PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 2, Pages 463–473
S 0002-9939(99)05075-3
Article electronically published on September 27, 1999

OBLIQUE PROJECTIONS, BIORTHOGONAL RIESZ BASES AND MULTIWAVELETS IN HILBERT SPACES

WAI-SHING TANG

(Communicated by David R. Larson)

ABSTRACT. In this paper, we obtain equivalent conditions relating oblique projections to biorthogonal Riesz bases and angles between closed linear subspaces of a Hilbert space. We also prove an extension theorem in the biorthogonal setting, which leads to biorthogonal multiwavelets.

1. INTRODUCTION

In recent developments in wavelet analysis, many significant results have been obtained using the powerful tool of Fourier transform on the Hilbert space $L^2(\mathbf{R}^d)$ of square integrable complex-valued functions on $\mathbf{R}^d$. However, it is not clear whether analogous results are still valid in domains, other than $L^2(\mathbf{R}^d)$, where Fourier transform is no longer available.

In an effort to get a better understanding of the problem, in a series of papers ([5], [9] and [6]) we try to extract the essence of the underlying problem by considering wavelets in a *general* Hilbert space. This paper is another step in this direction. A recent paper [3] by Dai and Larson is in the same spirit, though with different emphasis.

One of the earliest papers concerning wavelets in Hilbert spaces is that by J. B. Robertson [10], before the birth of modern wavelet theory. While our papers [9] and [6] can be viewed as an extension of the results in [10], the present paper is in turn an extension of [9] and [6] in the following respect. Theorem 2 in [10] leads to orthonormal wavelets associated with orthonormal multiresolutions, whereas an analogous result, [6, Theorem 2.5] (see also [9, Theorem 3.2]), leads to prewavelets. One of our main results in this paper, namely Theorem 3.6, leads to biorthogonal wavelets associated with biorthogonal multiresolutions.

We recall some definitions and set up some notation. Throughout this paper, H denotes a complex Hilbert space. A sequence $\{v_n\}$ in H is a *Riesz basis* for its closed linear span $V := \overline{span}\{v_n\}$ if there exist positive constants A and B such that

$$A \sum |a_n|^2 \le \| \sum a_n v_n \|^2 \le B \sum |a_n|^2, \quad \forall \{a_n\} \in \ell^2(\mathbf{Z}). \tag{1.1}$$

Two sequences $\{v_n\}$ and $\{\tilde{v}_n\}$ in H are *biorthogonal* if

$$\langle v_n, \tilde{v}_m \rangle = \delta_{n,m} \quad \forall n, m\,. \tag{1.2}$$

Received by the editors March 23, 1998.
2000 *Mathematics Subject Classification.* Primary 46C99, 47B99, 46B15.
Key words and phrases. Riesz basis, biorthogonal system, oblique projection, multiwavelets.

Chapters 1 and 4 of [12] provide much information on these concepts.

If V and W are closed linear subspaces of H such that $V \cap W = \{0\}$ and $V + W = H$, then we write $H = V \oplus W$ and call this a *direct sum*. In this case, we can define a map $P : H \longrightarrow H$ by

$$P(v + w) = v, \quad v \in V, w \in W, \tag{1.3}$$

and call P the projection (sometimes *oblique* projection) of H on V along W. For the special case when $W = V^{\perp}$, the orthogonal complement of V in H, we shall call P the *orthogonal* projection of H on V, which is related to the orthogonal direct sum $V \oplus^{\perp} V^{\perp}$. Theorem 2.3, a main result in this paper, gives equivalent conditions relating oblique projections to biorthogonal Riesz bases and angles between closed linear subspaces of H.

We now give a summary of the contents of this paper. In Section 2, we discuss various relations between oblique projections and biorthogonal Riesz bases. In Section 3, in the biorthogonal setting we obtain an extension theorem (Theorem 3.6), which leads to the existence of biorthogonal multiwavelets. The final section describes briefly the construction of biorthogonal wavelets associated with biorthogonal scaling vectors.

2. Oblique projections and biorthogonality

Let us first state a probably folk result on the vector sum of two closed linear subspaces of a Hilbert space. We leave its proof to the reader.

Theorem 2.1. *Let V and W be closed linear subspaces of a complex Hilbert space H. The following conditions are equivalent:*

(i) $\sup\{|\langle v, w\rangle| : v \in V, w \in W, \|v\| = \|w\| = 1\} < 1.$

(ii) *There exists a positive constant C such that*

$$\|v + w\|^2 \geq C(\|v\|^2 + \|w\|^2), \quad v \in V, w \in W.$$

(iii) *$V + W$ is closed in H, and $V \cap W = \{0\}$.*

Moreover, if $V \cap W = \{0\}$, and X and Y are Riesz bases for V and W respectively, then (i), (ii) *and* (iii) *are each equivalent to:*

(iv) *$X \cup Y$ is a Riesz basis for $\overline{V + W}$.*

The proof of (iii)$\Longrightarrow$(i) in the above theorem is implicit in [4, pp. 339–340]. Note that there exist closed linear subspaces V and W of a Hilbert space H for which $V \cap W = \{0\}$, but $V + W$ is not closed in H (see [7, pp. 28–29]).

