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MINIMALLY ALMOST PERIODIC TOTALLY DISCONNECTED GROUPS

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Abstract. In this paper we prove that every closed noncompact group G of isometries of a homogeneous tree which acts transitively on the tree boundary contains a normal closed cocompact subgroup G' which is minimally almost periodic. Moreover we prove that G' is a topologically simple group.

1. Introduction

Let X be a homogeneous tree of finite order $q+1 \geq 3$. We denote by $\mathrm{Aut}(X)$ the locally compact group of all isometries of X with respect to the natural distance of X ($d(x,y)$ is the length of the unique geodesic connecting x to y). We refer the reader to [2] for undefined notions and terminology. We fix $x_0 \in X$; then the sets $X^+ = \{x \in X\colon d(x,x_0) \text{ is even}\}$ and $X^- = \{x \in X\colon d(x,x_0) \text{ is odd}\}$ are the equivalence classes of the relation "$d(x,y)$ is an even number". Therefore this partition of X into the sets X^+ and X^- is independent of the choice of x_0. If G is a closed noncompact subgroup of $\mathrm{Aut}(X)$ acting transitively on the tree boundary Ω, then either G acts transitively on X or G has exactly the orbits X^+ and X^- [4, Prop. 2, pg. 143]. In particular if G has two orbits X^+ and X^-, then every closed noncompact subgroup of G acting transitively on Ω has the same orbits of G. A notable example of this type is the subgroup $\mathrm{Aut}^+(X)$ of $\mathrm{Aut}(X)$ generated by all rotations of X. More generally, let G be a closed subgroup of $\mathrm{Aut}(X)$ acting transitively on X and Ω. Then the subgroup G^+ generated by all rotations of G is an open normal subgroup of G of index 2 acting transitively on Ω and having two orbits (X^+ and X^-) on X. In [8] J. Tits has proved that $\mathrm{Aut}^+(X)$ is an algebraically simple group. Furthermore, J. Tits proved that the group G^+ is algebraically simple for a larger class of groups with property (P) (see [8, 4.2, pg. 197]).

Let G be a locally compact group; then G is said to be minimally almost periodic (briefly: m.a.p.) if every finite-dimensional unitary representation is trivial. This is equivalent to the fact that there is no continuous almost periodic function except constant functions.

In the present paper we consider the class $\mathcal{G}$ of all closed subgroups G of $\mathrm{Aut}(X)$ with the following properties: G acts transitively on Ω and G has two orbits on X. We prove that every group $G \in \mathcal{G}$ contains one and only one nontrivial normal closed subgroup $G' \in \mathcal{G}$ which is m.a.p., cocompact and topologically simple. This

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implies that a group $G \in \mathcal{G}$ is m.a.p. if and only if G is topologically simple. If G acts transitively on X and Ω, then $G^+ \in \mathcal{G}$, therefore also G contains one and only one topologically simple m.a.p. subgroup in $\mathcal{G}$.

2. The results

Let G be a closed subgroup of $\mathrm{Aut}(X)$. Let H be a closed subgroup of G. Let $v \in X$ be a fixed vertex of X and $K_v = \{g \in G : g(v) = v\}$. K_v is a compact open subgroup of G.

Proposition 1. *The space G/H is compact if and only if the orbit $G(v)$ is the union of finitely many orbits of H; that is, there exist $x_1, x_2, \ldots, x_n \in G(v)$, such that $G(v) = H(x_1) \cup H(x_2) \cup \cdots \cup H(x_n)$.*

Proof. Let $\{K_v gH\}$ be the partition of G into the double cosets $K_v gH$ for $g \in G$. Since $K_v gH$ is an open set of G and $p(K_v gH) = p(K_v g)$ is a compact open subset of G/H for every $g \in G$ (where $p : G \to G/H$ is the canonical map), then it is easy to see that G/H is compact if and only if the partition $\{K_v gH\}_{g\in G}$ has only finitely many sets. Therefore the proposition follows from the fact that the map $\Lambda(K_v gH) = H(g^{-1}(v))$ is a bijective map of the double cosets of the partition onto the set of H-orbits contained in $G(v)$.

Remark. In particular Proposition 1 implies that if G acts transitively on X, then G/H is a compact space if and only if H has finitely many orbits on X.

Definition 1. Let $\mathcal{G}$ be the class of all closed subgroups G of $\mathrm{Aut}(X)$ with the following properties:

(1) G acts transitively on the tree boundary Ω,
(2) G has exactly two orbits on X, that is, X^+ and X^-.

