EVERY \((\lambda^+, \kappa^+)\)-REGULAR ULTRAFILTER IS \((\lambda, \kappa)\)-REGULAR

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Abstract. We prove the following:

Theorem A. If \(D\) is a \((\lambda^+, \kappa^+)\)-regular ultrafilter, then either
(a) \(D\) is \((\lambda, \kappa)\)-regular, or
(b) the cofinality of the linear order \(\prod D \langle \lambda, < \rangle\) is \(\text{cf} \kappa\), and \(D\) is \((\lambda, \kappa')\)-regular
for all \(\kappa' < \kappa\).

Corollary B. Suppose that 
(\(\kappa\) is singular, \(\kappa > \lambda\) and either \(\lambda\) is regular, or \(\text{cf} \kappa < \text{cf} \lambda\). Then every \((\lambda^{+n}, \kappa)\)-regular ultrafilter is \((\lambda, \kappa)\)-regular.

We also discuss some consequences and variations.

The notion of a \((\lambda, \kappa)\)-regular ultrafilter has been introduced by J. Keisler in [Kei]. An ultrafilter \(D\) is \((\lambda, \kappa)\)-regular iff there is a family of \(\kappa\) members of \(D\) such that the intersection of any \(\lambda\) members of the family is empty.

In [Kei] Keisler proved some cardinality results about ultraproducts taken modulo such ultrafilters. Further results were proved in the 70’s: for example, the following are theorems of ZFC:

(a) Every \((\lambda^+, \lambda^+)\)-regular ultrafilter is \((\lambda, \lambda)\)-regular ([CC], [KP]).
(b) If \(\lambda\) is singular, then every \((\lambda^+, \lambda^+)\)-regular ultrafilter is \((\lambda, \lambda^+)\)-regular [Ka]; moreover, it is either \((\text{cf} \lambda, \text{cf} \lambda)\)-regular or \((\lambda', \lambda^+)\)-regular for some \(\lambda' < \lambda\) ([CC], [KP]).
(c) If 
\(2^{\kappa} = \kappa^+\) and \(2^{\kappa^+} > \kappa^{++}\), then every \((\kappa^+, \kappa^+)\)-regular ultrafilter is \((\kappa, \kappa^+)\)-regular ([BK], [Ket]).

It was soon realized, however, that \((\text{ir})\)regular ultrafilters are connected with large cardinals, inner models, and combinatorial or reflection principles; this paved the way for significant applications to set theory, but seemed to dash any hope that other results besides (a)–(c) above can be proved in ZFC alone (see e.g. [KM] or [Lp1] for further references).

However, in [Lp1] (more than twenty years later) we proved some slight improvements of (b), as well as a “down from exponents” transfer result for \((\lambda, \lambda)\)-regularity; we also suggested the possibility that further results are theorems of ZFC. It is actually so; in this paper we prove the following generalization of (a):

Theorem 1. If \(n < \omega\), then every \((\lambda^{+n}, \kappa^{+n})\)-regular ultrafilter is \((\lambda, \kappa)\)-regular.
Theorem 1 solves a problem raised in [Lp1]. Actually, (the proof of) Theorem 1 has some further consequences:

**Corollary 1.** Suppose that \( \lambda \leq \kappa \), \( \lambda \) is a regular cardinal, and \( D \) is a \((\lambda^+, \kappa)\)-regular ultrafilter. Then the following are equivalent:

(i) \( D \) is \((\lambda, \kappa)\)-regular;
(ii) the cofinality of \( \prod_D \langle \lambda, < \rangle \) is \( \kappa \);
(iii) the cofinality of \( \prod_D \langle \lambda, < \rangle \) is different from \( \text{cf} \kappa \).

**Corollary 2** ([Lp1]). Suppose that \( \text{cf} \lambda \neq \text{cf} \kappa \). If \( D \) is \((\lambda^+, \kappa)\)-regular and not \((\text{cf} \lambda, \text{cf} \lambda)\)-regular, then \( D \) is \((\lambda, \kappa)\)-regular.