Corollary 2.2. *Let U_1 and U_2 be closed linear subspaces of H and $U_1 \cap U_2 = \{0\}$. If $U_1 + U_2$ is closed, then $V_1 + V_2$ is closed for any closed linear subspace V_i of U_i, $i = 1, 2$.*

The expression in condition (i) of Theorem 2.1 is closely related to the concept of the *angle* $\theta(V, W)$ $(0 \leq \theta(V, W) \leq \frac{\pi}{2})$ between two closed linear subspaces V and W of a Hilbert space H, which is defined by

$$\cos(\theta(V, W)) := \inf_{\substack{v \in V \\ \|v\|=1}} \|P_W v\|, \tag{2.1}$$

where P_W is the orthogonal projection of H on W. (See [1], [4, pp. 339-340] and [11] for related properties and engineering interpretations of $\theta(V, W)$.) It is easy to see that

$$\cos^2(\theta(V,W)) = 1 - \sup_{\substack{v\in V, w\in W^\perp \\ \|v\|=\|w\|=1}} |\langle v,w\rangle|^2 . \tag{2.2}$$

Note that in general $\theta(V,W)$ and $\theta(W,V)$ are not necessarily equal, but by (2.2) we always have $\theta(V,W)=\theta(W^\perp,V^\perp)$. If $V\oplus W^\perp = H$, then $\theta(V,W)=\theta(W,V)$.

Theorem 2.3. *Let V and $\tilde V$ be closed linear subspaces of H. The following conditions are equivalent:*

(i) $V\oplus \tilde V^\perp = H$.

(ii) $\tilde V\oplus V^\perp = H$.

(iii) *There exist Riesz bases $\{v_n\}$ and $\{\tilde v_n\}$ for V and $\tilde V$ respectively such that $\{v_n\}$ is biorthogonal to $\{\tilde v_n\}$.*

(iv) $\cos(\theta(V,\tilde V))>0$ *and* $\cos(\theta(\tilde V,V))>0$.

Proof. (iii)$\Longrightarrow$(ii): Suppose that $\{v_n : n\in \mathbf{Z}\}$ and $\{\tilde v_n : n\in\mathbf{Z}\}$ are Riesz bases for V and $\tilde V$ respectively such that

$$\langle v_n,\tilde v_k\rangle = \delta_{n,k}, \quad n,k\in\mathbf{Z}. \tag{2.3}$$

Then

$$V=\{f\in H : f=\sum a_n v_n, \sum |a_n|^2<\infty\} \tag{2.4}$$

and

$$\tilde V=\{g\in H : g=\sum b_n \tilde v_n, \sum |b_n|^2<\infty\}. \tag{2.5}$$

Let $g\in \widetilde V\cap V^\perp$. By (2.5) and (2.3),

$$g=\sum \langle g,v_n\rangle \tilde v_n = 0.$$

Hence $\widetilde V\cap V^\perp=\{0\}$.

Let $f\in H$. By the Riesz basis property of $\{v_n\}$ and $\{\tilde v_n\}$,

$$Pf := \sum\langle f,v_n\rangle\tilde v_n \tag{2.6}$$

is a well-defined vector in $\widetilde V$. By (2.3) and (2.6),

$$\langle f-Pf, v_k\rangle = 0,\quad k\in\mathbf{Z}.$$

Hence $f-Pf\in V^\perp$, and so

$$f=Pf+(f-Pf)\in \widetilde V + V^\perp.$$

Therefore $\widetilde V+V^\perp = H$.

(ii)$\Longrightarrow$(iii): Suppose that $\widetilde V\oplus V^\perp = H$. Let $\{v_n\}$ and $\{u_n\}$ be dual Riesz bases for V, i.e., $\{v_n\}$ and $\{u_n\}$ are both Riesz bases for V and

$$\langle v_n, u_k\rangle = \delta_{n,k},\quad \forall n,k.$$

Recall that ([12], p. 185 and p. 188) if $F: V\longrightarrow V$ is the *frame operator* defined by

$$F(v)=\sum\langle v,v_n\rangle v_n\,,\quad v\in V, \tag{2.7}$$

then

$$u_n = F^{-1}(v_n),\quad \forall n. \tag{2.8}$$

Consider the map $P_V : H \longrightarrow H$ defined by

$$P_V f := \sum \langle f, v_n \rangle u_n\,, \quad f \in H. \tag{2.9}$$

Then P_V is the orthogonal projection of H on V. Let $G := P_V|_{\widetilde{V}}$ be the restriction of P_V to $\widetilde{V}$. If $f \in \widetilde{V}$ and $G(f) = 0$, then $f \in \widetilde{V} \cap V^{\perp} = \{0\}$. So G is injective, and

$$G(\widetilde{V}) = P_V(\widetilde{V}) = P_V(\widetilde{V} + V^{\perp}) = P_V(H) = V.$$

Hence G is an invertible bounded operator from $\widetilde{V}$ onto V. Define

$$\tilde{v}_n = G^{-1}(u_n), \quad \forall n. \tag{2.10}$$

Then $\{\tilde{v}_n\}$ is a Riesz basis of $\widetilde{V}$, and for every n,

$$u_n = G(\tilde{v}_n) = \sum_k \langle \tilde{v}_n, v_k \rangle u_k.$$

Hence $\langle \tilde{v}_n, v_k \rangle = \delta_{n,k}$, $\forall n, k$. Therefore (ii)$\Longleftrightarrow$(iii).

Interchanging the roles of V and $\widetilde{V}$, we also have (i)$\Longleftrightarrow$(iii).