If $W \subseteq K_v$ is a subset acting transitively on Ω and g is a translation of even step, then the closed subgroup generated by W and g is in $\mathcal{G}$. On the other hand, as observed in the introduction, if G acts transitively on Ω and on X, then $G^+ \in \mathcal{G}$. The reader is referred to [2, pg. 31–32, 133–134] for examples. By [4, Prop. 2, pg. 143] it follows that if $G \in \mathcal{G}$ and H is a closed noncompact subgroup of G acting transitively on Ω, then $H \in \mathcal{G}$.

Lemma 1. *Let G be in the class $\mathcal{G}$, and let H be a closed nontrivial normal subgroup of G; then $H \in \mathcal{G}$. In particular G/H is a compact group. If in addition H is open, then G/H is a finite group.*

Proof. First we observe that H is not compact. In fact since G has two orbits on X, then G contains no inversion (an inversion interchanges X^+ and X^-). Therefore every compact subgroup of G fixes a vertex $v \in X$ [2, Theorem 5.2, pg. 12]. But if $H \subseteq K_v$, then $H = gHg^{-1} \subseteq gK_vg^{-1} = K_{g(v)}$ for all $g \in G$, which means that H fixes X^+ or X^- and so H fixes X. This is impossible because H is not trivial. We prove now that H acts transitively on Ω. Since H is not trivial, then there exist $h \in H$ and $\omega \in \Omega$ such that $\omega \neq h(\omega)$. Let ω' be an end of Ω such that $\omega' \neq \omega$ and $\omega' \neq h(\omega)$. Since G acts doubly transitively on Ω (see [2, pg. 29–30]) there exists $g \in G$ such that $g(\omega) = \omega$ and $g(h(\omega)) = \omega'$. Therefore $ghg^{-1}(\omega) = \omega'$ and $\omega' \in H(\omega)$ because $ghg^{-1} \in H$. So $H(\omega) = \Omega$. The lemma follows from [2, Prop. 10.2, pg. 27] and Proposition 1.

Lemma 2. *Let G be in the class $\mathcal{G}$; let $\{H_n\}$ be a sequence of open normal subgroups of G. Then $\bigcap_{n=1}^{\infty} H_n$ is a nontrivial subgroup of G.*

Proof. The proof is similar to the proof of [1, Prop. 16.4.4, pg. 302]. We suppose, on the contrary, that $\bigcap_{n=1}^{\infty} H_n$ is trivial. Replacing, if necessary, H_n by $H_1 \cap H_2 \cap \cdots \cap H_n$ we may assume that $H_{n+1} \subseteq H_n$. G contains a translation w [2, Th. 8.1, p. 20]; let K_v be the stability subgroup of a vertex v; let $U = K_v \cup wK_v \cup K_v w^{-1}$. Therefore U is a compact open symmetric neighborhood of the identity e of G. Since $U^n \subseteq U^{n+1}$, then $\bigcup_{n=1}^{\infty} U^n$ is a noncompact open subgroup of G, in fact the subgroup of G generated by U. Since $K_v \subseteq U$, then $\bigcup_{n=1}^{\infty} U^n$ acts transitively on Ω [4, Prop. 1, pg. 143]. Therefore $\bigcup_{n=1}^{\infty} U^n \in \mathcal{G}$ [4, Prop. 2, pg. 143], that is, $\bigcup_{n=1}^{\infty} U^n$ has the same orbits of G. The fact that $K_v \subseteq \bigcup_{n=1}^{\infty} U^n$ implies that $G = \bigcup_{n=1}^{\infty} U^n$. The sequence $H_n \cap U^3$ is a sequence of compact open subsets of U^3 such that $\bigcap_{n=1}^{\infty}(H_n \cap U^3) = \{e\}$. Since $e \in U \subseteq U^3$, it follows that there exists m such that $H_m \cap U^3 \subseteq U$. This implies that $H = H_m \cap U^3$ is a compact open subgroup of G. We prove now that H is a normal subgroup of G. Indeed, if $t \in U$ and $h \in H \subseteq U$, then $tht^{-1} \in U^3 \cap H_m = H$ and so $tHt^{-1} \subseteq H$. But U is symmetric and $tHt^{-1} = H$ for every $t \in U$. Since $G = \bigcup_{n=1}^{\infty} U^n$, then $gHg^{-1} = H$ for every $g \in G$. As observed in the first part of the proof of Lemma 1, this is impossible because G is not discrete and H is not trivial.

