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THE DYNKIN SYSTEM GENERATED BY BALLS IN $\mathbb{R}^d$ CONTAINS ALL BOREL SETS

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ABSTRACT. We show that for every $d \in \mathbb{N}$ each Borel subset of the space $\mathbb{R}^d$ with the Euclidean metric can be generated from closed balls by complements and countable disjoint unions.

Let X be a nonempty set and $\mathcal{S} \subset 2^X$. Following [B, p. 8] we say that $\mathcal{S}$ is *a Dynkin system* if

(D1) $X \in \mathcal{S}$,
(D2) $A \in \mathcal{S} \Rightarrow X \setminus A \in \mathcal{S}$,
(D3) if $A_n \in \mathcal{S}$ are pairwise disjoint, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$.

Some authors use the name *σ-class* instead of Dynkin system. The smallest Dynkin system containing a system $\mathcal{T} \subset 2^X$ is denoted by $\mathcal{D}(\mathcal{T})$. Let P be a metric space. The system of all closed balls in P (of all Borel subsets of P, respectively) will be denoted by Balls(P) (Borel(P), respectively).

We will deal with the problem of whether

$$\mathcal{D}(\mathrm{Balls}(P)) = \mathrm{Borel}(P). \qquad (\star)$$

One motivation for such a problem comes from measure theory. Let μ and ν be finite Radon measures on a metric space P having the same values on each ball. Is it true that $\mu = \nu$? If $\mathcal{D}(\mathrm{Balls}(P)) = \mathrm{Borel}(P)$, then obviously $\mu = \nu$. If P is a Banach space, then $\mu = \nu$ again (Preiss, Tišer [PT]). But Preiss and Keleti ([PK]) showed recently that $(\star)$ is false in infinite-dimensional Hilbert spaces.

We prove the following result.

Theorem 1. *Let $d \in \mathbb{N}$, and let $\mathbb{R}^d$ be equipped with the Euclidean metric. Then $\mathcal{D}(\mathrm{Balls}(\mathbb{R}^d)) = \mathrm{Borel}(\mathbb{R}^d)$.*

This theorem was partially proved by Olejček ([O]), who proved it for $d = 2,\ 3$ (the case $d = 1$ is easy and well-known). Several of Olejček's ideas will also be used in our proof of the general statement. In fact, we prove a slightly more general result:

Each Borel subset of the space $\mathbb{R}^d$ (with the Euclidean metric) can be generated from closed balls by countable monotone unions, countable monotone intersections and countable disjoint unions.

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Each open ball is a countable increasing union of closed balls; we could use the open balls as well.

The paper [PK] contains a remark on the result, proved independently by Jackson and Mauldin, that ($\star$) holds even for every Banach space of finite dimension.

We start with definitions of auxiliary notions. Let X be a nonempty set. We say that $\mathcal{S} \subset 2^X$ is *a monotone system* if the following hold:

(M1) if $A_1 \subset A_2 \subset A_3 \subset \dots,\quad A_n \in \mathcal{S}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$,
(M2) if $A_1 \supset A_2 \supset A_3 \supset \dots,\quad A_n \in \mathcal{S}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{S}$.

We say that $\mathcal{S} \subset 2^X$ is a *$\mathcal{D}^\star$-system* if $\mathcal{S}$ satisfies (M1), (M2) and (D3). The smallest $\mathcal{D}^\star$-system containing $\mathcal{T} \subset 2^X$ is denoted by $\mathcal{D}^\star(\mathcal{T})$. We say that $A \subset \mathbb{R}^d$ is *a sphere of dimension* t if there exists a closed ball $B \subset \mathbb{R}^d$ with a center $x \in \mathbb{R}^d$ and an affine subspace $V \subset \mathbb{R}^d$ of dimension $t+1$ containing x such that $A = V \cap \partial B$. We define

$$\begin{aligned} \mathcal{S}_t = \{A \subset \mathbb{R}^d;\ & A \text{ is a Borel set, which can be covered by} \\ & \text{countably many spheres of dimension } t\}. \end{aligned}$$

Let $A,\ B \subset \mathbb{R}^d$. We say that A is a subset of B modulo $\mathcal{S}_t$ (the notation $A \subset B \bmod \mathcal{S}_t$) if $A \setminus B \in \mathcal{S}_t$. We say that $A = B \bmod \mathcal{S}_t$ if $A \subset B \bmod \mathcal{S}_t$ and $B \subset A \bmod \mathcal{S}_t$. The Euclidean open ball with center $x \in \mathbb{R}^d$ and radius $r > 0$ is denoted by $B(x,r)$.

