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ON COMPACT SUBSETS IN COECHELON SPACES OF INFINITE ORDER

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Abstract. For coechelon spaces $k_\infty(v)$ of infinite order it is proved that every compact subset of $k_\infty(v)$ is contained in a closed absolutely convex hull of some null sequence if and only if the matrix v is regularly decreasing.

In connection with the study of some interesting problems on Montel maps which are closely connected to the classical Grothendieck question on completeness of regular LB–spaces [PB, Problem 13.8.6] and the problem of bornologicity of $C(K,E)$ with E an LB–space [S, Chapter IV], Dierolf and Domański [DD2, Example 3.1] gave an example of a coechelon LB–Montel space of infinite order which has compact sets not contained in closed absolutely convex hulls of any null sequence. Consequently, the well–known characterization of compact sets in Fréchet spaces [J, Theorem 9.4.2] turns out to be not generally true in the LB setting.

The purpose of this note is to show that for coechelon spaces $k_\infty(v)$ of order ∞ the condition *v regularly decreasing* [BMS, Definition 3.1] is necessary and sufficient for every compact set to be contained in a closed absolutely convex hull of some null sequence.

For more information on Montel maps and related questions the reader is referred to [DD1], [DD2], [DD3] and [D].

In what follows we recall some notation.

Let E be a Fréchet space with a fundamental system of seminorms $(\| \ \|_n)_n$; then the inductive dual E'_i is defined to be $\operatorname{ind}_n E'_n$, where the E_n are the completions of the normed spaces $(E/\ker\| \ \|_n, \| \ \|_n)$. It is known that algebraically $E'_i = E'$, the inclusion map $E'_i \hookrightarrow E'_\beta$ is continuous and E'_i is the bornological space associated with E'_β, i.e., $E'_i = (E', \beta(E', E''))$ [J, Theorem 13.4.2] (E' and E'_β denote the topological dual and the strong dual of E, resp.).

We also recall that an LB–space $E = \operatorname{ind}_n E_n$ is called *boundedly retractive* (respectively, *compactly regular*) if, and only if, for each bounded (respectively, compact) subset B of E there is $n \in \mathbf{N}$ such that $B \subset E_n$ and E_n and E induce the same topology on B (see [PB, Definitions 8.5.32–(ii), –(iii)]). It is clear that a boundedly retractive LB–space is compactly regular. On the other hand, in [N] it is proved that these conditions are also equivalent.

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Let $a = (a_n(i))_n$ and $v = (v_n(i))_n$ be matrices on an index set I, with $v_n(i) \geq v_{n+1}(i) > 0$ and $a_n(i) = \dfrac{1}{v_n(i)}$ for all $i \in I$ and $n \in \mathbf{N}$. The Köthe echelon space of order 1 (associated to a) and the Köthe coechelon space of order ∞ (associated to v) are defined by

$$\lambda_1(a) := \left\{ (x_i)_{i\in I} : \|(x_i)_i\|_n^1 := \sum_{i\in I} a_n(i)\,|x_i| < +\infty \text{ for all } n \in \mathbf{N} \right\}$$

and

$$k_\infty(v) := \left\{ (x_i)_{i\in I} : \|(x_i)_i\|_n^\infty := \sup_{i\in I} v_n(i)\,|x_i| < +\infty \text{ for some } n \in \mathbf{N} \right\},$$

respectively. Clearly, $k_\infty(v) = \operatorname{ind}_n l_\infty(v_n)$ ($l_\infty(w) := \{(x_i)_{i\in I} : \sup_{i\in I} w(i)|x_i| < +\infty\}$ for any positive function w on I); also, $k_\infty(v) = (\lambda_1(a))'_i$ (see [BMS, Corollary 2.8]) and hence it is a regular and complete LB–space.

If $\bar{V}$ is the maximal Nachbin family of positive functions on I associated to v which is defined by

$$\bar{V} := \left\{ \bar{v}\colon I \to [0, +\infty[\; : \sup_{i\in I} \frac{\bar{v}(i)}{v_n(i)} < +\infty \text{ for all } n \in \mathbf{N} \right\},$$

then, by [BMS, Theorem 2.7], $(\lambda_1(a))'_\beta$ coincides algebraically and topologically with

$$K_\infty\left(\bar{V}\right) := \operatorname{proj}_{\bar{v}\in\bar{V}} l_\infty(\bar{v}).$$

So, algebraically $k_\infty(v) = K_\infty\left(\bar{V}\right)$, the inclusion map $k_\infty(v) \hookrightarrow K_\infty\left(\bar{V}\right)$ is continuous and they have the same bounded sets. In particular, since every compact set in $K_\infty\left(\bar{V}\right) = (\lambda_1(a))'_\beta$ is separable and metrizable by [CO, Corollary 1.2], the topology of $K_\infty\left(\bar{V}\right)$ coincides with the one of $k_\infty(v)$ on these sets (cf. [PB, Proposition 8.3.12]); hence, every compact set in $K_\infty\left(\bar{V}\right)$ is also a compact set in $k_\infty(v)$.

