

Suppose that I is generated minimally by homogeneous elements $f_1, \cdots, f_k$ in S of degrees $p_1, \cdots, p_k$ respectively. Let $R = k[X_1, \cdots, X_d, T_1, \cdots, T_k]$ with bigrading defined by $\deg(X_i) = (1, 0)$ and $\deg(T_j) = (p_j, 1)$. The natural map $R \to S[It]$ defined by $X_i \mapsto x_i$ and $T_j \mapsto f_j t$ is then a surjective homomorphism of bigraded rings. In particular, $S[It]$ is a cyclic bigraded R-module.

For a bigraded R-module $M = \bigoplus_{p,n\in\mathbb{N}} M_{(p,n)}$, define $M^{(n)}$ to be the graded S-module $\bigoplus_{p\in\mathbb{N}} M_{(p,n)}$ where x_i acts as X_i with its obvious grading. Note that $S[It]^{(n)} \cong I^n$. The assignment $M \mapsto M^{(n)}$ is an exact functor. For $a, b \in \mathbb{N}$, define the twisted module $M(-a, -b)$ by $M(-a, -b)_{(p,n)} = M_{(p-a,n-b)}$. The crucial observation used in the proof is that

$$R(-a, -b)^{(n)} \cong R^{(n-b)}(-a) \cong \bigoplus_{l_1+\cdots+l_k=n-b} S(-l_1p_1 - \cdots - l_kp_k - a)$$

as graded S-modules.

Theorem 1. *Let k be a field and $S = k[x_1, \cdots, x_d]$ graded as usual. Let $R = k[X_1, \cdots, X_d, T_1, \cdots, T_k]$ with bigrading defined by $\deg(X_i) = (1, 0)$ and $\deg(T_j) = (p_j, 1)$ for some $p_j \in \mathbb{N}$. For a finitely generated bigraded R-module M, there exists a constant Q so that $reg(M^{(n)}) \leq Pn+Q$ for all $n \geq 1$ where $P = \max\{p_1, \cdots, p_k\}$.*

Proof. By a bigraded version of Hilbert's syzygy theorem, the R-module M has a bigraded free resolution of the form

$$0 \to F_k \to \cdots \to F_1 \to F_0 \to M \to 0$$

where $F_i = \bigoplus_{j=1}^{t_i} R(-a_{ij}, -b_{ij})$.

Applying the functor $(\ \cdot\ )^{(n)}$ to this resolution yields a graded free S-resolution of $M^{(n)}$ from which an upper bound on its regularity can be read off. Since

$$F_i^{(n)} \cong \bigoplus_{j=1}^{t_i} \bigoplus_{l_1+\cdots+l_k=n-b_{ij}} S(-l_1p_1 - \cdots - l_kp_k - a_{ij}),$$

the maximal twist in F_i is $\max_j\{P(n - b_{ij}) + a_{ij}\}$ where $P = \max\{p_1, \cdots, p_k\}$. Hence $\mathrm{reg}(M^{(n)}) \leq Pn + Q$ with $Q = \max_{i,j}\{a_{ij} - Pb_{ij} - i\}$. □

As a matter of notation, for a graded S-module N, by $\theta(N)$ we will denote the maximal degree of a minimal generator of N. Equivalently, $\theta(N) = \mathrm{reg}(N/\mathfrak{m}N)$. For the definition and properties of integral closures of modules, see [Res].

Corollary 2. *Let $S = k[x_1, \cdots, x_d]$ and N be a finitely generated graded S-module with $\theta(N) = P$. Let $F_n(N)$ denote any one of:*

(1) $\mathrm{Sym}_n(N)$ *— the nth symmetric power of N.*

(2) $\underline{\mathrm{S}_n(N)}$ — *the nth symmetric power of N modulo S-torsion.*
(3) $\overline{(\mathrm{S}_n(N))}$ — *the integral closure of the module* $\mathrm{S}_n(N)$.

Then, there exists Q *so that* $reg(F_n(N)) \le Pn + Q$ *for all* $n \in \mathbb{N}$.

Proof. Let N be generated by minimal generators in degrees $p_1 \le \cdots \le p_k = P$ and let $R = k[X_1, \cdots, X_d, T_1, \cdots, T_k]$ bigraded as before. Then, $M = \bigoplus_{n\in\mathbb{N}} F_n(N)$ is naturally a finitely generated, bigraded R-module with $M^{(n)} \cong F_n(N)$. Now appeal to Theorem 1. □

Recall that an ideal $J \subseteq I$ is said to be a reduction of I if for some $n \in \mathbb{N}$ we have that $I^n = JI^{n-1}$. We will denote by $\rho(I)$ the minimum of $\theta(J)$ over all graded reductions J of I. Clearly, $\mathrm{reg}(I) \ge \theta(I) \ge \rho(I)$.

Corollary 3. *Let* I *be a graded ideal of* $S = k[x_1, \cdots, x_d]$ *and let* $P = \rho(I)$. *Then there exists a constant* Q *so that* $\mathrm{reg}(I^n) \le Pn + Q$ *for all* $n \in \mathbb{N}$.