Lemma 2. *Let X be a nonempty set, and let $\mathcal{V} \subset 2^X$ be a system closed with respect to finite unions and finite intersections. Then the smallest monotone system containing $\mathcal{V}$ is closed with respect to countable unions and countable intersections.*

Proof. We define

$$\begin{aligned} \mathcal{M}(\mathcal{V}) &= \text{the smallest monotone system containing } \mathcal{V}, \\ \mathcal{M}_1 &= \{A \in \mathcal{M}(\mathcal{V});\ \forall B \in \mathcal{V}:\ A \cup B \in \mathcal{M}(\mathcal{V}) \text{ and } A \cap B \in \mathcal{M}(\mathcal{V})\} \qquad \text{and} \\ \mathcal{M}_2 &= \{A \in \mathcal{M}(\mathcal{V});\ \forall B \in \mathcal{M}(\mathcal{V}):\ A \cup B \in \mathcal{M}(\mathcal{V}) \text{ and } A \cap B \in \mathcal{M}(\mathcal{V})\}. \end{aligned}$$

It is easy to see that $\mathcal{M}_1$ and $\mathcal{M}_2$ are monotone systems. We also have $\mathcal{V} \subset \mathcal{M}_1 \subset \mathcal{M}(\mathcal{V})$. Hence $\mathcal{M}_1 = \mathcal{M}(\mathcal{V})$ and $\mathcal{V} \subset \mathcal{M}_2 \subset \mathcal{M}(\mathcal{V})$. This gives $\mathcal{M}(\mathcal{V}) = \mathcal{M}_2$ and therefore $\mathcal{M}(\mathcal{V})$ is closed with respect to finite unions and finite intersections. Thus $\mathcal{M}(\mathcal{V})$ is closed with respect to countable unions and countable intersections. □

Lemma 3. *Each Dynkin system is a $\mathcal{D}^\star$-system.*

Lemma 4. *Each Borel subset of a metric space P can be generated from open subsets of P by countable monotone unions and countable monotone intersections.*

The proof of Lemma 3 is obvious and will be omitted. Lemma 4 is well-known (see [K, p. 344] and Lemma 2).

Lemma 5. *Let $m \in \mathbb{N}$, let $L \subset \mathbb{R}^d$ be a closed set, let $t \in \mathbb{N},\ t \le d$, and let G be a subset of L. Suppose that there exists a sequence of systems of closed balls $\{\mathcal{B}_n\}_{n=1}^{\infty}$ such that*

(i) *$\mathcal{B}_n$ is a union of m systems such that each of them is disjoint,*
(ii) *the center of each ball from $\bigcup_{n=1}^{\infty} \mathcal{B}_n$ is in L,*
(iii) *$L \cap \bigcup \mathcal{B}_n \subset G$ and $G = L \cap \bigcup \mathcal{B}_n \bmod \mathcal{S}_{t-1}$ for every $n \in \mathbb{N}$,*

(iv) $\forall n, n' \in \mathbb{N},\ n' > n\ \forall C \in \mathcal{B}_n\ \forall C' \in \mathcal{B}_{n'} : C' \subset C$ *or* $C' \cap C = \emptyset$,

(v) $\lim_{n\to+\infty} \sup\{\operatorname{diam} C;\ C \in \mathcal{B}_n\} = 0$.

Then there exists $\tilde{G} \in \mathcal{D}^\star(\mathrm{Balls}(\mathbb{R}^d))$ *with* $\tilde{G} \subset G$ *and* $\tilde{G} = G \bmod \mathcal{S}_{t-1}$.