**Corollary 3.** If \( D \) is \((\lambda^+, \kappa)\)-regular and \((\text{cf} \lambda, \text{cf} \kappa)\)-regular, then \( D \) is \((\lambda, \kappa)\)-regular.

**Corollary 4.** Suppose that \( D \) is \((\lambda^+, \kappa)\)-regular and \((\lambda^+, \kappa')\)-regular. If \( \text{cf} \lambda = \text{cf} \lambda' \) and \( \text{cf} \kappa \neq \text{cf} \kappa' \), then \( D \) is either \((\lambda, \kappa)\)-regular or \((\lambda', \kappa')\)-regular.

The following improves [BK, Theorem 1.3 and Corollaries 1.8 and 2.5]:

**Proposition.** Suppose that \( D \) is a \((\lambda^+, \kappa)\)-regular ultrafilter, and that the cofinality of the linear order \( \prod_D \langle \lambda, < \rangle \) is different from \( \text{cf} \kappa \). Then \( D \) is \((\lambda, \kappa)\)-regular.

Notice that, trivially, the cofinality of \( \prod_D \langle \lambda, < \rangle \) equals the cofinality of \( \prod_D \langle \text{cf} \lambda, < \rangle \).

In order to discuss some further results, let us introduce some notation and definitions (see [CK] for basics about models and ultrafilters).

\( S_\mu(X) \) denotes the set of all subsets of \( X \) of cardinality \( < \mu \). If \( X \) is a well-ordered set and \( \alpha \) is an ordinal, \( S_\alpha(X) \) is the set of all subsets of \( X \) which have order type \( < \alpha \) (notice that the two definitions are consistent).

Throughout this paper, we shall use the following reformulation of \((\lambda, \kappa)\)-regularity. As Benda and Ketonen [BK] noticed, it makes sense even when \( \lambda \) is allowed to be an ordinal: the next corollary deals with such a situation. See [Lp1] for further comments.

An ultrafilter \( D \) is \((\beta, \kappa)\)-regular iff in the ultrapower \( \prod_D \langle S_\beta(\kappa), \subseteq, \{ \alpha \} \rangle \) there is an element \( x \) such that \( \{ \alpha \} \subseteq x \), for every \( \alpha \in \kappa \) (equivalently, for \( \kappa \)-many \( \alpha \)-s).

**Corollary 5.** Let \( D \) be an ultrafilter, and let \( \kappa \) be a cardinal. Let \( \gamma \) be the least ordinal such that \( D \) is \((\gamma, \kappa)\)-regular. Then either:

(i) \( \gamma \) is a limit ordinal and \( D \) is \((\text{cf} \gamma, \text{cf} \gamma)\)-regular, or
(ii) \( \gamma = \delta + 1 \), \( \delta \) is a limit ordinal and the cofinality of \( \prod_D \langle \delta, < \rangle \) is \( \text{cf} \kappa \). Hence either \( D \) is \((\text{cf} \delta, \text{cf} \delta)\)-regular and \( \text{cf} \kappa > \text{cf} \delta \), or \( \text{cf} \kappa = \text{cf} \delta \) and \( D \) is not \((\text{cf} \delta, \text{cf} \delta)\)-regular.

**Theorem 2.** There is a finite expansion \( A^+ \) of the model \( \langle S_{\lambda+n}(\mu+n), V, \subseteq, \{ \alpha \} \rangle_{\alpha \in \mu+n} \) (where \( V(a) \iff a \in S_\lambda(\mu) \)) such that whenever \( B \equiv A^+ \) and there is \( b \in B \) such that \( |\{ \alpha \in \mu+n \mid B \models \{ \alpha \} \subseteq b \}| = \mu+n \), then there is \( b' \in B \) such that \( V(b') \) and \( |\{ \alpha \in \mu \mid B \models \{ \alpha \} \subseteq b' \}| = \mu \).