Let $\widehat{G}$ be the set of equivalence classes of unitary continuous irreducible representations of G. If G is a totally disconnected group, then $\operatorname{Ker}\pi$, the kernel of the representation π, is a normal open subgroup of G for every unitary continuous finite dimensional representation π [2, Prop. 1.2, pg. 86].

Definition 2. For $G \in \mathcal{G}$, we define:

$$\mathcal{A}(G) = \{H \colon H \text{ is a nontrivial normal closed subgroup of } G\},$$
$$\mathcal{B}(G) = \{H \colon H \text{ is a normal open subgroup of } G\},$$
$$\mathcal{C}(G) = \{\operatorname{Ker}\pi \colon \pi \in \widehat{G} \text{ and } \dim\pi < +\infty\}.$$

We have $\mathcal{C}(G) \subseteq \mathcal{B}(G) \subseteq \mathcal{A}(G)$. If $G \in \mathcal{G}$ and $H \in \mathcal{A}(G)$, then G/H is a compact group. In particular, if $p \colon G \to G/H$ is the natural homomorphism, then $p \circ \pi$ is a finite dimensional irreducible representation of G for every $\pi \in (G/H)\widehat{}$. Therefore $\operatorname{Ker}(p \circ \pi) \in \mathcal{C}(G)$ and $\bigcap_{\pi \in (G/H)\widehat{}} \operatorname{Ker}(p \circ \pi) = H$. This means that

$$\bigcap_{H \in \mathcal{C}} H = \bigcap_{H \in \mathcal{B}} H = \bigcap_{H \in \mathcal{A}} H.$$

We put $G' = \bigcap_{H \in \mathcal{C}} H = \bigcap_{H \in \mathcal{B}} H = \bigcap_{H \in \mathcal{A}} H$. G' is a closed normal subgroup of G and, by Lemma 2, $G' \neq \{e\}$, hence $G' \in \mathcal{G}$. In fact G is separable; therefore there exists a sequence $K_{n+1} \subseteq K_n$ of compact open subgroups of G which is a basis of neighborhoods of the identity of G. Let H_n be the open normal subgroup of G generated by K_n. G/H_n is finite and every $H \in \mathcal{B}(G)$ contains H_n for n sufficiently large. This proves that $\mathcal{B}(G)$ is finite or countable. In particular $G' \neq \{e\}$ by Lemma 2.

We summarize the above facts in the following proposition.

Proposition 2. *Let $G' = \bigcap_{H \in \mathcal{A}} H$; then $G' \in \mathcal{A}(G)$. In particular $G' \subseteq H$ for every $H \in \mathcal{A}(G)$. Moreover $G' = \bigcap_{H \in \mathcal{C}} H = \bigcap_{H \in \mathcal{B}} H = \bigcap_{H \in \mathcal{A}} H$.*

Theorem. *Let $G \in \mathcal{G}$, and let G' be as in Proposition 2. Then $G' \in \mathcal{G}$. G' is a topologically simple group.*

Proof. The fact that $G' \in \mathcal{G}$ follows from Lemma 1. Let H be a nontrivial closed normal subgroup of G'. We prove now that $H = G'$. G' is a normal subgroup of G, therefore, for every $g \in G$, gHg^{-1} is a nontrivial closed normal subgroup of G'. Proposition 2 implies that $G'' \subseteq gHg^{-1}$ for every $g \in G$ where $G'' = (G')'$ (we recall that if $G \in \mathcal{G}$, then $G' \in \mathcal{G}$ and $G'' \neq \{e\}$). In particular $\{e\} \neq G'' \subseteq \bigcap_{g \in G} gHg^{-1}$. This means that $\bigcap_{g \in G} gHg^{-1}$ is a normal closed nontrivial subgroup of G, and so $G' \subseteq \bigcap_{g \in G} gHg^{-1} \subseteq H \subseteq G'$. Hence $H = G'$ and theorem follows.

Remarks. 1) G' is topologically simple, therefore $G'' = G'$.

2) The compact group G/G' is the compact group associated with G in the sense of [1, Th. 16.1.1, pg. 296] and the canonical surjection $p\colon G \to G/G'$ is the canonical morphism of G [1, Th. 16.1.1, pg. 296]. In particular a bounded continuous function f on G is almost periodic iff f is G'-invariant. As π varies among all finite dimensional representations of G/G' (as π varies in $(G/G')\widehat{\ }$), $p \circ \pi$ describes all finite dimensional representations of G (all finite dimensional irreducible representations of G).