Proof. Let $\mathcal{B}_n = \bigcup_{k=1}^m \mathcal{B}_n^k$, where each $\mathcal{B}_n^k$ is a disjoint system of closed balls. Put

$$\mathcal{D}_{n,j}^k = \{C \in \mathcal{B}_n^k;\ C \not\subset \bigcup_{i=1}^{k-1} \bigcup_{s=j}^{\infty} \bigcup \mathcal{B}_s^i\}, \qquad D_{n,j}^k = \bigcup \mathcal{D}_{n,j}^k,$$

for every $k \in \{1,\dots,m\},\ j, n \in \mathbb{N}$, and

$$\tilde{G} = \bigcup_{k=1}^m \bigcup_{j=1}^\infty \bigcap_{l=1}^\infty \bigcup_{p=l}^\infty D_{p,j}^k, \qquad Z = G \setminus \bigcap_{n=1}^\infty \bigcup \mathcal{B}_n.$$

Using (iii) we obtain $Z \in \mathcal{S}_{t-1}$. Let $x \in G \setminus Z$. Then there exist a sequence $(C_n)_{n=1}^\infty$ of closed balls, an increasing sequence $(p_n)_{n=1}^\infty$ of natural numbers, $k \in \mathbb{N}$ and $j \in \mathbb{N}$ such that

- $x \in C_n \in \mathcal{B}_{p_n}^k$ for every $n \in \mathbb{N}$,
- $x \notin \bigcup_{i=1}^{k-1} \bigcup_{s=j}^\infty \bigcup \mathcal{B}_s^i$.

It is easy to verify that $x \in \bigcap_{l=1}^\infty \bigcup_{p=l}^\infty D_{p,j}^k$. Thus we have $G \subset \tilde{G} \bmod \mathcal{S}_{t-1}$. On the other hand the conditions (ii) and (v) imply that $\tilde{G} \subset L$, since L is closed. Using this and the condition (iii) we obtain

$$\tilde{G} = L \cap \tilde{G} \subset L \cap \left(\bigcup_{n=1}^\infty \bigcup \mathcal{B}_n \right) = \bigcup_{n=1}^\infty \left(L \cap \bigcup \mathcal{B}_n \right) \subset G.$$

Now we want to prove that $\tilde{G} \in \mathcal{D}^\star(\mathrm{Balls}(\mathbb{R}^d))$. At first we prove that

$$\left(\bigcup_{j=1}^\infty \bigcap_{l=1}^\infty \bigcup_{p=l}^\infty D_{p,j}^u \right) \cap \left(\bigcup_{j=1}^\infty \bigcap_{l=1}^\infty \bigcup_{p=l}^\infty D_{p,j}^v \right) = \emptyset \text{ whenever } u \neq v.$$

We may assume that $u < v$. Suppose on the contrary that

$$x \in \left(\bigcup_{j=1}^\infty \bigcap_{l=1}^\infty \bigcup_{p=l}^\infty D_{p,j}^u \right) \cap \left(\bigcup_{j=1}^\infty \bigcap_{l=1}^\infty \bigcup_{p=l}^\infty D_{p,j}^v \right).$$

This means that there exist $j \in \mathbb{N}$ and $j' \in \mathbb{N}$ such that $x \in \bigcap_{l=1}^\infty \bigcup_{p=l}^\infty D_{p,j}^u$ and $x \in \bigcap_{l'=1}^\infty \bigcup_{p'=l'}^\infty D_{p',j'}^v$. There exist $p > j'$ and a ball $C \in \mathcal{D}_{p,j}^u$ with $x \in C$. There exist $p' > p$ and a ball $C' \in \mathcal{D}_{p',j'}^v$ with $x \in C'$. But this and (iv) imply that $C' \subset C \subset \bigcup \mathcal{B}_p^u$. This is a contradiction with $C' \in \mathcal{D}_{p',j'}^v$.