The statement of Theorem 2 reads \( \text{alm}(\lambda+n, \mu+n) \Rightarrow \text{alm}(\lambda, \mu) \), in the notation of [Lp2]. There we also discuss the interest of such results. Notice that, in view of the reformulation of \((\lambda, \mu)\)-regularity we have given, Theorem 2 is stronger than Theorem 1.
Finally, let us mention that \((\lambda, \kappa)-regular\) ultrafilters have found other applications both in general topology and in the model theory for abstract logics (see [\text{Lp1}, \text{Lp2}] for further references).

A notice to the reader wanting to consult the literature on regular ultrafilters: results like (a)–(c) above are sometimes stated in equivalent forms in terms of the closely related notions of \(\lambda\)-decomposability, \(\lambda\)-descending incompleteness and uniformity. We shall not need those notions in the present paper, but, for the convenience of the reader, we give below a "table of correspondence".

For every cardinal \(\lambda\) and every ultrafilter \(D\), the following hold:

(i) \(D\) is \(\lambda\)-descendingly incomplete iff it is \((\text{cf}\, \lambda, \text{cf}\, \lambda)\)-regular.
(ii) If \(D\) is \(\lambda\)-decomposable, then \(D\) is \((\text{cf}\, \lambda, \text{cf}\, \lambda)\)-regular.
(iii) If \(D\) is \((\text{cf}\, \lambda, \text{cf}\, \lambda)\)-regular, then \(D\) is \((\lambda, \lambda)\)-regular.
(iv) \(D\) is \(\lambda\)-decomposable iff there is a \(D'\) uniform on \(\lambda\) and \(D' \leq D\) in the Rudin Keisler (pre)-order.

Moreover, if \(\lambda\) is regular, then,

(v) if \(D\) is \((\lambda, \lambda)\)-regular, then \(D\) is \(\lambda\)-decomposable.

By the above statements, for \(\lambda\) a regular cardinal, the notions of \((\lambda, \lambda)\)-regularity, \(\lambda\)-descending incompleteness and \(\lambda\)-decomposability are all equivalent. In particular, the notion of \((\lambda, \kappa)\)-regularity encompasses all the above notions, except for \(\lambda\)-decomposability when \(\lambda\) is singular.

Moreover, it is trivial that if \(D' \leq D\) in the Rudin Keisler order, and \(D'\) is \((\lambda, \kappa)\)-regular, then \(D\) is \((\lambda, \kappa)\)-regular, also.

These facts make it possible, for example, to reformulate (a) together with the second statement in (b) as: every uniform ultrafilter over \(\lambda^+\) is either \(\lambda\)-descendingly incomplete or \((\lambda', \lambda^+)\)-regular for some \(\lambda' < \lambda\).

We now prove the stated results. The statement in the title of the paper is a consequence of Theorem A in the abstract (when \(\kappa\) is a successor cardinal). Theorem 1 follows now from a simple induction. Corollary 4 is immediate from the proposition.

**Proof of the Proposition.** For every \(x \in S_{\lambda^+}(\kappa)\), let \(G(x, \beta) (\beta < \lambda)\) be a sequence of subsets of \(x\) such that:

(i) if \(\beta \leq \beta' < \lambda\), then \(G(x, \beta) \subseteq G(x, \beta')\);
(ii) \(|G(x, \beta)| \leq |\beta|\), for every \(\beta < \lambda\);
(iii) \(\bigcup_{\beta < \lambda} G(x, \beta) = x\).

Consider the model \(A = \langle S_{\lambda^+}(\kappa), \subseteq, U, <, G, \{\alpha\}_{\alpha \in \kappa} \rangle\), where \(\langle U, < \rangle = \langle \lambda, < \rangle\) and \(G\) is the above function from \(A \times U\) to \(A\).