(i)$\Longrightarrow$(iv): Suppose that (i) holds. Since $V + \widetilde{V}^{\perp}$ is closed and $V \cap \widetilde{V}^{\perp} = \{0\}$, by Theorem 2.1,

$$\sup\{|\langle v, w \rangle| : v \in V, w \in \widetilde{V}^{\perp}, \|v\| = \|w\| = 1\} < 1.$$

Hence by (2.2), $\cos(\theta(V, \widetilde{V})) > 0$. Since (ii) also holds, interchanging the roles of V and $\widetilde{V}$ in the above argument, $\cos(\theta(\widetilde{V}, V)) > 0$, too.

(iv)$\Longrightarrow$(i): Suppose that (iv) holds. Since $\cos(\theta(V, \widetilde{V})) > 0$,

$$\sup\{|\langle v, w \rangle| : v \in V, w \in \widetilde{V}^{\perp}, \|v\| = \|w\| = 1\} < 1.$$

By Theorem 2.1, $V + \widetilde{V}^{\perp}$ is closed and $V \cap \widetilde{V}^{\perp} = \{0\}$. Since $\cos(\theta(\widetilde{V}, V)) > 0$ also, by the above argument, $\widetilde{V} + V^{\perp}$ is closed and $\widetilde{V} \cap V^{\perp} = \{0\}$. Hence

$$V + \widetilde{V}^{\perp} = (V + \widetilde{V}^{\perp})^{\perp\perp} = (V^{\perp} \cap \widetilde{V})^{\perp} = H.$$

□

Let $U = (U_1, ..., U_d)$ be an ordered d-tuple of distinct unitary operators on a Hilbert space H such that $U_k U_j = U_j U_k$, $k, j = 1, \ldots, d$. We shall use the multi-index notation $U^m = U_1^{m_1} \cdots U_d^{m_d}$ for $m = (m_1, ..., m_d) \in \mathbf{Z}^d$, with the convention that U_j^0 is the identity operator on H, $j = 1, ..., d$. We also assume that U^m is the identity operator only if $m = 0$.

Corollary 2.4. *Suppose that one of the conditions in Theorem 2.3 holds for the closed linear subspaces V and $\widetilde{V}$ of H.*

(a) *Given a Riesz basis $\{v_n\}$ for V, there exists a Riesz basis $\{\tilde{v}_n\}$ for $\tilde{V}$ such that $\{v_n\}$ is biorthogonal to $\{\tilde{v}_n\}$.*

(b) *If $\{U^n w_j : n \in \mathbf{Z}^d, j = 1, \ldots, r\}$ is a Riesz basis for V for some positive integer r, and*

$$U_k(\widetilde{V}) \subset \widetilde{V}, \quad k = 1, \ldots, d,$$

then there exist $\tilde{w}_1, \ldots, \tilde{w}_r$ in $\widetilde{V}$ such that $\{U^n \tilde{w}_j : n \in \mathbf{Z}^d, j = 1, \ldots, r\}$ is a Riesz basis for $\widetilde{V}$, and it is biorthogonal to $\{U^n w_j : n \in \mathbf{Z}^d, j = 1, \ldots, r\}$.

Proof. We follow the proof of the implication (ii)$\Longrightarrow$(iii) in Theorem 2.3, which gives (a) directly.

For (b), suppose now that V has a Riesz basis of the form $\{U^n w_j : n \in \mathbf{Z}^d, j = 1, \ldots, r\}$. By (2.7), F commutes with each U_k, $k = 1, \ldots, d$. Using this observation and (2.8), the dual Riesz basis in V is given by $\{U^n z_j : n \in \mathbf{Z}^d, j = 1, \ldots, r\}$, where $z_j = F^{-1}(w_j), j = 1, \ldots, r$. Likewise, by (2.9), G commutes with each U_k, $k = 1, \ldots, d$. Using (2.10), $\{U^n \tilde{w}_j : n \in \mathbf{Z}^d, j = 1, \ldots, r\}$ is then the desired Riesz basis for $\widetilde{V}$, where $\tilde{w}_j = G^{-1}(z_j), j = 1, \ldots, r$. □

Corollary 2.5 ([1, Theorem 3.2 (iii)]). *Let T be a unitary operator on H. Let V and W be closed linear subspaces of H, and let $\{T^n v_j : n \in \mathbf{Z}, j = 1, \ldots, r\}$ and $\{T^n w_j : n \in \mathbf{Z}, j = 1, \ldots, r\}$ be Riesz bases for V and W respectively, for some positive integer r. Then the oblique projection of H on V along $W^\perp$ is well defined (i.e., $V \oplus W^\perp = H$) if and only if* $\cos(\theta(V, W)) > 0$.

Proof. This follows from Theorem 2.3 and the result [1, Theorem 3.2(i)] that for such V and W, $\cos(\theta(V, W)) = \cos(\theta(W, V))$. □

Corollary 2.6. *Let $V, \tilde{V}, W$ and $\tilde{W}$ be closed linear subspaces of H, and let $V \perp \tilde{W}$ and $W \perp \tilde{V}$. Let $\{v_n\}, \{\tilde{v}_n\}, \{w_n\}$, and $\{\tilde{w}_n\}$ be Riesz bases for $V, \tilde{V}, W$ and $\tilde{W}$ respectively, let $\{v_n\}$ be biorthogonal to $\{\tilde{v}_n\}$, and let $\{w_n\}$ be biorthogonal to $\{\tilde{w}_n\}$. Then $V \cap W = \{0\}, V + W$ is closed, and $\{v_n\} \cup \{w_n\}$ is a Riesz basis for $V \oplus W$.*

