KD∞ IS A CS-ALGEBRA

S. K. JAIN, P. KANWAR, S. MALIK, AND J. B. SRIVASTAVA

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Abstract. In this paper, it is shown that the group algebra KD∞ is right CS if and only if char(K) ≠ 2. Moreover, when char(K) ≠ 2, then KD∞ is also CS as a module over its center.

1. Introduction

Rings whose complement right ideals are direct summands are called right CS-rings. The class of CS-rings includes selfinjective rings, continuous rings etc. and have been of interest to many authors. However, there is hardly any literature on CS-group algebras. It is well known that the group algebra KG, where K is a field, is selfinjective if and only if G is a finite group. But the group algebra KG may be CS without the finiteness condition on the group G. For example if G is a torsion-free solvable-by-finite group, then KG is an Ore domain and hence is a CS-algebra. On the other hand if G is a finite group, then the group ring RG over the ring R = Mn(Z) is not CS for any n ≥ 1. It is, therefore, of interest to study when a given group algebra is CS. In this paper we study the group algebra S = KD∞ over a field K for its being CS or not. It is proved that S is a right CS-algebra if and only if char(K) ≠ 2 (Theorem 3.6). It is further shown that the center Z(S) of S is a Dedekind domain (Lemma 3.8) and that S is also a CS-module over Z(S) (Theorem 3.9).

2. Notation and preliminaries

Throughout, unless otherwise stated, K will denote a field and D∞, the infinite dihedral group, that is, the group generated by two elements a and b where a is of infinite order, b is of order 2 and ab = ba−1. A module will always mean a right unital module. A nonzero submodule N of a module M is said to be essential in M, denoted by N ⊲ M, if, for every nonzero submodule L of M, L ∩ N ≠ 0. N is called closed in M if N has no proper essential extensions in M. A module M is said to be CS if every nonzero submodule of M is essential in a summand of M, or equivalently, if every closed submodule of M is a summand of M. CS-modules are also commonly known as extending modules ([1]). A ring R is called right CS if it is CS as a right module over itself.

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If \( A \) is an algebra over a ring \( R \), then an element \( u \in A \) is called integral over \( R \) if it satisfies a polynomial equation with coefficients in \( R \) and leading coefficient 1. \( A \) is called integral if all its elements are integral.

### 3. Group ring \( KD_\infty \)

Throughout this section, \( S \) will denote the group algebra \( KD_\infty \), and \( R \) will denote the group algebra \( KA \) where \( A = \langle a \rangle \). It is well known that \( KA \) is a PID ([5], Exercise 2, p.28). Also, by ([6], Theorem 5.1 and [3], Proposition 9, p.165), \( S \) is a prime PI-ring. We begin with some lemmas which will be useful to prove our main result. Our first lemma is well known. We state it here without proof for convenience.

**Lemma 3.1** ([1], Corollary 12.8). For a commutative domain \( R \), the following are equivalent:

(a) \( R \) is a Prüfer domain.

(b) \( (R \oplus R)_R \) is a CS-module.

**Lemma 3.2.** \( S \) is CS as a right \( R \)-module.

**Proof.** Since \( S = R \oplus Rb \cong R \times R \) and \( R \) is a Prüfer domain, the result follows by Lemma 3.1.

**Lemma 3.3.** If \( \text{char}(K) \neq 2 \) and if \( U \) is a right ideal of \( S \) such that \( S_R = U_R \oplus X_R \) for some \( R \)-submodule \( X \) of the right \( R \)-module \( S_R \), then there exists a right ideal \( V \) of \( S \) such that \( S = U \oplus V \).

**Proof.** Let \( \pi_1, \pi_2 \) be the projections of \( S_R \) onto \( U_R \) and \( X_R \) respectively. Define \( \gamma : S \to S \) by \( \gamma(s) = \frac{1}{2}[\pi_2(s) + \pi_2(sb)b] \). Since \( \text{char}(K) \neq 2 \), \( \gamma \) is well-defined. Clearly, \( \gamma(s_1 + s_2) = \gamma(s_1) + \gamma(s_2) \). Also,

\[
\gamma(sa) = \frac{1}{2}[\pi_2(sa) + \pi_2(sb)b] = \frac{1}{2}[\pi_2(s)a + \pi_2(sba^{-1})b]
\]

\[
= \frac{1}{2}[\pi_2(s)a + \pi_2(sb)a^{-1}b] = \frac{1}{2}[\pi_2(s)a + \pi_2(sb)ba] = \gamma(s)a.
\]

Similarly, \( \gamma(sb) = \gamma(s)b \). Thus \( \gamma \in \text{Hom}_S(S,S) \). Let \( V = \gamma(S) \). Then \( V \) is a right ideal of \( S \). We will prove that \( S = U \oplus V \). So, let \( s \in S \). Write \( s = (s - \gamma(s)) + \gamma(s) \).