Finally, we recall that the matrix v is called *regularly decreasing* [BMS, Definition 3.1] if, and only if, given $n \in \mathbf{N}$ there is $m \geq n$ such that

$$\forall I_0 \subset I \text{ with } \inf_{i\in I_0} \frac{v_m(i)}{v_n(i)} > 0, \text{ we have also } \inf_{i\in I_0} \frac{v_k(i)}{v_n(i)} > 0 \text{ for all } k \geq m.$$

By [BMS, Theorem 3.4] v is regularly decreasing if, and only if, $\lambda_1(a)$ is quasi-normable if, and only if, $k_\infty(v)$ is a boundedly retractive LB–space and hence if, and only if, it is a compactly regular LB–space.

For all undefined notation we refer to [PB] and [BMS].

We are now able to state and prove our result.

Theorem. *Let $k_\infty(v)$ be a coechelon space of order ∞. Then the following conditions are equivalent:*

(i) *v is regularly decreasing;*

(ii) *every compact set in $k_\infty(v)$ is contained in a closed absolutely convex hull of some null sequence.*

Proof. We establish that (i) implies (ii). Let K be a compact set in $k_\infty(v)$. Since v is regularly decreasing, $k_\infty(v)$ is boundedly retractive. Thus K is contained in $l_\infty(v_n)$ for some $n \in \mathbf{N}$ and $l_\infty(v_n)$ and $k_\infty(v)$ induce the same topology on K; hence, it is a compact subset of $l_\infty(v_n)$. Since $l_\infty(v_n)$ is a Banach space, by [J, Theorem 9.4.2]

K is contained in a closed absolutely convex hull of some null sequence in $l_\infty(v_n)$ and the result follows.

Next, we show that (ii) implies (i). Suppose that v is not regularly decreasing. Then:

$$\exists n_o \in \mathbf{N}\ \forall m > n_0\ \exists J \subset I \text{ and } \exists k > m \text{ such that}$$

$$\inf_{i\in J} \frac{v_m(i)}{v_{n_0}(i)} > 0 \text{ and } \inf_{i\in J} \frac{v_k(i)}{v_{n_0}(i)} = 0.$$

Proceeding by induction as in [V, p. 232], we can find a sectional subspace of $k_\infty(v)$ which is isomorphic to $k_\infty(\tilde{v})$, with $\tilde{v} = (\tilde{v}_n(i,j))_n$ a matrix on $\mathbf{N}\times\mathbf{N}$ satisfying:

$$\tilde{v}_1(i,j) = 1,\ \forall i,\ j \in \mathbf{N}; \tag{1}$$

$$\lim_{j\to\infty} \tilde{v}_{n+1}(n,j) = 0,\ \forall n \in \mathbf{N}; \tag{2}$$

$$\forall n \in \mathbf{N}\ \forall i \geq n\ \exists \alpha_{n,i} > 0\ \inf_{j\in\mathbf{N}} \tilde{v}_n(i,j) > \alpha_{n,i}. \tag{3}$$

Since $k_\infty(\tilde{v})$ is complemented in $k_\infty(v)$ as it is easy to see, we can restrict our attention to $k_\infty(\tilde{v})$ ($k_\infty(v)$ satisfies (ii) $\Rightarrow$ every complemented subspace of $k_\infty(v)$ satisfies (ii)). For technical reasons we modify the weights by defining $v_n^\star(i,j) := i^{-2}\tilde{v}_n(i,j)$. Clearly, $k_\infty(\tilde{v})$ is isomorphic to $k_\infty(v^\star)$, with $v^\star = (v_n^\star(i,j))_n$. Moreover, $v^\star$ satisfies the following conditions:

$$v_1^\star(i,j) = i^{-2},\ \forall i,\ j \in \mathbf{N}; \tag{a}$$

$$\lim_{j\to\infty} v_{n+1}^\star(n,j) = 0,\ \forall n \in \mathbf{N}; \tag{b}$$

$$\forall n \in \mathbf{N}\ \forall i \geq n\ \exists \alpha_{n,i} > 0\ \inf_j v_n^\star(i,j) > i^{-2}\alpha_{n,i}. \tag{c}$$

We will show that the set $C := \left\{(x_{ij})_{i,j\in\mathbf{N}} : \sup_{i,j\in\mathbf{N}} |x_{ij}|\, i^{-1} \leq 1\right\}$ is a compact set in $k_\infty(v^\star)$ which is not contained in the closed absolutely convex hull of any null sequence (clearly, $C \subset B_1$, where B_1 denotes the closed unit ball of $l_\infty(v_1^\star)$).