Proof. By Theorem 2.3, $V \cap \widetilde{V}^\perp = \{0\}$, and $V + \widetilde{V}^\perp = H$. In particular, $V + \widetilde{V}^\perp$ is closed. Since $W \subset \widetilde{V}^\perp$, we have $V \cap W = \{0\}$, and by Corollary 2.2, $V + W$ is closed. Hence the desired result follows from Theorem 2.1. □

3. An extension theorem and biorthogonal multiwavelets

Let $V_0, \widetilde{V}_0, V_1$ and $\widetilde{V}_1$ be closed linear subspaces of a Hilbert space H, let $V_0 \subset V_1$, $\widetilde{V}_0 \subset \widetilde{V}_1$, and for $i = 0, 1$, let V_i and $\widetilde{V}_i$ have biorthogonal Riesz bases. By Theorem 2.3,

$$V_0 \cap \widetilde{V}_0^\perp = \{0\}, \quad V_0 + \widetilde{V}_0^\perp = H, \tag{3.1}$$

and

$$V_1 \cap \widetilde{V}_1^\perp = \{0\}, \quad V_1 + \widetilde{V}_1^\perp = H. \tag{3.2}$$

Define

$$W_0 = V_1 \cap \widetilde{V}_0^\perp \tag{3.3}$$

and

$$\widetilde{W}_0 = \widetilde{V}_1 \cap V_0^\perp. \tag{3.4}$$

We collect below some properties of W_0 and $\widetilde{W}_0$.

Proposition 3.1. *W_0 and $\widetilde{W}_0$ have the following properties:*

(i) *W_0 is the unique closed linear subspace of H satisfying the conditions $V_1 = V_0 \oplus W_0$ and $W_0 \perp \widetilde{V}_0$.*

(ii) $\widetilde{W}_0^\perp = V_0 + \widetilde{V}_1^\perp$.

(iii) $W_0 \oplus \widetilde{W}_0^\perp = H$.

(iv) *If P, Q and R are the projection on V_1 along $\widetilde{V}_1^\perp$, the projection on V_0 along $\widetilde{V}_0^\perp$, and the projection on W_0 along $\widetilde{W}_0^\perp$ respectively, then $P = Q + R$.*

Proof. We omit the proof of (i), which follows from some simple arguments using (3.1) and (3.3).

Since $V_0 \subset V_1$, by (3.2) and Corollary 2.2, $V_0 + \widetilde{V}_1^{\perp}$ is closed. Hence

$$\widetilde{W}_0^{\perp} = (\widetilde{V}_1 \cap V_0^{\perp})^{\perp} = \overline{\widetilde{V}_1^{\perp} + V_0} = V_0 + \widetilde{V}_1^{\perp}.$$

Now $W_0 \cap \widetilde{W}_0^{\perp} = (V_1 \cap \widetilde{V}_0^{\perp}) \cap (V_0 + \widetilde{V}_1^{\perp})$. Let $x \in V_1 \cap \widetilde{V}_0^{\perp}$ and $x = v + w$ for some v in V_0 and w in $\widetilde{V}_1^{\perp}$. Then w is in $\widetilde{V}_0^{\perp}$, and $v = x - w \in \widetilde{V}_0^{\perp} \cap V_0 = \{0\}$. Hence $x = w \in V_1 \cap \widetilde{V}_1^{\perp} = \{0\}$. Thus $W_0 \cap \widetilde{W}_0^{\perp} = \{0\}$, and

$$W_0 + \widetilde{W}_0^{\perp} = W_0 + (V_0 + \widetilde{V}_1^{\perp}) = V_1 + \widetilde{V}_1^{\perp} = H.$$

Finally, (iv) is an easy consequence of the above properties. □

We recall some notation and terminology from [6]. Let $U := (U_1, ..., U_d)$ be an ordered d-tuple of commuting distinct unitary operators on a Hilbert space H. For a subset S of H, let $\langle S \rangle$ denote the closed linear span of S, and

$$U^{\mathbf{Z}^d}(S) := \{U^n s : n \in \mathbf{Z}^d, s \in S\}.$$

If $V = \{v_1, ..., v_r\}$ and $W = \{w_1, ..., w_p\}$ are finite subsets of H such that

$$\{\langle v_k, U^n w_j \rangle\}_{n \in \mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d), \quad k = 1, \dots, r,\ j = 1, \dots, p, \tag{3.5}$$

then the function $\Phi_{V,W}$ defined almost everywhere on $\mathbf{R}^d$ by

$$\Phi_{V,W}(\theta) := \Big(\sum_{n \in \mathbf{Z}^d} \langle v_k, U^n w_j \rangle e^{in \cdot \theta} \Big)_{1 \le k \le r,\, 1 \le j \le p} \tag{3.6}$$

is an $r \times p$ matrix function with entries in $L^2([0, 2\pi)^d)$. It is easy to verify the following properties:

$$(\Phi_{V,W}(\theta))^* = \Phi_{W,V}(\theta); \tag{3.7}$$

$$V \perp U^{\mathbf{Z}^d}(W) \iff \Phi_{V,W}(\theta) = 0 \ a.e.; \tag{3.8}$$

and if $r = p$, then

(3.9)

$$\langle U^n v_k, U^m w_j \rangle = \delta_{n,m} \delta_{k,j},\ k, j = 1, \dots, r,\ n, m \in \mathbf{Z}^d \iff \ \Phi_{V,W}(\theta) = I_r \ a.e.,$$

where I_r is the $r \times r$ identity matrix.