3) Obviously G' is open in G iff G has only finitely many classes of finite dimensional irreducible representations.

Corollary 1. *Let $G \in \mathcal{G}$; then the following are equivalent.*

1) *G is m.a.p.*
2) *G is topologically simple.*
3) *$G = G'$.*

Proof. The corollary follows from Remark 2) above and the fact that G is topologically simple iff $G = G'$, that is, iff $(G/G')\widehat{\ }$ is trivial.

Corollary 2. *Let G be a closed subgroup of* $\mathrm{Aut}(X)$ *acting transitively on X and Ω. Let $G^+ = \{g \in G\colon d(x, g(x))$ is even$\}$. Then $G^+ \in \mathcal{G}$ is topologically simple iff G has exactly two (classes of) unitary irreducible finite dimensional representations, that is, the trivial character ($\chi(g) = 1$ for every $g \in G$) and the character χ^+ (where $\chi^+(g) = 1$ for every $g \in G^+$ and $\chi^+(g) = -1$ for every $g \notin G^+$).*

Proof. Because G/G^+ is the cyclic group of order 2, then an elementary argument of induced representations proves that the set of irreducible finite dimensional representations of G is $\{\chi, \chi^+\}$ iff every irreducible finite dimensional representation of G^+ is trivial. Therefore Corollary 2 follows from Corollary 1.

Remarks. 1) If G acts transitively on X and Ω and it satisfies the property (P) of Tits [8], then $G^+ = G'$.

2) Let $\mathbb{Q}_p$ be the field of the p-adic numbers; the group $\mathbf{PGL}(2, \mathbb{Q}_p)$ can be embedded into the group of all isometries of some homogeneous tree in such a way that it acts transitively on X and Ω [7]. In this case $\mathbf{PGL}(2, \mathbb{Q}_p)^+$ is not simple but

$$(\mathbf{PGL}(2, \mathbb{Q}_p))' = \mathbf{PSL}(2, \mathbb{Q}_p) = [\mathbf{PGL}(2, \mathbb{Q}_p), \mathbf{PGL}(2, \mathbb{Q}_p)]$$

where [,] means the commutator subgroup. The finite dimensional irreducible representations of $\mathbf{PGL}(2, \mathbb{Q}_p)$ are in fact characters [3, Prop. 2.7, pg. 31].

3) For $G = \mathrm{Aut}(X)$ or $G = \mathbf{PGL}(2, \mathbb{Q}_p)$ we have that $G' = [G, G]$ and G' is open in G. This means that every finite dimensional irreducible representation of G is a character and the set of characters must be finite. For $G = \mathbf{PGL}(2, \mathbb{Q}_p)$ this is a consequence of the fact that every open normal subgroup of $\mathbf{PGL}(2, \mathbb{Q}_p)$ contains the group $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ (see the first part of the proof of [3, Prop. 2.7, pg. 31]).

4) Let G be a closed subgroup of $\mathrm{Aut}(X)$ acting transitively on X and Ω, and let $\omega \in \Omega$. We define B^G_ω as the subgroup of all rotations of G such that $g(\omega) = \omega$. In [5] we consider the group B^G_ω for $G = \mathrm{Aut}(X)$ and we prove that B^G_ω is minimally almost periodic. A curious fact is that if B^G_ω is minimally almost periodic for a general G acting transitively on X and Ω, then G^+ is topologically simple. This is a consequence of the following claim: if an open normal subgroup H of G^+ contains B^G_ω, then $H = G^+$. We prove now, briefly, the claim. Since H is normal and $B^G_\omega \subseteq H$, it follows that $gB^G_\omega g^{-1} = B^G_{g(\omega)} \subseteq gHg^{-1} = H$ for every $g \in G^+$. Hence $B^G_\sigma \subseteq H$ for every $\sigma \in \Omega$ because G^+ acts transitively on Ω. We recall that G^+ is the subgroup generated by all rotations of G; therefore it is enough to prove that $K_v \subseteq H$ for every $v \in X$. As observed $K_v \cap B^G_\sigma \subseteq H$ for every $v \in X$ and $\sigma \in \Omega$. Let k be in K_v and σ, $\sigma' \in \Omega$ such that $k(\sigma) = \sigma'$. The subgroup H is in $\mathcal{G}$ and so $H \cap K_v$ acts transitively on Ω [4, Prop. 1, pg. 143]; therefore there exists $h \in H \cap K_v$ such that $h(\sigma') = \sigma$. This means that $hk \in B^G_\sigma \subseteq H$ and $k \in H$.

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