Now it is sufficient to show that

$$\bigcup_{j=1}^\infty \bigcap_{l=1}^\infty \bigcup_{p=l}^\infty D_{p,j}^k \in \mathcal{D}^\star(\mathrm{Balls}(\mathbb{R}^d)) \text{ for every } k = 1,\dots,m.$$

We fix $k \in \{1,\dots,m\}$ and observe that the system $\mathcal{V} = \{\bigcup \mathcal{X};\ \mathcal{X} \subset \bigcup_{n=1}^\infty \mathcal{B}_n^k\}$ is contained in the system of all disjoint unions of closed balls in $\mathbb{R}^d$ and is closed with respect to finite intersections and finite unions. Moreover, $D_{p,j}^k \in \mathcal{V}$ for every

$p, j \in \mathbb{N}$. Using Lemma 2 we obtain that the set $\bigcup_{j=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^k$ is contained in the smallest monotone system containing $\mathcal{V}$ and hence also

$$\bigcup_{j=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^k \in \mathcal{D}^\star(\mathrm{Balls}(\mathbb{R}^d)).$$

□

Lemma 6. *Let $L \subset \mathbb{R}^d$ be an affine subspace of dimension t or a sphere of dimension t. Let $G \subset L$ be relatively open in L. Then there exist $m \in \mathbb{N}$ and a sequence of systems of closed balls $\{\mathcal{B}_n\}_{n=1}^{\infty}$ satisfying the conditions (i) – (v) in Lemma 5.*

Proof. At first we prove the following Claim.

Claim. Let $L \subset \mathbb{R}^d$ be as in Lemma 6, $H \subset L$ be relatively open in L and $\varepsilon > 0$. Then there exist $m \in \mathbb{N}$ (depending only on the dimension of $\mathbb{R}^d$) and a system of closed balls $\mathcal{A}$ such that

(a) $\mathcal{A}$ is a union of m systems such that each of them is disjoint,
(b) each ball from $\mathcal{A}$ has its center in L,
(c) $L \cap \bigcup \mathcal{A} = H$,
(d) each ball from $\mathcal{A}$ has its diameter less than ε,
(e) for every $x \in H$ there exists $r > 0$ such that $B(x, r)$ intersects only finitely many elements from $\mathcal{A}$.

Proof of the Claim. Let $\mathcal{U}$ be a system of all closed balls $B \subset \mathbb{R}^d$ such that the center of B is in H, $B \cap L \subset H$ and $\min\{\frac{1}{2}\operatorname{dist}(\mathit{center}(B), L \setminus H), \frac{1}{4}\varepsilon\} \le \operatorname{diam} B < \varepsilon$. Since H is relatively open in L, we have $L \cap \bigcup \mathcal{U} = H$.

We use Besicovitch's theorem ([Z, p. 9]) to obtain $m \in \mathbb{N}$ (depending only on the dimension of $\mathbb{R}^d$) and disjoint systems $\mathcal{A}_1, \dots, \mathcal{A}_m$ such that $\bigcup_{k=1}^m \mathcal{A}_k \subset \mathcal{U}$ and $L \cap (\bigcup_{k=1}^m \bigcup \mathcal{A}_k) = H$. It is easy to check that $\mathcal{A} = \bigcup_{k=1}^m \mathcal{A}_k$ satisfies (a) – (e) and the proof of the Claim is over. □

We will construct the desired systems $\mathcal{B}_n,\ n \in \mathbb{N}$. Put $\varepsilon = 1$ and $H = G$. Using the Claim we obtain a system of closed balls $\mathcal{B}_1$, which satisfies (a) – (e) and hence also the conditions (i'_l) – (vi'_l) for $l = 1$ below.