Let $\bar{V}$ be the maximal Nachbin family of positive functions on $\mathbf{N}\times\mathbf{N}$ associated to $v^\star$. Since every compact set in $K_\infty\left(\bar{V}\right)$ is also a compact set in $k_\infty(v^\star)$, we have only to show that C is compact in $K_\infty\left(\bar{V}\right)$, i.e., C is compact in $l_\infty(\bar{v})$ for every $\bar{v} \in \bar{V}$.

Now, let $\bar{v} \in \bar{V}$. By (a) and (b), it follows that:

$$\lim_{i\to\infty} i \sup_{j\in\mathbf{N}} \bar{v}(i,j) = 0 \text{ and } i\bar{v}(i,j) \leq c < +\infty,\ \forall i,\ j \in \mathbf{N}; \tag{4}$$

$$\lim_{j\to\infty} n\bar{v}(n,j) = 0,\ \forall n \in \mathbf{N}. \tag{5}$$

Next, let $(x^r)_r$ be a sequence of C. Then, for each i and $j \in \mathbf{N}$, $\sup_r i^{-1}\left|x_{ij}^r\right| \leq 1$. This implies that (first by passing to subsequences and then by diagonalization) there are a sequence $x = (x_{ij})_{i,j\in\mathbf{N}}$ in $\mathbf{R}$ and a subsequence $\left(x^{r'}\right)_{r'}$ of $(x^r)_r$ so that, for each i and $j \in \mathbf{N}$,

$$\lim_{r'\to\infty} i^{-1}\left|x_{ij}^{r'} - x_{ij}\right| = 0; \tag{6}$$

hence, for each i and $j \in \mathbf{N}$,

$$i^{-1}\left|x_{ij}\right| \leq 1 \quad \text{and} \quad i^{-1}\left|x_{ij}^{r'} - x_{ij}\right| \leq 2. \tag{7}$$

Clearly, $x \in C$. We check that $\left(x^{r'}\right)_{r'}$ converges to x in $l_\infty(\bar{v})$. Given any $\varepsilon > 0$, by (4), there is $i_0 \in \mathbf{N}$ such that

$$\sup_{j\in\mathbf{N}} i\bar{v}(i,j) < \varepsilon/6 \quad \text{for all } i \geq i_0.$$

It follows, by (7), that

$$\begin{aligned}\sup_{i,\ j\in\mathbf{N}}\left|x_{ij}^{r'} - x_{ij}\right|\bar{v}(i,j) &\leq \sup_{i<i_0}\sup_{j\in\mathbf{N}}\left|x_{ij}^{r'} - x_{ij}\right|\bar{v}(i,j) + \sup_{i\geq i_0}\sup_{j\in\mathbf{N}} i^{-1}\left|x_{ij}^{r'} - x_{ij}\right| i\bar{v}(i,j)\\ &\leq \sup_{i<i_0}\sup_{j\in\mathbf{N}}\left|x_{ij}^{r'} - x_{ij}\right|\bar{v}(i,j) + 2\sup_{i\geq i_0}\sup_{j\in\mathbf{N}} i\bar{v}(i,j)\\ &\leq \sup_{i<i_0}\sup_{j\in\mathbf{N}}\left|x_{ij}^{r'} - x_{ij}\right|\bar{v}(i,j) + \varepsilon/3.\end{aligned} \tag{8}$$

for every $r' \in \mathbf{N}$. On the other hand, by (5), $\lim_{j\to\infty} i\bar{v}(i,j) = 0$ for $i = 1, \dots, i_0 - 1$ and hence there is $j_0 \in \mathbf{N}$ such that $i\bar{v}(i,j) < \varepsilon/6$ for each $i = 1, \dots, i_0 - 1$ and $j \geq j_0$. This implies by (4), (7) and (8) that, for each $r' \in \mathbf{N}$,

