SOLVING THE $p$-LAPLACIAN ON MANIFOLDS

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Abstract. We prove in this paper that the equation $\Delta_p u + h = 0$ on a $p$-hyperbolic manifold $M$ has a solution with $p$-integrable gradient for any bounded measurable function $h : M \to \mathbb{R}$ with compact support.

1. Introduction

The $p$-Laplacian of a function $f$ on a connected oriented Riemannian manifold without boundary $M$ is defined by $\Delta_p f = \text{div}(|\nabla f|^{p-2}\nabla f)$; it is the Euler-Lagrange operator associated with the functional $\int_M |\nabla f|^p$.

A function $u \in W^{1,p}_{\text{loc}}(M)$ is said to be a weak solution to the equation

$\Delta_p u + h = 0 \quad (1)$

if for all $\psi \in C^1_0(M)$ one has

$\int_M \langle |\nabla u|^{p-2}\nabla u, \nabla \psi \rangle = \int_M h \psi.$

We introduce the $p$-Dirichlet space $L^{1,p}(M)$ of functions $u \in W^{1,p}_{\text{loc}}(M)$ admitting a weak gradient such that $\int_M |\nabla u|^p < \infty$.

In [2], the following result has been proved:

Theorem 1. Suppose that $M$ is $p$-parabolic, and let $h \in L^1(M)$ be a function such that $\int_M h \neq 0$. Then (1) has no weak solution $u \in L^{1,p}(M)$.

The goal of this paper is to prove the following result in the converse direction.

Theorem 2. Suppose that $M$ is a $p$-hyperbolic manifold $(1 < p < \infty)$ and that $h \in L^\infty(M)$ has compact support. Then (1) has a weak solution $u \in L^{1,p}(M)$. Moreover $u$ is of class $C^{1,\alpha}$ on each compact set (where $\alpha \in (0,1)$ may depend on the compact set).

The notion of $p$-hyperbolic and $p$-parabolic manifolds will be recalled below (see also [6]). As an example, the euclidean space $\mathbb{R}^n$ is $p$-hyperbolic if and only if $p < n$.

Remark. If $M = \mathbb{R}^n$ with $1 < p < n$ and $h \geq 0$, then equation (1) (and in fact a more general eigenvalue problem) is solved in [1].

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2. Preliminaries on $p$-hyperbolicity

**Definition.** Let $(M, g)$ be a connected Riemannian manifold, and $K \subset M$ a compact set. For $1 < p < \infty$, the $p$-capacity of $K$ is defined by

$$
\text{Cap}_p(K) := \inf \left\{ \int_M |\nabla u|^p : u \in C_0^1(M), \, u \geq 1 \text{ on } K \right\}.
$$

The manifold $M$ is said to be $p$-parabolic if $\text{Cap}_p(K) = 0$ for all compact subsets $K \subset M$ and $p$-hyperbolic otherwise. It is a well known fact that, in a $p$-hyperbolic manifold, the $p$-capacity of any compact set with non empty interior is always positive (see e.g. [6]).

Let $D \subset M$ be a non empty bounded domain. We introduce the Banach space $E^p = E^p(D, M)$ of functions $u \in W^{1,p}_{\text{loc}}(M)$ such that

$$
\|u\|^p_E := \int_D |u|^p dx + \int_M |\nabla u|^p dx < \infty.
$$

We denote by $E^p_0$ the closure of $C_0^1(M)$ in $E^p$.

**Lemma 1.** If $M$ is $p$-parabolic, then $1 \in E^p_0$.

**Proof.** By hypothesis $\text{Cap}_p(D) = 0$; hence for all $\epsilon > 0$, there exists a function $u \in C_0^1(M)$ such that $u \equiv 1$ on $D$ and $\int_M |\nabla u|^p dx < \epsilon$. Thus we have

$$
\|1 - u\|^p_E := \int_D |1 - u|^p dx + \int_M |\nabla u|^p dx = \int_M |\nabla u|^p dx \leq \epsilon.
$$

It follows that $1 \in E^p_0$. \hfill $\Box$

The next lemma is the well known Poincaré inequality.

