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BLASCHKE PRODUCTS AND EXPANDING MAPS OF THE CIRCLE

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ABSTRACT. For an analytic map of the closed unit disk onto itself that leaves the boundary circle invariant, we give necessary and sufficient conditions for the map of the boundary to be expanding.

Let B be an analytic map from the closed unit disk onto itself that maps the boundary circle into itself. Suppose B is expanding when restricted to the boundary. In this case, it is known that there is a fixed point in the interior of the disk. However the converse is not true. In this note we will give a couple of necessary and sufficient conditions for B to be expanding on the boundary circle, one of which gives an estimate on the derivative of the expanding map. A map B as above is a constant multiple of a finite Blaschke product.

Let $D_r = \{z \in \mathbf{C}\colon\ |z| < r\}$ and $C_r = \{z \in \mathbf{C}\colon\ |z| = r\}$. Let $a = \{a_1, \ldots, a_n\}$. The finite Blaschke product B_a is defined by the formula $B_a(z) = \prod_{i=1}^{n} \frac{z-a_i}{1-\overline{a}_i z}$, where $B_a(D_1) = D_1$ and $B_a(C_1) = C_1$. Let τ_a be the restriction of B_a to C_1. We will give conditions on B_a so that τ_a is expanding and an estimate for $|\tau_a'|$. Compare with Martin [2], where a sufficient condition for τ_a expanding is given.

Theorem 1. *The following three conditions are equivalent.*

i) $|\tau_a'| > 1$ *for* $z \in C_1$.
ii) *For some* $r_1 < 1$, $B_a(D_{r_1})$ *is contained in the interior of* D_{r_1}.
iii) *For all* $\lambda \in C_1$, $\lambda \cdot B_a$ *has a fixed point in the interior of* D_1.

Theorem 2. i) *Let* $M_r = \max |B_a(z)|$ *for* $z \in C_r$. *Suppose* $M_{r_1} < r_1$ *is as in condition ii) of Theorem 1. Let* $k = \log M_{r_1} / \log r_1$. *Then* $|\tau_a'(z)| \geq k > 1$ *for* $z \in C_1$.

ii) *Let* $A = \{r\colon\ B_a(D_r)$ *is contained in the interior of* $D_r\}$. *Then* A *is an interval with 1 as an endpoint.*

Proof of Theorem 1. i) $\Rightarrow$ ii). For $z \in C_1$, $B_a'(z) = \tau_a'(z)$. Therefore, the derivative of B_a normal to C_1 is greater than 1. Since C_1 is invariant by B_a, for r sufficiently close to 1, $B_a(D_r) \subset$ interior of D_r.

ii) $\Rightarrow$ iii). This follows from the Brouwer fixed point theorem and the fact that $\lambda \cdot B_a$ satisfies ii) whenever B_a does.

iii)$\Rightarrow$i). B_a has $n+1$ fixed points on the Riemann sphere. Note that $B(\frac{1}{\overline{z}})\overline{B(z)} = 1$ for all $z \in \mathbf{C}$. If B_a has a fixed point in the interior of D_1, then it has one in

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the exterior of D_1. Hence there are at most $n-1$ fixed points on C_1. Since τ_a has topological degree n, B_a must have at least $n-1$ and hence exactly $n-1$ fixed points on C_1. The same is true for $\lambda \cdot B_a$ for $\lambda \in C_1$. Let $\tilde{\tau}_a\colon\ R \Rightarrow R$ be a lift of τ_a to the universal covers. Then $\tilde{\tau}_a(u) - u +$ constant restricted to a fundamental domain of the covering must intersect $n-1$ different lifts of $1 \in C_1$. This implies $\tilde{\tau}_a(u) - u +$ constant must have a positive derivative. If the derivative were zero at some point, then $\tilde{\tau}_a(u) - u + \text{constant} = 0$ would have a multiple root for some constant and hence, for some $\lambda \in C_1$, $\lambda \cdot B_a$ would have more than $n-1$ fixed points in C_1. If $\tilde{\tau}_a(u) - u$ has a positive derivative, then $|\tilde{\tau}'(u)| > 1$ for all u and hence $|\tau_a'| > 1$ for all $z \in C_1$. □

Proof of Theorem 2. Let $r_1 \le r \le r_2 = 1$. Hadamard's Three Circle theorem asserts that $M(r) \le M(r_1)^a M(r_2)^{1-a}$, where $a = \log(r_2/r)/\log(r_2/r_1)$; see Alhfor's [1]. Suppose $M_{r_1} < r_1$. Let $k = \log(M_{r_1})/\log(r_1) > 1$. Let $r_2 = 1$ in Hadamard's inequality. Since $M_{r_2} = 1$ we obtain $M_r \le r^k$ for $r_1 \le r \le 1$. Note that $M_1 = 1$ and $(r^k)' = k$ when $r = 1$. Therefore the derivative of B_a normal to C_1 is greater than or equal to k. Therefore $|B_a'(z)| \ge k$ for $z \in C_1$ and hence $|\tau_a'(z)| \ge k$. This proves part i). Since $r^k < r$ for $r < 1$ we see that $M_r < r$. This shows that A is an interval which completes the proof. □

Remark 1. Theorem 2. i) shows that in Theorem 1, ii) ⇒ i).

Remark 2. In Theorem 1, condition iii) is equivalent to a certain discriminant being negative for all $\lambda \in C_1$. However, it does not appear any easier to verify the discriminant condition than verifying $|\tau_a'| > 1$ directly. The discriminant is obtained as follows. There is a linear fractional transformation of $\mathbf{C}$ that takes the real line onto C_1. Conjugating $\lambda \cdot B_a$ by this map gives a rational function f/g with real coefficients. This function has either exactly one pair of complex conjugate fixed points or none, depending on whether or not condition iii) is satisfied. The discriminant of $f(z) - zg(z)$ is negative if and only if there is one pair of complex conjugate fixed points for f/g.

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