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CAUCHY CONDITION FOR THE CONVERGENCE IN CATEGORY

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ABSTRACT. It is well known that the sequence $\{f_n\}_{n\in N}$ of real measurable functions converges in measure to some measurable function f if and only if $\{f_n\}_{n\in N}$ is fundamental in measure.

In this note we introduce the notion of sequence fundamental in category in this manner such that the sequence $\{f_n\}_{n\in N}$ of real functions having the Baire property converges in category to some function f having the Baire property if and only if $\{f_n\}_{n\in N}$ is fundamental in category.

1

Let (X, S, m) be a finite measure space and let $\{f_n\}_{n\in N}$ be a sequence of real $\mathcal{S}$-measurable functions defined on X. We say that $\{f_n\}_{n\in N}$ converges in measure to an $\mathcal{S}$-measurable function f if and only if $m(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) \underset{n\to\infty}{\to} 0$ for each $\varepsilon > 0$. We say that $\{f_n\}_{n\in N}$ is fundamental in measure if and only if $m(\{x \in X : |f_n(x) - f_k(x)| \geq \varepsilon\}) \underset{n,k\to\infty}{\to} 0$ for each $\varepsilon > 0$. It is well known that the convergence in measure can be described without the notion of measure, using only the σ-ideal of sets of measure zero. This fact was used in [W] to define the notion of the convergence in category. Recall that $\{f_n\}_{n\in N}$ converges in measure to f if and only if for each subsequence $\{f_{n_m}\}_{m\in N}$ of $\{f_n\}_{n\in N}$ there exists a subsequence $\{f_{n_{m_p}}\}_{p\in N}$ which converges to f a.e. A moment of reflection shows that the fact that $\{f_n\}_{n\in N}$ is fundamental in measure also can be described without the notion of measure. Namely, a sequence $\{f_n\}_{n\in N}$ is fundamental in measure if and only if for all increasing sequences $\{n_k\}_{k\in N}$ and $\{m_k\}_{k\in N}$ of natural numbers the sequence $\{f_{n_k} - f_{m_k}\}_{k\in N}$ converges in measure to a function which is a.e. equal to zero. This observation will be used to define the notion of a sequence which is fundamental in category.

2

Let $X = [0, 1]$, let $\mathcal{S}$ be the σ-algebra of sets having the Baire property and let $\mathcal{I}$ be the σ-ideal of sets of the first category. We shall say that some property holds $\mathcal{I}$-almost everywhere (abbr. $\mathcal{I}$-a.e.) on X if and only if the set of all points which do not have this property belongs to $\mathcal{I}$. Let $\mathcal{M}$ denote the family of all real,

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$\mathcal{S}$-measurable (i.e. having the Baire property), $\mathcal{I}$-a.e. finite and $\mathcal{I}$-a.e. defined on $[0,1]$ functions.

We shall say that a sequence $\{f_n\}_{n\in N}$ of functions from $\mathcal{M}$ converges in category to some function $f \in \mathcal{M}$ if and only if for each subsequence $\{f_{n_m}\}_{m\in N}$ of $\{f_n\}_{n\in N}$ there exists a subsequence $\{f_{n_{m_p}}\}_{p\in N}$ which converges to f $\mathcal{I}$-a.e. We shall use the denotation $f_n \underset{n\to\infty}{\overset{\mathcal{I}}{\to}} f$. We shall say that a sequence $\{f_n\}_{n\in N}$ of functions from $\mathcal{M}$ is fundamental in category if and only if for all increasing sequences of natural numbers $\{n_k\}_{k\in N}$ and $\{m_k\}_{k\in N}$ the sequence $\{f_{n_k} - f_{m_k}\}_{k\in N}$ converges in category to a function which is $\mathcal{I}$-a.e. equal to zero.