**Lemma 2.** Let $D$ be any bounded regular domain in a Riemannian manifold $M$ and $1 \leq p < \infty$. Then there exists a constant $A$ such that

$$
\left( \int_D |u - u_D|^p dx \right)^{1/p} \leq A \left( \int_D |\nabla u|^p dx \right)^{1/p}
$$

for all $u \in W^{1,p}_{\text{loc}}(M)$, where $u_D = \frac{1}{\text{vol}(D)} \int_D u dx$ is the mean value of $u$ on $D$.

A reference is [3, Lemma 3.8]. \hfill $\Box$

Combining this lemma with Hölder’s (or Jensen’s) inequality, we obtain

**Corollary 1.** There exists a constant $c = c_D$ such that

$$
\left( \int_D |u - u_D| dx \right)^{1/p} \leq c_D \left( \int_M |\nabla u|^p dx \right)^{1/p}
$$

for all $u \in W^{1,p}_{\text{loc}}(M)$.

**Proposition 1.** Suppose that $M$ is $p$-hyperbolic and let $D \subset M$ be as in Lemma 2. Then there exists a constant $C_1$ such that for all $u \in E^p_0$

$$
\int_D |u| dx \leq C_1 \left( \int_M |\nabla u|^p dx \right)^{1/p}.
$$


Proof. Suppose that such a constant does not exist. Then for all \( \varepsilon > 0 \) it is possible to find a function \( u \in E^p_0 \) such that
\[
\int_D |u| \, dx = \text{vol}(D) \quad \text{and} \quad \| \nabla u \|_{L^p(M)} \leq \varepsilon .
\]
We may also assume \( u \geq 0 \) (else replace \( u \) by \( |u| \)). From Corollary 1 one gets
\[
\int_D |u - 1| \, dx \leq c_D \varepsilon . (3)
\]
Let us now choose a ball \( B \subset \subset D \) and a function \( \psi \in C^1_0(M) \) such that \( 0 \leq \psi \leq \frac{1}{2} \), \( \text{supp}(\psi) \subset D \) and \( \psi \equiv \frac{1}{2} \) on \( B \), and define the function \( v \in E^p_0 \) by \( v = 2 \max\{u; \psi\} \).
Observe first that \( v \geq 1 \) on \( B \), and define the sets
\[
A := \{ x \in D | \psi(x) \geq u(x) \} \quad \text{and} \quad A' := \{ x \in D | |u(x) - 1| \geq \frac{1}{2} \} .
\]
We have \( A \subset A' \) and by (3) we have \( \frac{1}{2} \text{vol}(A') \leq c_D \varepsilon \); thus
\[
\text{vol}(A) \leq 2c_D \varepsilon . (4)
\]
Now we have almost everywhere
\[
\nabla v = \begin{cases} 2\nabla u & \text{on } M \setminus A, \\ 2\nabla \psi & \text{on } A; \end{cases}
\]
in particular
\[
|\nabla v| \leq 2|\nabla u| + 2\chi_A|\nabla \psi| \quad \text{a.e.}
\]
from which one deduces
\[
\| \nabla v \|_{L^p(M)} \leq 2 \| \nabla u \|_{L^p(M)} + 2 \sup |\nabla \psi| \left( \text{vol}(A) \right)^{1/p} . (5)
\]
From (4) and (5) one obtains
\[
\| \nabla v \|_{L^p(M)} \leq \left( 2\varepsilon + 2 \sup |\nabla \psi| \left( 2c_D \varepsilon \right)^{1/p} \right) .
\]
Since \( v \geq 1 \) on \( B \) and \( \varepsilon \) is arbitrary, one deduces that \( \text{Cap}_p(B) = 0 \), which contradicts the fact that \( M \) is \( p \)-hyperbolic.