Thus, by (iii) above, for every \(\alpha \in \kappa\), \(A\) satisfies

\[\forall x (\{\alpha\} \subseteq x \Rightarrow \exists w (U(w) \wedge \{\alpha\} \subseteq G(x, w))).\]

Let \(D\) be a \((\lambda^+, \kappa)\)-regular ultrafilter; let us work in \(B = \prod_D A\); and let \(y\) witness the \((\lambda^+, \kappa)\)-regularity of \(D\). By (\(\ast\)), and since \(y\) witnesses that \(D\) is \((\lambda^+, \kappa)\)-regular, for every \(\alpha \in \kappa\), there is a \(w_\alpha\) in \(B\) such that, in \(B\), \(\{\alpha\} \subseteq G(y, w_\alpha)\).

For every \(w \in B\), let \(X_w = \{\alpha \mid B \models \{\alpha\} \subseteq G(y, w)\}\). Notice that:

(I) The union of all the \(X_w\)'s \((w \in B\rangle\) is \(\kappa\), by the above argument.
(II) If \(w \leq w' \in B\), then \(X_w \subseteq X_{w'}\), because of clause (i) (and elementarity).
It is enough to show that there is \( w \in B \mid U \) such that \( |X_w| = \kappa \). If this happens, then \( G(y, w) \) witnesses the \((\mu, \kappa)\)-regularity of \( D \) (because of (ii)), \( G(y, w) \) is in \( \prod_D S_\lambda(\kappa) \).

Suppose by contradiction that

\[(\text{III}) \ |X_w| < \kappa, \text{ for all } w \in B \mid U.\]

Easy cofinality arguments then show that (I)–(III) imply that \( \text{cf}(B \mid U, <) = \kappa \), contradicting the hypothesis.

**Proof of Theorem A.** If (a) fails, then the proposition implies that \( \text{cf} \prod_D (\lambda, <) \) is \( \kappa \).

If \( \kappa' < \kappa \), then \( D \) is trivially \((\lambda^+, \kappa')\)-regular. If \( \kappa' \neq \kappa \), then the proposition (applied to \((\lambda^+, \kappa')\)-regularity) implies that \( D \) is \((\lambda, \kappa')\)-regular.

But this is enough, since for every \( \kappa'' < \kappa \) there is \( \kappa' \) such that \( \kappa' \neq \kappa \) and \( \kappa'' < \kappa \).

**Proof of Corollary 1.** (i)\( \Rightarrow \) (ii) is standard (e.g. [BK, p. 233]).

(ii)\( \Rightarrow \) (iii) is trivial.

(iii)\( \Rightarrow \) (i) is immediate from the proposition.

**Remark.** The hypothesis \( \lambda \) regular is necessary in Corollary 1. If \( \mu \) is strongly compact, then there exists a \( \mu \)-complete \((\mu, \mu^{+\omega+\omega})\)-regular ultrafilter \( D \). If we take \( \lambda = \mu^{+\omega} \) and \( \kappa = \mu^{+\omega+\omega} \), then condition (i) holds, but (ii) and (iii) fail.

We wonder whether some generalization of Corollary 1 is possible when \( \lambda \) is singular.

Notice that Corollary 1 and Theorem A imply that if \( \lambda \) is regular, and \( D \) is \((\lambda^+, \kappa)\)-regular, then \( \text{cf} \prod_D (\lambda, <) \geq \kappa \).

**Proof of Corollary B.** An induction on \( n \) shows that it is enough to prove the particular case when \( n = 1 \). So assume that \( \kappa \) is singular, \( \kappa > \lambda \) and \( D \) is \((\lambda^+, \kappa)\)-regular.

Case 1: \( \lambda \) is regular. By Theorem A, \( D \) is \((\lambda, \kappa')\)-regular for all \( \kappa' < \kappa \). By, e.g., Corollary 1, the cofinality of \( \prod_D (\lambda, <) \) is \( > \kappa' \), for all \( \kappa' < \kappa \). Since \( \kappa \) is singular, the cofinality of \( \prod_D (\lambda, <) \) must be \( > \kappa \); in particular, it is \( \neq \text{cf} \kappa \). Hence, case (b) in Theorem A cannot occur.