A deeper result is the following [6, Theorem 2.1]:

Theorem 3.2. *If a finite subset $V = \{v_1, ..., v_r\}$ of H satisfies*

$$\sum_{n \in \mathbf{Z}^d} |\langle v_k, U^n v_j \rangle|^2 < \infty, \quad k, j = 1, \dots, r, \tag{3.10}$$

then $U^{\mathbf{Z}^d}(V)$ is a Riesz basis for $\langle U^{\mathbf{Z}^d}(V) \rangle$ if and only if there exist positive constants C_1 and C_2 such that

$$C_1 \le \Phi_{V,V}(\theta) \le C_2 \tag{3.11}$$

for almost every $\theta \in \mathbf{R}^d$.

Assume now that $Y = \{y_1, ..., y_s\}$ and $\widetilde{Y} = \{\tilde{y}_1, ..., \tilde{y}_s\}$ are finite subsets of H such that $U^{\mathbf{Z}^d}(Y)$ and $U^{\mathbf{Z}^d}(\widetilde{Y})$ are Riesz bases for $\langle U^{\mathbf{Z}^d}(Y)\rangle$ and $\langle U^{\mathbf{Z}^d}(\widetilde{Y})\rangle$ respectively, and $U^{\mathbf{Z}^d}(Y)$ and $U^{\mathbf{Z}^d}(\widetilde{Y})$ are biorthogonal, i.e.,

$$\langle U^n y_k, U^m \tilde{y}_j\rangle = \delta_{n,m}\delta_{k,j}, \quad k, j = 1, \dots, s, \quad n, m \in \mathbf{Z}^d. \tag{3.12}$$

In particular, since $U^{\mathbf{Z}^d}(Y)$ and $U^{\mathbf{Z}^d}(\widetilde{Y})$ are Bessel sequences in H (see [12, pp. 154–155]), the entries of the matrix functions $\Phi_{V,Y}$ and $\Phi_{V,\widetilde{Y}}$ are L^2-functions for any finite subset V of H. By (3.9) and (3.12),

$$\Phi_{Y,\widetilde{Y}} = \Phi_{\widetilde{Y},Y} = I_s \ \ a.e., \tag{3.13}$$

where I_s is the $s \times s$ identity matrix, and we have the biorthogonal expansions

$$f = \sum_{j=1}^{s} \sum_{n\in\mathbf{Z}^d} \langle f, U^n\tilde{y}_j\rangle U^n y_j \quad f \in \langle U^{\mathbf{Z}^d}(Y)\rangle\,, \tag{3.14}$$

and

$$g = \sum_{j=1}^{s} \sum_{n\in\mathbf{Z}^d} \langle g, U^n y_j\rangle U^n \tilde{y}_j, \quad g \in \langle U^{\mathbf{Z}^d}(\widetilde{Y})\rangle\,. \tag{3.15}$$

We also have the following results.

Proposition 3.3. *Let $V = \{v_1, ..., v_r\}$ and $W = \{w_1, ..., w_p\}$ be finite subsets of H satisfying condition* (3.5)*.*

(i) *If $V \subset \langle U^{\mathbf{Z}^d}(Y)\rangle$, then*

$$\Phi_{V,W} = \Phi_{V,\widetilde{Y}}\Phi^*_{W,Y}\,, \quad \Phi_{W,V} = \Phi_{W,Y}\Phi^*_{V,\widetilde{Y}} \tag{3.16}$$

and

$$\Phi_{V,Y} = \Phi_{V,\widetilde{Y}}\Phi_{Y,Y}. \tag{3.17}$$

(ii) *If $V \subset \langle U^{\mathbf{Z}^d}(\widetilde{Y})\rangle$, then*

$$\Phi_{V,W} = \Phi_{V,Y}\Phi^*_{W,\widetilde{Y}}\,, \quad \Phi_{W,V} = \Phi_{W,\widetilde{Y}}\Phi^*_{V,Y}. \tag{3.18}$$

(iii) *If $V \subset \langle U^{\mathbf{Z}^d}(Y)\rangle$ and condition* (3.10) *holds, then*

$$\Phi_{V,V} = \Phi_{V,Y}\Phi^*_{V,\widetilde{Y}} = \Phi_{V,\widetilde{Y}}\Phi^*_{V,Y} = \Phi_{V,\widetilde{Y}}\Phi_{Y,Y}\Phi^*_{V,\widetilde{Y}}\,. \tag{3.19}$$

Proof. By (3.14), for $\nu \in \mathbf{Z}^d$, $k = 1, \dots, r$, $\ell = 1, \dots, p$,

$$\langle v_k, U^\nu w_\ell\rangle = \sum_{j=1}^{s} \sum_{n\in\mathbf{Z}^d} \langle v_k, U^n\tilde{y}_j\rangle\langle y_j, U^{\nu-n}w_\ell\rangle.$$

Hence $\Phi_{V,W} = \Phi_{V,\widetilde{Y}}\Phi_{Y,W} = \Phi_{V,\widetilde{Y}}\Phi^*_{W,Y}$. The second equality in (3.16) follows by taking adjoints. For (3.17), take $W = Y$. (ii) is proved similarly using (3.15), and (iii) follows directly from (i). □

We should emphasis that our present setting is more general than that of [6]. There we assumed that $\langle U^{\mathbf{Z}^d}(Y)\rangle = \langle U^{\mathbf{Z}^d}(\widetilde{Y})\rangle$, but here we don't. Hence some of the results in [6] may not be valid in this new setting.