Proof. Let J be a graded reduction of I with $\theta(J) = P$. As above, map a bigraded polynomial ring R onto $S[Jt]$. Since J is a reduction of I, $S[It]$ is a finitely generated bigraded $S[Jt]$-module and hence also a finitely generated bigraded R-module. Apply Theorem 1 to this module. □

In order to improve the inequality of the corollary to an asymptotic equality, we first linearly bound $\mathrm{reg}(I^n)$ below by simply bounding $\theta(I^n)$.

Proposition 4. *Let* I *be a graded ideal of* $S = k[x_1, \cdots, x_d]$ *and let* $P = \rho(I)$. *Then* $\theta(I^n) \ge Pn$ *for all* $n \in \mathbb{N}$.

Proof. Let $p \in \mathbb{N}$ be largest so that there exists $f \in I$ of degree p with $f^n \notin \mathfrak{m}I^n$ for all $n \in \mathbb{N}$. Hence I^n has a minimal generator in degree pn for every n and so $\theta(I^n) \ge pn$ for all $n \in \mathbb{N}$. It suffices to show that $p \ge P$ or equivalently that there exists a graded reduction J of I with $\theta(J) \le p$.

Choose a minimal generating set $f_1, \cdots, f_k$ of I of degrees $p_1 \le \cdots \le p_k$ respectively so that $f_j^n \notin \mathfrak{m}I^n$ for all n and $p_i > p_j = p$ for $i > j$. Set $J = (f_1, \cdots, f_j)$ and $K = (f_{j+1}, \cdots, f_k)$. Clearly J is a graded ideal with $\theta(J) = p$ and we claim that J is a reduction of I. This will complete the proof.

From the definition of p it follows easily that, for some $n \in \mathbb{N}$, $K^n \subseteq \mathfrak{m}I^n$. Then $I^n = (J + K)^n = J(J + K)^{n-1} + K^n \subseteq JI^{n-1} + \mathfrak{m}I^n$. By Nakayama's lemma, it follows that J is a reduction of I. □

Theorem 5. *Let* I *be a graded ideal of* $S = k[x_1, \cdots, x_d]$ *and let* $P = \rho(I)$. *Then there exists a constant* $M \in \mathbb{N}$ *so that, for all* $n \in \mathbb{N}$ *sufficiently large,* $\mathrm{reg}(I^n) = Pn + M$.

Proof. By Proposition 4, $\mathrm{reg}(I^n) = Pn + Q_n$ for non-negative integers Q_n. We will show that the sequence Q_n is eventually constant.

Choose a graded reduction $J = (f_1, \cdots, f_k)$ of I where f_i is homogeneous of degree p_i and $\theta(J) = P$. Let R be the bigraded ring $k[X_1, \cdots, X_d, T_1, \cdots, T_k]$ mapping onto $S[Jt]$ as before. Consider the Koszul complex of the bigraded R-module $S[It]$ with respect to $T_1, \cdots, T_k$. All homology modules of this complex are annihilated by a power of $(T_1, \cdots, T_k)$ and hence, for all sufficiently large n, applying the functor $(\,\cdot\,)^{(n)}$ yields an exact complex of graded S-modules:

$$0 \to I^{n-k}(-p_1 - p_2 - \cdots - p_k) \to \cdots \to I^{n-1}(-p_1) \oplus \cdots \oplus I^{n-1}(-p_k) \to I^n \to 0.$$

This complex may be used to construct a resolution of I^n given resolutions of $I^{n-1}, \cdots, I^{n-k}$ and it follows from this construction that, for all n sufficiently large, $\mathrm{reg}(I^n) \leq \max\{\mathrm{reg}(I^{n-1}) + \max_i\{p_i\}, \mathrm{reg}(I^{n-2}) + \max_{i<j}\{p_i+p_j\} - 1, \cdots, \mathrm{reg}(I^{n-k}) + p_1 + \cdots + p_k - (k-1)\}$. Since $P = \max_i\{p_i\}, 2P \geq \max_{i<j}\{p_i + p_j\}$ etc., this implies that $Q_n \leq \max\{Q_{n-1}, Q_{n-2} - 1, \cdots, Q_{n-k} - (k-1)\}$.

For $n > k$, define $M_n = \max\{Q_{n-1}, \cdots, Q_{n-k}\}$. Then for all sufficiently large n, the sequence M_n is a non-increasing sequence of non-negative integers and therefore eventually constant with value, say, M. The sequence Q_n is bounded above for all large n by M. For sufficiently large n, if some $Q_n < M$, it follows that all successive Q's are also less than M. But then, M_n would also be less than M for all large n. The contradiction shows that the sequence Q_n is also eventually constant with value M. □

Remarks. (1) The theorem should be compared with [BrtEinLzr, Proposition 1] and its refinements in [Brt]. Explicit determination of a Q as in Corollary 3 seems to involve fairly subtle techniques. On the other hand, it may be possible to find the M of Theorem 5 in the spirit of the methods of this paper.