Case 2: \( \text{cf} \kappa < \text{cf} \lambda \). Then case (b) in Theorem A cannot occur, since the cofinality of \( \prod_D (\lambda, <) \) is \( \text{cf} \lambda \) if \( D \) is not \((\text{cf} \lambda, \text{cf} \lambda)\)-regular, and is \( > \text{cf} \lambda \) otherwise. Hence, we are in case (a) and we are done.

**Proof of Corollary 2.** \( D \) is \((\text{cf} \lambda, \text{cf} \lambda)\)-regular iff the cofinality of \( \prod_D (\text{cf} \lambda, <) \) is different from \( \text{cf} \lambda \) (e.g., by Corollary 1). So it is enough to apply the proposition.

**Proof of Corollary 3.** If \( \kappa \leq \lambda \), the result is trivial. The case \( \kappa < \text{cf} \lambda \) is covered by Corollary B. Otherwise, by Corollary 1, the cofinality of \( \prod_D (\text{cf} \lambda, <) \) is \( > \kappa \).

Now apply the proposition.

**Proof of Theorem 2.** It is enough to prove the case \( n = 1 \).

The proofs of the proposition and of Theorem A have been devised to work as a proof for Theorem 2.

Add to the model \( A \) the relations and functions necessary in order to carry over the proof of the proposition in the two cases \( \kappa = \mu^+ \) and \( \kappa = \mu \). Notice that \( U = \lambda \) is the same in both cases.
Let $B$ be as in the statement of Theorem 2. The cofinality of $\langle B|_U, < \rangle$ is different from $\text{cf} \kappa$ for at least one of the choices $\kappa = \mu^+$ and $\kappa = \mu$. Now it is enough to apply the arguments in the proof of the proposition, for the appropriate choice of $\kappa$.

Of course, a generalization of Theorem A can be given along the lines of the above proof. We can obtain the separation in cases (a) and (b); we can also obtain that, in case (b), the cofinality of $\langle U, < \rangle$ becomes $\text{cf} \kappa$, in $B$ (but notice that, when $\kappa$ is singular, we need $\text{cf} \kappa$ symbols in the expansion $A^+$). Corollaries 2 and 4, too, can be generalized.

We can also generalize Corollaries 1, 3 and B in this fashion, but some technicalities are needed in the proof: we have to add Skolem functions to $A^+$, and consider the substructure of $B$ generated by $b$ (seemingly, $\kappa^+$ symbols are needed in $A^+$).

Proof of Corollary 5. (i) Let $\gamma$ be limit. For every $x \in S_\Gamma(\kappa)$ let $F(x)$ be the order type of $x$. Consider an appropriate expansion of $S_\Gamma(\kappa)$, and take its ultrapower under $D$. If $y$ witnesses $(\gamma, \kappa)$-regularity, then $\beta < F(y)$, for every $\beta < \gamma$, since, otherwise, $D$ would be $(\beta + 1, \kappa)$-regular. This easily implies that $D$ is $(\text{cf} \gamma, \text{cf} \gamma)$-regular.

Essentially, this is the argument used in [Lp2, Corollary 1.4].

(ii) If $\gamma = \delta + 1$, then it is easy to show that $\delta$ is limit.

A variation on the proof of the proposition shows that the cofinality of $\prod_D(\delta, <)$ is $\text{cf} \kappa$. Just consider $U$ to be $\langle \delta, < \rangle$ and for $\beta < \delta$ let $G(x, \beta)$ be the initial subset of $x$ of order type $\beta$. Now observe that the case corresponding to (a) in Theorem A cannot occur since $\gamma$ is the least ordinal for which $D$ is $(\gamma, \kappa)$-regular.

The last remark follows from the fact that the cofinality of $\prod_D(\delta, <)$ is equal to the cofinality of $\prod_D(\text{cf} \delta, <)$.

REFERENCES


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