Using Theorem 3.2 and Proposition 3.3 above, the proof of [9, Proposition 3.3] can be carried over verbatim for our present setting to give

Proposition 3.4. *Let $V = \{v_1, ..., v_r\} \subset \langle U^{\mathbf{Z}^d}(Y)\rangle$, where $r \leq s$. The following conditions are equivalent:*

(i) $U^{\mathbf{Z}^d}(V)$ *is a Riesz basis for* $\langle U^{\mathbf{Z}^d}(V)\rangle$.

(ii) *There exist positive constants A and B such that*

$$A \leq \Phi_{V,Y}\Phi_{V,Y}^* \leq B \quad a.e.$$

(iii) *There exist positive constants $\widetilde{A}$ and $\widetilde{B}$ such that*

$$\widetilde{A} \leq \Phi_{V,\widetilde{Y}}\Phi_{V,\widetilde{Y}}^* \leq \widetilde{B} \quad a.e.$$

We record one more useful result from [6, Theorem 2.4] (see also [9, Theorem 3.1]) that can still be applied here.

Theorem 3.5. *Let $V = \{v_1, ..., v_r\}$ and $Y = \{y_1, ..., y_s\}$ be finite subsets of H and suppose that $U^{\mathbf{Z}^d}(V)$ and $U^{\mathbf{Z}^d}(Y)$ are Riesz bases for $\langle U^{\mathbf{Z}^d}(V)\rangle$ and $\langle U^{\mathbf{Z}^d}(Y)\rangle$ respectively. If $\langle U^{\mathbf{Z}^d}(V)\rangle \subset \langle U^{\mathbf{Z}^d}(Y)\rangle$ and $r = s$, then $\langle U^{\mathbf{Z}^d}(V)\rangle = \langle U^{\mathbf{Z}^d}(Y)\rangle$.*

We now state and prove the main result of this section, which is an extension of [6, Theorem 2.5] to the biorthogonal setting.

Theorem 3.6. *Let $X = \{x_1, ..., x_r\}$, $\widetilde{X} = \{\tilde{x}_1, ..., \tilde{x}_r\}$, $Y = \{y_1, ..., y_s\}$ and $\widetilde{Y} = \{\tilde{y}_1, ..., \tilde{y}_s\}$ be finite subsets of H. Let $U^{\mathbf{Z}^d}(X), U^{\mathbf{Z}^d}(\widetilde{X}), U^{\mathbf{Z}^d}(Y)$ and $U^{\mathbf{Z}^d}(\widetilde{Y})$ be Riesz bases for their closed linear spans $V_0, \widetilde{V}_0, V_1$ and $\widetilde{V}_1$ respectively, $U^{\mathbf{Z}^d}(X)$ biorthogonal to $U^{\mathbf{Z}^d}(\widetilde{X})$, $U^{\mathbf{Z}^d}(Y)$ biorthogonal to $U^{\mathbf{Z}^d}(\widetilde{Y})$, and*

$$V_0 \subset V_1, \quad \widetilde{V}_0 \subset \widetilde{V}_1.$$

Let $W_0 = V_1 \cap \widetilde{V}_0^{\perp}$ and $\widetilde{W}_0 = \widetilde{V}_1 \cap V_0^{\perp}$. If $r < s$, then

(i) *there exists a subset $\Gamma := \{z_1, ..., z_{s-r}\}$ of W_0 such that $U^{\mathbf{Z}^d}(\Gamma)$ is a Riesz basis for W_0 and $U^{\mathbf{Z}^d}(X \cup \Gamma)$ is a Riesz basis for V_1, and*

(ii) *there exists a subset $\widetilde{\Gamma} := \{\tilde{z}_1, ..., \tilde{z}_{s-r}\}$ of $\widetilde{W}_0$ such that $U^{\mathbf{Z}^d}(\widetilde{\Gamma})$ is a Riesz basis for $\widetilde{W}_0$ and $U^{\mathbf{Z}^d}(\widetilde{X} \cup \widetilde{\Gamma})$ is a Riesz basis for $\widetilde{V}_1$, and $U^{\mathbf{Z}^d}(\widetilde{\Gamma})$ is biorthogonal to $U^{\mathbf{Z}^d}(\Gamma)$.*

Proof. Since the $r \times s$ matrix $\Phi_{\widetilde{X},Y}(\theta)$ has at most rank r, choose an $(s-r) \times s$ matrix $Y(\theta)$ such that

$$Y(\theta)\Phi_{\widetilde{X},Y}(\theta)^* = 0 \tag{3.20}$$

and

$$Y(\theta)Y(\theta)^* = I_{s-r} \ . \tag{3.21}$$

(The measurability of one such function Y is assured by some standard arguments in measure theory; see, e.g., [8, Lemma 2.4].) Let the (k, j)-entry of $Y(\theta)$ be $Y_{k,j}(\theta),\ k = 1, ..., s-r,\ j = 1, ..., s$. By (3.21),

$$\sum_{j=1}^{s} |Y_{k,j}(\theta)|^2 = 1, \quad k = 1, ..., s-r,$$

and so all $Y_{k,j}$ are bounded functions in $L^2([0, 2\pi)^d)$. For $k = 1, ..., s-r,\ j = 1, ..., s$, let

$$Y_{k,j}(\theta) = \sum_{n \in \mathbf{Z}^d} a_{k,j}(n)e^{in\cdot\theta} \ , \tag{3.22}$$

where $\{a_{k,j}(n)\}_{n\in\mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d)$, and let

$$z_k = \sum_{j=1}^{s} \sum_{n\in\mathbf{Z}^d} a_{k,j}(n) U^n y_j \ . \tag{3.23}$$