Since

\[
s - \gamma(s) = s - \frac{1}{2}[\pi_2(s) + \pi_2(sb)b] = \frac{1}{2}[(s - \pi_2(s)) + (s - \pi_2(sb)b)]
\]

\[
= \frac{1}{2}[(s - \pi_2(s)) + (sb - \pi_2(sb))b] = \frac{1}{2}[\pi_1(s) + \pi_1(sb)b],
\]

and \( U \) is a right ideal of \( S \), we have \( s - \gamma(s) \in U \). Thus \( S = U \oplus V \). Also since for every \( s \in S \), \( s - \gamma(s) \in U \), we have \( \gamma(s - \gamma(s)) = 0 \), that is, \( \gamma(s) = \gamma^2(s) \) for every \( s \in S \).

To prove \( U \cap V = (0) \), let \( x \in U \cap V \). Then \( x = \gamma(s) \) for some \( s \in S \) and \( \gamma(x) = 0 \). Thus \( \gamma^2(s) = 0 \) and consequently, \( x = \gamma(s) = \gamma^2(s) = 0 \), as desired.

**Lemma 3.4.** If \( U \) is a closed right ideal of \( S \), then \( U \) is a closed submodule of the right \( R \)-module \( S_R \).
Proof. Suppose \( x \in cl(U_R) \), the closure of \( U_R \) in \( S_R \). Then \( x \in S \) and \( xE \subset U \) for some essential right ideal \( E \) of \( R \). Consequently, \( x(ES) \subset US \subset U \). Also \( ES \) is an essential right ideal of \( S \) ([5], Exercise 27, p. 467). Thus \( x \in cl(U_S) = U \), because \( U \) is a closed right ideal of \( S \). This completes the proof.

**Lemma 3.5.** If \( \text{char}(K) = 2 \), then \( S \) has no nontrivial idempotents.

Proof. For \( \alpha = \sum k_i a^i \in R \), let \( \alpha^* = \sum k_i a^{-i} \). Since \( ab = ba^{-1} \), it follows that \( \alpha^* = \sum k_i a^{-i} \). Now if \( \alpha + b\beta \in S \) is a nontrivial idempotent in \( S \), then using \( ab = ba^{-1} \) and \((\alpha+b\beta)^2 = \alpha+b\beta \) we get \( \alpha^2 + \beta^2 \beta = \alpha + \beta \). Thus \( \alpha^2 + \beta^2 \beta = \alpha + \beta \). Since \( R \) is a PID and \( \alpha + b\beta \) is an idempotent in \( S \), \( \beta \neq 0 \). Consequently, using \( R \) is a domain, the relation \( \alpha^2 + \beta^2 \beta = \alpha + \beta \) yields \( \alpha^* + \alpha = 1 \). Since \( \alpha \in R \), \( \alpha = \sum k_i a^i \) where \( k_i \in K \). Since \( \alpha^* + \alpha = 1 \) and \( \text{char}(K) = 2 \), we have \( \alpha = 0 \), a contradiction. Thus \( S \) has no nontrivial idempotents.

**Theorem 3.6.** \( S \) is a right CS-ring if and only if \( \text{char}(K) \neq 2 \).

Proof. First assume that \( \text{char}(K) \neq 2 \). Let \( U \) be a closed right ideal of \( S \). By Lemma 3.4, \( U_R \) is a closed submodule of the right \( R \)-module \( S_R \). Since \( S_R \) is CS (Lemma 3.2), \( U_R \) is a direct summand of \( S_R \). But then by Lemma 3.3, \( U \) is a summand of \( S \). Hence \( S \) is a right CS-ring. Conversely, let \( S \) be right CS. If possible, let \( \text{char}(K) = 2 \). By Lemma 3.5, \( S \) has no nontrivial idempotents. Since \( S \) is right CS, every nonzero right ideal of \( S \) is essential in \( S \). Thus \( S \) and hence the right maximal quotient ring \( Q_{\text{max}}(S) \) of \( S \) is uniform. Since \( S \) is right nonsingular, it follows that \( Q_{\text{max}}(S) \) is a division ring. Hence \( S \) is a domain, a contradiction because \((1 + b)^2 = 0 \) and \( 1 + b \neq 0 \). Thus \( \text{char}(K) \neq 2 \).

In what follows \( Z(S) \) will denote the center of the ring \( S \). Unless otherwise stated \( \text{char}(K) \neq 2 \) and \( e = \frac{1}{2} + \frac{1}{2} b \). Notice that \( e \) is an idempotent in \( S \). For \( \alpha = \sum k_i a^i \), we will write \( \alpha^* = \sum k_i a^{-i} \). In the following lemma we determine \( Z(S) \).