(2) Following [SnbGot], say that a graded ideal I has a linear resolution if its regularity is equal to the degree of each of its minimal generators. In an earlier version of this paper I had the following proof that Chandler's conjecture is equivalent to the statement: If I has a linear resolution, so do all powers of I.

Proof. Clearly, Chandler's conjecture implies that statement. Conversely suppose that powers of ideals with linear resolutions also have linear resolutions. Let I be an arbitrary graded ideal of S with $\mathrm{reg}(I) = r$. By [SnbGot, Proposition 1.1], the graded ideal $I_{\geq r}$ ($= I \cap \mathfrak{m}^r$) has a linear resolution. Hence, so does $(I_{\geq r})^n$ for all $n \in \mathbb{N}$. Since $r \geq \theta(I)$, $(I_{\geq r})^n = I^n \cap \mathfrak{m}^{rn}$. By [SnbGot, Proposition 1.1] again, $\mathrm{reg}(I^n) \leq rn = n.\,\mathrm{reg}(I)$. □

Subsequently, I was made aware of an example, attributed to Terai in [Cnc], of a monomial ideal with linear resolution whose square does not have a linear resolution in characteristic different from 2. Thus, Chandler's conjecture is false. A recent preprint of Bernd Sturmfels [Str] gives a characteristic free such example.

(3) The main result of this paper has been independently obtained in [CtkHrzTrn]. This paper also studies the regularity of the saturations of powers of an ideal.

(4) The referee has suggested that the bound

$$\mathrm{reg}(I^n) \leq \max\{\mathrm{reg}(I^{n-1}) + \max_i\{p_i\}, \cdots, \mathrm{reg}(I^{n-k}) + p_1 + \cdots + p_k - (k-1)\},$$

in the proof of Theorem 5 follows easily by inductively applying the lemma: If $0 \to A \to B \to C \to 0$ is an exact sequence, then $\mathrm{reg}(C) \leq \max\{\mathrm{reg}(B), \mathrm{reg}(A) - 1\}$.

Acknowledgements

I am very grateful to Irena Swanson for her preprint which kindled my interest in this question. I also thank Karen Chandler for her preprint, Dan Katz for his lectures on regularity, Radha Mohan and Shreedhar Inamdar for helpful conversations and the referee for a meticulous reading.

References

[BrtEinLzr] Aaron Bertram, Lawrence Ein, Robert Lazarsfeld, *Vanishing theorems, a theorem of Severi, and the equations defining projective varieties*, Journal of the American Mathematical Society **4** (1991), 587-602. MR **92g:**14014

[Brt] Aaron Bertram, *An application of a log version of the Kodaira vanishing theorem to embedded projective varieties*, Preprint alg-geom/9707001.

[Chn] Karen A. Chandler, *Regularity of the powers of an ideal*, Communications in Algebra **25 (12)** (1997), 3773-3776. CMP 98:04

[Cnc] Aldo Conca, *Hilbert function and resolutions of the powers of the ideal of the rational normal curve*, Preprint.

[CncHrzTrnVll] Aldo Conca, Jurgen Herzog, Ngo Viet Trung, Giuseppe Valla, *Diagonal subalgebras of bigraded algebras and embeddings of blow-ups of projective spaces*, American Journal of Mathematics **119 (4)** (1997), 859-902. CMP 97:17

[CtkHrzTrn] Dale Cutkosky, Jurgen Herzog, Ngo Viet Trung, *Asymptotic behaviour of the Catelnuovo-Mumford regularity*, Preprint.

[Snb] David Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, 1995. MR **97a:**13001

[SnbGot] David Eisenbud, Shiro Goto, *Linear free resolutions and minimal multiplicity*, Journal of Algebra **88** (1984), 89-133. MR **85f:**13023

[GrmGmgPtt] A. V. Geramita, A. Gimigliano, Y. Pitteloud, *Graded Betti numbers of some embedded rational n-folds*, Mathematische Annalen **301** (1995), 363-380. MR **96f:**13022

[HoaTrn] Le Tuan Hoa, Ngo Viet Trung, *On the Castelnuovo-Mumford regularity and the arithmetic degree of monomial ideals*, Mathematische Zeitschrift, To appear.

[Res] D. Rees, *Reduction of modules*, Mathematical Proceedings of the Cambridge Philosophical Society **101** (1987), 431-448. MR **88a:**13001

[SmtSwn] Karen E. Smith, Irena Swanson, *Linear bounds on growth of associated primes*, Communications in Algebra, **25** (1997), 3071-3079. CMP 97:17

[Str] Bernd Sturmfels, *Four counterexamples in combinatorial algebraic geometry*, Preprint.

[Swn] Irena Swanson, *Powers of ideals: Primary decompositions, Artin-Rees lemma and regularity*, Mathematische Annalen **307** (1997), 299-313. MR **97j:**13005

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