We may sum up our results so far in

**Theorem 3.** The following conditions are equivalent:

(a) \( M \) is \( p \)-hyperbolic;

(b) There exists a constant \( C_2 \) such that for all \( u \in E^p_0 \) one has
\[
\| u \|_{L^p(D)} \leq C_2 \cdot \| \nabla u \|_{L^p(M)} ;
\]

(c) \( 1 \notin E^p_0 \).

Proof. The implication (b) \( \Rightarrow \) (c) is obvious and (c) \( \Rightarrow \) (a) is Lemma 1.
Let us write $u$ as $u = (u - u_D) + u_D$; using Proposition 1 and Lemma 2, we see that

$$
\|u\|_{L^p(D)} \leq \|u - u_D\|_{L^p(D)} + \|u_D\|_{L^p(D)}
$$

$$
\leq A \left( \int_D |\nabla u|^p \, dx \right)^{1/p} + (\text{Vol}(D))^{1/p} \int_D |u| \, dx
$$

$$
\leq A \left( \int_D |\nabla u|^p \, dx \right)^{1/p} + (\text{Vol}(D))^{(1-p)/p} \left( \int_M |\nabla u|^p \, dx \right)^{1/p}
$$

$$
\leq C_2 \left( \int_M |\nabla u|^p \, dx \right)^{1/p}.
$$

This proves $(a) \Rightarrow (b)$. \hfill \Box

### 3. Proof of Theorem 2

We first choose a regular bounded domain $D \subset M$ such that $\text{supp}(h) \subset D$. We then define a functional $\mathcal{J} : E^p_0 \to \mathbb{R}$ by

$$
\mathcal{J}(u) = \frac{1}{p} \left( \int_M |\nabla u|^p \, dx \right) - \int_M h u \, dx.
$$

The manifold $M$ being $p$-hyperbolic, we have

$$
\mathcal{J}(u) \geq \frac{1}{p} \|\nabla u\|_{L^p(M)}^p - \left| \int_M h u \, dx \right|
$$

$$
\geq \frac{1}{p} \|\nabla u\|_{L^p(M)}^p - \|h\|_{L^\infty} \cdot \|u\|_{L^1(D)}
$$

$$
\geq \frac{1}{p} \|\nabla u\|_{L^p(M)}^p - C_1 \|h\|_{L^\infty} \cdot \|\nabla u\|_{L^p(M)},
$$

where $C_1$ is the constant of Proposition 1. Since the function $g(x) = |x|^p - ax$ of the real variable $x$ is bounded below, we conclude that the functional $\mathcal{J}$ is bounded below on the space $E^p_0$.

Set $m := \inf \{ \mathcal{J}(u) | u \in E^p_0 \}$, and let $\{u_i\} \subset E^p_0$ be a minimizing sequence for $\mathcal{J}$ (i.e. $\mathcal{J}(u_i) \to m$). Then from the inequality above, one deduces that $\{u_i\}$ is a bounded sequence in $E^p_0$. Since $E^p_0$ is a reflexive Banach space, this sequence contains a weakly convergent subsequence (still denoted by $\{u_i\}$). Let us denote by $u^*$ the weak limit of $\{u_i\}$. By the compactness of the embedding $E^p_0 \subset L^1(D)$, we may assume that $\{u_i\}$ converges strongly in $L^1(D)$, in particular

$$
\int_D h u_i \to \int_D h u^*.
$$

By Theorem 3, $\|\nabla u\|_{L^p(M)}$ is an equivalent norm on $E^p_0$; hence by the weak lower semi-continuity of the norm on $E^p_0$ we have

$$
\|\nabla u^*\|_{L^p(M)} \leq \liminf_{i \to \infty} \|\nabla u_i\|_{L^p(M)}.
$$
From (6) and (7) one deduces that \( J(u^*) \leq \lim_{i \to \infty} \inf J(u_i) = m \); hence \( J(u^*) = m \). By the usual arguments from variational calculus, one deduces that \( u^* \) is a weak solution to (1).

The \( C^{1,\alpha} \) regularity follows from Theorem 1 in [5].

\[ \square \]

Remark. We have in fact solved (1) in the space \( E_0^p \subset L^{1,p}(M) \).

References


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