(i'_l) For every $n = 1, \dots, l$, $\mathcal{B}_n$ is a union of m systems such that each of them is disjoint,
(ii'_l) the center of each ball from $\bigcup_{n=1}^l \mathcal{B}_n$ is in L,
(iii'_l) $L \cap \bigcup \mathcal{B}_n \subset G$ and $G = L \cap \bigcup \mathcal{B}_n \bmod \mathcal{S}_{t-1}$ for every $n = 1, \dots, l$,
(iv'_l) $\forall n, n' \in \{1, \dots, l\},\ n' > n\ \forall C \in \mathcal{B}_n\ \forall C' \in \mathcal{B}_{n'} :\ C' \subset C$ or $C' \cap C = \emptyset$,
(v'_l) $\sup\{\operatorname{diam} C;\ C \in \mathcal{B}_n\} \le 1/n$ for every $n = 1, \dots, l$,
(vi'_l) for every $n = 1, \dots, l$ and for every $x \in \bigcup \mathcal{B}_n$ there exists $r > 0$ such that $B(x, r)$ intersects only finitely many elements from $\mathcal{B}_n$.

Suppose that we have defined systems $\mathcal{B}_n,\ n = 1, \dots, l$, satisfying ($\mathrm{i}'_l$) – ($\mathrm{vi}'_l$). Put

$$H = \Big(\bigcup \mathcal{B}_l \setminus \bigcup\{\partial B;\ B \in \mathcal{B}_l\}\Big) \cap L \qquad \text{and} \qquad \varepsilon = \frac{1}{l+1}.$$

The condition (vi'_l) gives that H is a relatively open subset of L. Applying the Claim to H and ε we obtain a system of closed balls $\mathcal{B}_{l+1}$.

Let B be a closed ball with the center in L. Then $\partial B \cap L$ is a sphere of dimension $t-1$ or a singleton or an empty set since we use Euclidean balls. This implies that $\bigcup\{\partial B;\ B \in \mathcal{B}_l\} \cap L \in \mathcal{S}_{t-1}$. Thus we have

$$L \cap \bigcup \mathcal{B}_l = L \cap \bigcup \mathcal{B}_{l+1} \bmod \mathcal{S}_{t-1}.$$

Since $G = L \cap \bigcup \mathcal{B}_l \bmod \mathcal{S}_{t-1}$ we conclude $G = L \cap \bigcup \mathcal{B}_{l+1} \bmod \mathcal{S}_{t-1}$. The remaining part of condition (iii$'_{l+1}$) and conditions (i$'_{l+1}$) – (ii$'_{l+1}$), (iv$'_{l+1}$) – (vi$'_{l+1}$) are easy to check. □

Proposition 7. *Let $d \in \mathbb{N}$, and let $\mathbb{R}^d$ be equipped with the Euclidean metric. Then $\mathcal{D}^\star(\mathrm{Balls}(\mathbb{R}^d)) = \mathrm{Borel}(\mathbb{R}^d)$.*

Proof. We will prove by induction over t the following Claim, which implies Proposition 7.

Claim. Let $L \subset \mathbb{R}^d$ be an affine subspace of $\mathbb{R}^d$ of dimension t or a sphere of dimension t and let $B \subset L$ be a Borel set. Then $B \in \mathcal{D}^\star(\mathrm{Balls}(\mathbb{R}^d))$.

Let $t = 0$. Then the Claim obviously holds since B is empty or a singleton or a two point set. Suppose that we have proved the statement for $t = p-1$. We will deal with the case $t = p$. If $G \subset L$ is a relatively open set in L, then Lemmas 5 and 6 show that $G = G_1 \cup G_2$, where $G_1 \cap G_2 = \emptyset$, $G_1 \in \mathcal{D}^\star(\mathrm{Balls}(\mathbb{R}^d))$ and $G_2 \in \mathcal{S}_{p-1}$. The induction hypothesis easily implies that $G_2 \in \mathcal{D}^\star(\mathrm{Balls}(\mathbb{R}^d))$. Thus each relatively open subset of L is contained in $\mathcal{D}^\star(\mathrm{Balls}(\mathbb{R}^d))$. This fact and Lemma 4 show that each Borel subset of L is contained in $\mathcal{D}^\star(\mathrm{Balls}(\mathbb{R}^d))$. Thus the Claim is proved. □

Proof of Theorem 1. Theorem 1 follows immediately from Proposition 7 and Lemma 3. □

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