Then z_k is in V_1, and by (3.22), (3.23) and (3.6),

$$Y(\theta) = \Phi_{\Gamma,\widetilde{Y}}(\theta) \quad a.e., \tag{3.24}$$

where $\Gamma := \{z_1, ..., z_{s-r}\}$. By (3.16) and (3.20),

$$\Phi_{\Gamma,\widetilde{X}} = \Phi_{\Gamma,\widetilde{Y}} \Phi^*_{\widetilde{X},Y} = 0 \quad a.e.$$

Hence by (3.8), $\Gamma \perp U^{\mathbf{Z}^d}(\widetilde{X})$. By (3.21) and (3.24),

$$\Phi_{\Gamma,\widetilde{Y}} \Phi^*_{\Gamma,\widetilde{Y}} = I_{s-r} \quad a.e.$$

Hence by Proposition 3.4, $U^{\mathbf{Z}^d}(\Gamma)$ is a Riesz basis for $\langle U^{\mathbf{Z}^d}(\Gamma)\rangle$.

By Proposition 3.1, $V_0 \oplus W_0 = V_1$, which is closed in H. Since

$$\langle U^{\mathbf{Z}^d}(\Gamma)\rangle \subset V_1 \cap \widetilde{V}_0^{\perp} = W_0,$$

using Corollary 2.2, $V_0 + \langle U^{\mathbf{Z}^d}(\Gamma)\rangle$ is also closed in H and $V_0 \cap \langle U^{\mathbf{Z}^d}(\Gamma)\rangle = \{0\}$. Hence by Theorem 2.1, $U^{\mathbf{Z}^d}(X \cup \Gamma)$ is a Riesz basis for $V_0 + \langle U^{\mathbf{Z}^d}(\Gamma)\rangle$. Since

$$\langle U^{\mathbf{Z}^d}(X \cup \Gamma)\rangle = V_0 + \langle U^{\mathbf{Z}^d}(\Gamma)\rangle \subset V_1 = \langle U^{\mathbf{Z}^d}(Y)\rangle$$

and $\#(X \cup \Gamma) = \#(Y) = s$, by Theorem 3.5 we have

$$V_0 \oplus \langle U^{\mathbf{Z}^d}(\Gamma)\rangle = V_1 = V_0 \oplus W_0.$$

Since $\langle U^{\mathbf{Z}^d}(\Gamma)\rangle \subset W_0$, we then actually have $\langle U^{\mathbf{Z}^d}(\Gamma)\rangle = W_0$, and as proven above, $U^{\mathbf{Z}^d}(X \cup \Gamma)$ is a Riesz basis for V_1. This completes the proof of (i).

For each $k = 1, ..., d$, since U_k is unitary, $U_k(V_1) = V_1$, and $U_k(\widetilde{V}_0) = \widetilde{V}_0$, we have $U_k(\widetilde{W_0}) = \widetilde{W_0}$. By Proposition 3.1, $W_0 \oplus \widetilde{W_0}^{\perp} = H$. Hence the results of (ii) follow from (i), Corollary 2.4 and Theorem 2.1. □

Remark 3.7. Letting $V_0 = \widetilde{V}_0$ and $V_1 = \widetilde{V}_1$ in Theorem 3.6, we recover the results of [6, Theorem 2.5]. In this case, $W_0 = \widetilde{W_0}$, and V_1 is the *orthogonal* direct sum of V_0 and W_0.

4. Biorthogonal wavelets

We follow the notation in Section 3. Let $U := (U_1, ..., U_d)$ be an ordered d-tuple of commuting distinct unitary operators on a Hilbert space H. Let $X = \{x_1, ..., x_r\}$ and $\widetilde{X} = \{\tilde{x}_1, ..., \tilde{x}_r\}$ be finite subsets of H such that $U^{\mathbf{Z}^d}(X)$ and $U^{\mathbf{Z}^d}(\widetilde{X})$ are biorthogonal Riesz bases for their closed linear spans V_0 and $\widetilde{V}_0$ respectively. Now suppose that there is a unitary operator D on H such that

$$V_0 \subset V_1 := D(V_0), \quad \widetilde{V}_0 \subset \widetilde{V}_1 := D(\widetilde{V}_0) \tag{4.1}$$

and

$$U^n D = D U^{Mn}, \quad n \in \mathbf{Z}^d \ , \tag{4.2}$$

where M is a $d \times d$ matrix with integer entries and $|det(M)| > 1$. Under this setting, Theorem 3.6 yields analogous results to parts (1) and (2) of [6, Theorem

3.1], leading to the existence of biorthogonal multiwavelets in the Hilbert space setting. A special case of this is when $H = L^2(\mathbf{R}^d)$,

$$(U_k f)(x) = f(x - e_k),$$

where $e_k = (\delta_{k,j})_{j=1,\dots,d}$, $k = 1, \dots, d$, and

$$(Df)(x) = |det(M)|^{\frac{1}{2}} f(Mx),$$

for x in $\mathbf{R}^d$ and f in $L^2(\mathbf{R}^d)$. We shall not go into this any further. Instead, let us consider the special case when $d = 1, r = 1$ and $s = 2$.