**Lemma 3.7.** For any field \( K \), \( Z(S) = \{ \alpha \in R \mid \alpha = \alpha^* \} \).

Proof. Clearly, \( \{ \alpha \in R \mid \alpha = \alpha^* \} \subset Z(S) \). To prove the reverse inclusion, let \( s = \alpha + b \beta \in Z(S) \). Then \( sx = xs \) for every \( x \in S \). In particular, \( sa = as \) and \( sb = bs \). Now \( sa = as \) gives \( \alpha a = \alpha a + a\beta b \). Since \( \alpha a = \alpha a \), we get \( \beta a^{-1} b = a\beta b \), that is, \( (a^2 - 1) \beta = 0 \). Thus, \( \beta = 0 \). Consequently, \( s = \alpha \in R \). Again as \( sb = bs \), we have \( \alpha b = \alpha b = \alpha^* b \). Thus \( \alpha = \alpha^* \) and the proof is complete.

**Lemma 3.8.** \( Z(S) \) is a Dedekind domain.

Proof. Clearly, \( Z(S) \cong eZ(S) = eSe \). Since \( S \) is right noetherian, \( eSe \) is right noetherian ([7], Lemma 2.7.12). Thus \( Z(S) \) is right noetherian. Since \( S \) is a prime \( PI \) ring, by ([4], Corollary 6.14, p.467), \( S \) is a finitely generated \( Z(S)\)-module. Let \( S = s_1 Z(S) + s_2 Z(S) + ... + s_k Z(S) \) and let for \( 1 \leq i \leq k \), \( s_i = \alpha_i + \beta_i b \) where \( \alpha_i, \beta_i \in R \). Then \( R \oplus R \beta = (\alpha_1 + \beta_1 b) Z(S) + (\alpha_2 + \beta_2 b) Z(S) + ... + (\alpha_k + \beta_k b) Z(S) \). Consequently, \( R = \alpha_1 Z(S) + \alpha_2 Z(S) + ... + \alpha_k Z(S) \), that is, \( R \) is a finitely generated noetherian \( Z(S)\)-module. Thus by ([2], Theorem 17) \( R \) is integral over \( Z(S) \), that is, \( R \cap Z(S) \) is an integral extension of \( Z(S) \).

Let \( L \) denote the quotient field of \( Z(S) \). We will show that \( R \cap L = Z(S) \). Let \( \gamma \in R \cap L \). Then \( \gamma = \alpha \beta^{-1} \) for some \( \alpha, \beta \in Z(S) \). Thus \( \alpha = \gamma \beta \). Since
\( \alpha, \beta \in Z(S), \alpha = \alpha^*, \beta = \beta^* \). Hence \( \alpha = \alpha^* = (\gamma \beta)^* = \beta^* \gamma^* = \beta \gamma^* \) so that \( \gamma^* = \alpha \beta^{-1} = \gamma \). Since \( \gamma \in R \), we have \( \gamma \in Z(S) \). Hence \( R \cap L = Z(S) \). Since \( R \) is a Dedekind domain, it follows, by ([8], Theorem 20, p.283), that \( Z(S) \) is a Dedekind domain.

**Theorem 3.9.** \( R \) and \( S \) are CS as right \( Z(S) \)-modules.

**Proof.** Clearly, \( Z(S) \cap aZ(S) = (0) \). Further, since \( a^{-n} = (a^n + a^{-n}) - a^n \) and \( a^n = a^{n-1}(a + a^{-1}) - a^{n-2} \) we have \( R = Z(S) \oplus aZ(S) \cong Z(S) \times Z(S) \). Also as \( S = R \oplus bR \), we have \( S \cong Z(S) \times Z(S) \times Z(S) \times Z(S) \). By Theorem 3.8, \( Z(S) \) is a Dedekind domain. The result now follows by Lemma 3.1.

**Remark 1.** The uniform dimension of \( R \) as a \( Z(S) \)-module is 2 and that of \( S \) as a \( Z(S) \)-module is 4.

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**REFERENCES**


Department of Mathematics, Ohio University, Athens, Ohio 45701
E-mail address: jain@math.ohiou.edu
E-mail address: pkanwar@math.ohiou.edu
Current address, P Kanwar: Division of Mathematics and Computer Science, Truman State University, Kirksville, Missouri 63501

Department of Mathematics, Hindu College, Delhi-110007, India
E-mail address: sbm@csec.ernet.in

Department of Mathematics, Indian Institute of Technology, New Delhi-110016, India
E-mail address: jbsrivas@maths.iitd.ernet.in

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