$$\begin{aligned}\sup_{i,\ j\in\mathbf{N}}\left|x_{ij}^{r'} - x_{ij}\right|\bar{v}(i,j) \leq& \sup_{i<i_0}\sup_{j<j_0} i^{-1}\left|x_{ij}^{r'} - x_{ij}\right| i\bar{v}(i,j)\\ &+ \sup_{i<i_0}\sup_{j\geq j_0} i^{-1}\left|x_{ij}^{r'} - x_{ij}\right| i\bar{v}(i,j) + \varepsilon/3\\ \leq& c\sup_{i<i_0}\sup_{j<j_0} i^{-1}\left|x_{ij}^{r'} - x_{ij}\right| + 2\sup_{i<i_0}\sup_{j\geq j_0} i\bar{v}(i,j) + \varepsilon/3\\ \leq& c\sup_{i<i_0}\sup_{j<j_0} i^{-1}\left|x_{ij}^{r'} - x_{ij}\right| + 2\varepsilon/3,\end{aligned} \tag{9}$$

where by (6) $\lim_{r'\to\infty}\sup_{i<i_0}\sup_{j<j_0} i^{-1}\left|x_{ij}^{r'} - x_{ij}\right| = 0$ and hence there is $r_0' \in \mathbf{N}$ such that, for each $r' \geq r_0'$,

$$\sup_{i<i_0}\sup_{j<j_0} i^{-1}\left|x_{ij}^{r'} - x_{ij}\right| < \varepsilon/3c.$$

Thus, by (9), we get that

$$\sup_{i,\ j\in\mathbf{N}}\left|x_{ij}^{r'} - x_{ij}\right|\bar{v}(i,j) < \varepsilon$$

for all $r' \geq r_0'$.

Since ε is arbitrary, this means that $\left(x^{r'}\right)_{r'}$ converges to x in $l_\infty(\bar{v})$, where $x \in C$. We can therefore conclude that C is a compact subset of $l_\infty(\bar{v})$. Since $\bar{v}$ is also arbitrary, C is a compact subset of $K_\infty\left(\bar{V}\right)$ too.

It remains to show that C is not contained in the closed absolutely convex hull of any null sequence in $k_\infty(v^\star)$. For this we proceed in a similar way as in Example 3.1 [DD2], which is based on an idea of L. Frerick and J. Wengenroth.

Assume that

$$C \subset \overline{\operatorname{absconv}\{y_k : \ k \in \mathbf{N}\}} := C_1,$$

where $(y_k)_k \in c_0\left(k_\infty(v^\star)\right)$. Since $k_\infty(v^\star)$ is a regular LB–space, $C_1 \subseteq \rho B_n$ for some $n > 1$ and $\rho > 0$ (B_n denotes the closed unit ball of $l_\infty(v_n^\star)$).

Let $P\colon k_\infty(v^\star) \to k_\infty(v^\star)$ be the projection defined by

$$P\left((x_{ij})_{i,j}\right) = (z_{ij})_{i,j}\,, \quad z_{ij} := \begin{cases} 0, & \text{if } i \neq n; \\ x_{nj}, & \text{if } i = n. \end{cases}$$

We then have $P(C) \subseteq P(B_1) \subseteq P(B_n)$. By (c), we have that $P(B_n) \subseteq n\alpha_{n,n}^{-1} P(C)$. Put $D := P(C)$ and $E_D := \operatorname{span} D$ (clearly, (E_D, p_D) is a Banach space).

Next, we define a map

$$T\colon l^1 \to E_D, \quad T\left((a_k)_k\right) := \sum_{k=1}^{\infty} a_k P y_k.$$

Since $T(B_{l^1}) \subseteq \rho P(B_n) \subseteq \rho n \alpha_{n,n}^{-1} D$, T is continuous. Also, since $(Py_k)_k \in c_0\left(k_\infty(v^\star)\right)$, the restriction of T to the closed unit ball of l^1, B_{l^1}, endowed with the weak topology $\sigma\left(l^1, c_0\right)$ is continuous (see [PB, Lemma 3.2.11]) and hence $T(B_{l^1})$ is a compact absolutely convex subset of $k_\infty(v^\star)$. So, we have

$$T(B_{l^1}) \supseteq \overline{\operatorname{absconv}\{Py_k : \ k \in \mathbf{N}\}} \supseteq D$$

and hence T is open. Now, (E_D, p_D) is isomorphic to l^∞ and l^1 is separable, thereby obtaining a contradiction.

Thus, (ii) implies (i) and this completes the proof. □

As an immediate consequence, we obtain:

Remark. Every coechelon LB–Montel space $k_\infty(v)$ of order ∞ which is not a (DFS)–space (and hence $(k_\infty(v))'_\beta = \lambda_1(a)$ is not quasinormable) has compact subsets not contained in closed absolutely convex hulls of any null sequence and hence so does the coechelon LB–Montel space constructed by Grothendieck and Köthe and considered in Example 3.1 of [DD2].

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