Let T and D be unitary operators on a Hilbert space H such that

$$TD = DT^2 . \tag{4.3}$$

Let $T^{\mathbf{Z}}(\{\phi\})$ and $T^{\mathbf{Z}}(\{\tilde{\phi}\})$ be biorthogonal Riesz bases for their closed linear spans V_0 and $\widetilde{V}_0$ respectively such that (4.1) holds. For $j = 0, 1,$ let $\phi_j := DT^j \phi$ and $\tilde{\phi}_j := DT^j \tilde{\phi}$. Then

$$T^{\mathbf{Z}}(\{\phi_0, \phi_1\}) = \{DT^n \phi : n \in \mathbf{Z}\}$$

and

$$T^{\mathbf{Z}}(\{\tilde{\phi}_0, \tilde{\phi}_1\}) = \{DT^n \tilde{\phi} : n \in \mathbf{Z}\}$$

are biorthogonal Riesz bases for V_1 and $\widetilde{V}_1$ respectively. Since ϕ is in V_1 and $\tilde{\phi}$ is in $\widetilde{V}_1$,

$$\phi = \sum_{n \in \mathbf{Z}} \tilde{c}_n DT^n \phi \tag{4.4}$$

and

$$\tilde{\phi} = \sum_{n \in \mathbf{Z}} d_n DT^n \tilde{\phi} \tag{4.5}$$

for some sequences $\{\tilde{c}_n\}$ and $\{d_n\}$ in $\ell^2(\mathbf{Z})$.

Theorem 4.1. *Let*

$$\eta := \sum_{n \in \mathbf{Z}} (-1)^{-n+1} \overline{d_{-n+1}} DT^n \phi \tag{4.6}$$

and

$$\tilde{\eta} := \sum_{n \in \mathbf{Z}} (-1)^{-n+1} \overline{\tilde{c}_{-n+1}} DT^n \tilde{\phi} . \tag{4.7}$$

Then $T^{\mathbf{Z}}(\{\eta\})$ and $T^{\mathbf{Z}}(\{\tilde{\eta}\})$ are biorthogonal Riesz bases for $W_0 := V_1 \cap \widetilde{V}_0^{\perp}$ and $\widetilde{W}_0 := \widetilde{V}_1 \cap V_0^{\perp}$ respectively.

We omit the details here. The proof is a minor modification of the arguments in Section 4 of [9], by first establishing analogous results for our present biorthogonal setting.

Remark 4.2. If $H = L^2(\mathbf{R})$, $Tf(x) = f(x-1)$, and $Df(x) = \sqrt{2} f(2x)$, for x in $\mathbf{R}$ and f in $L^2(\mathbf{R})$, then Theorem 4.1 recovers the explicit formulae of the biorthogonal wavelets, constructed in [2], that are associated with biorthogonal scaling functions.

Acknowledgement

This research was supported by the Wavelets Strategic Research Programme, National University of Singapore, under a grant from the National Science and Technology Board and the Ministry of Education, Republic of Singapore. We would like to thank Professor S. L. Lee for his many helpful suggestions on this paper. After the completion of the first draft of this manuscript, Professor T. N. T. Goodman communicated orally to us that he and his co-authors had also obtained parts of Theorem 2.1 of this paper.

References

[1] A. Aldroubi, *Oblique projections in atomic spaces*, Proc. Amer. Math. Soc. 124 (1996), 2051-2060. MR **96i:**42020

[2] A. Cohen, I. Daubechies and J. C. Feauveau, *Biorthogonal bases of compactly supported wavelets*, Comm. Pure Appl. Math. XLV (1992), 485-560. MR **93e:**42044

[3] X. Dai and D. R. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Mem. Amer. Math. Soc. 134 (1998), no. 640. MR **98m:**47067

[4] I. C. Gohberg and M. G. Krein, *Introduction to The Theory of Linear Nonselfadjoint Operators,* Translations of Mathematical Monographs, Vol. 18, American Mathematical Society, Providence, 1969. MR **39:**7447

[5] T. N. T. Goodman, S. L. Lee and W. S. Tang, *Wavelets in wandering subspaces*, Trans. Amer. Math. Soc. 338 (1993), 639-654. MR **93j:**42017

[6] T. N. T. Goodman, S. L. Lee and W. S. Tang, *Wavelet bases for a set of commuting unitary operators*, Adv. Comput. Math. 1 (1993), 109-126. MR **94h:**42057

[7] P. R. Halmos, *Introduction to Hilbert Spaces and Spectral Multiplicity*, Chelsea, New York, 1951. MR **13:**563a

[8] R. Q. Jia and Z. W. Shen, *Multiresolution and wavelets*, Proc. Edinburgh Math. Soc. 37 (1994), 271-300. MR **95h:**42035

[9] S. L. Lee, H. H. Tan and W. S. Tang, *Wavelet bases for a unitary operator*, Proc. Edinburgh Math. Soc. 38 (1995), 233-260. MR **96g:**42019

[10] J. B. Robertson, *On wandering subspaces for unitary operators*, Proc. Amer. Math. Soc. 16 (1965), 233-236. MR **30:**5165

[11] M. Unser and A. Aldroubi, *A general sampling theory for non-ideal acquisition devices,* IEEE Trans. on Signal Processing 42 (1994), 2915-2925.

[12] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980. MR **81m:**42027

Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, 119260, Republic of Singapore

E-mail address: `mattws@math.nus.edu.sg`