Theorem 1. *If a sequence $\{f_n\}_{n\in N}$ of functions from $\mathcal{M}$ converges to a function $f \in \mathcal{M}$ in category, then $\{f_n\}_{n\in N}$ is fundamental in category.*

Proof. Let $\{n_k\}_{k\in N}$ and $\{m_k\}_{k\in N}$ be two increasing sequences of natural numbers. If $\{k_p\}_{p\in N}$ is an increasing sequence of natural numbers, then there exists a subsequence $\{k_{p_r}\}_{r\in N}$ such that $f_{n_{k_{p_r}}} \underset{r\to\infty}{\to} f$ except on a set of the first category. For the increasing sequence $\{k_{p_r}\}_{r\in N}$ there exists a subsequence $\{k_{p_{r_s}}\}_{s\in N}$ such that $f_{m_{k_{p_{r_s}}}} \underset{s\to\infty}{\to} f$ $\mathcal{I}$-a.e. Then $f_{n_{k_{p_{r_s}}}} - f_{m_{k_{p_{r_s}}}} \underset{s\to\infty}{\to} 0$ $\mathcal{I}$-a.e., which means that $\{f_{n_k} - f_{m_k}\}_{k\in N}$ converges to zero in category. □

To prove the inverse theorem first we shall study the behaviour of divergent sequences of functions from $\mathcal{M}$.

Observe that a sequence $\{f_n\}_{n\in N}$ of functions from $\mathcal{M}$ does not converge in category to any function $f \in \mathcal{M}$ if and only if for each $f \in \mathcal{M}$ there exists an increasing sequence $\{n_m\}_{m\in N}$ of natural numbers such that for each subsequence $\{n_{m_p}\}_{p\in N}$ of $\{n_m\}_{m\in N}$ the sequence $\{f_{n_{m_p}}\}_{p\in N}$ does not converge to f $\mathcal{I}$-a.e. The last expression is equivalent to the alternative of the following two conditions:

(1) there exists an increasing sequence $\{n_m\}_{m\in N}$ of natural numbers such that for each subsequence $\{n_{m_p}\}_{p\in N}$ of $\{n_m\}_{m\in N}$ the sequence $\{f_{n_{m_p}}\}_{p\in N}$ does not converge $\mathcal{I}$-a.e. to any function from $\mathcal{M}$,

(2) there exists a pair of functions $g_1, g_2 \in \mathcal{M}$ which are not equivalent (i.e. $\{x : g_1(x) \neq g_2(x)\} \notin \mathcal{I}$) and a pair of increasing sequences $\{n_k\}_{k\in N}$ and $\{m_k\}_{k\in N}$ of natural numbers such that $f_{n_k} \underset{k\to\infty}{\to} g_1$ $\mathcal{I}$-a.e. and $f_{m_k} \underset{k\to\infty}{\to} g_2$ $\mathcal{I}$-a.e.

Observe also that for the sequence $\{n_m\}_{m\in N}$ described in (1) we have two possibilities:

(1a) for each increasing subsequence $\{n_{m_p}\}_{p\in N}$ of $\{n_m\}_{m\in N}$

$$\{x \in X : \limsup_p f_{n_{m_p}}(x) - \liminf_p f_{n_{m_p}}(x) > 0\} \notin \mathcal{I}$$

or, what amounts to the same,

$$\{x \in X : \limsup_p f_{n_{m_p}}(x) - \liminf_p f_{n_{m_p}}(x) > \delta\} \notin \mathcal{I}$$

for some $\delta > 0$,

(1b) there exists a subsequence $\{n_{m_p}\}_{p\in N}$ of $\{n_m\}_{m\in N}$ for which

$$\{x \in X : \lim_{p\to\infty} f_{n_{m_p}}(x) = +\infty\} \cup \{x \in X : \lim_{p\to\infty} f_{n_{m_p}}(x) = -\infty\} \notin \mathcal{I}.$$

When (1a) is fulfilled we shall say that a sequence $\{f_{n_{m_p}}\}_{p\in N}$ diverges by oscillating (it means that no subsequence converges $\mathcal{I}$-a.e.; see [BG]).

Observe that all above remarks remain true for arbitrary σ-algebra $\mathcal{S}$ and σ-ideal $\mathcal{I} \subset \mathcal{S}$.

For infinite subsets M, M' of the set of all natural numbers, $M \subset^* M'$ will mean that $M - M'$ is a finite set.

Replacing the expressions "a.e.", "measure zero" and "measurability" by "except on a set of the first category", "first category" and "the Baire property" in the proofs of Proposition 2.4, Proposition 2.5, Theorem 2.6 and Corollary 2.7 from [BG], we obtain the following lemma.

Lemma 1. *Let $\{f_n\}_{n\in N}$ be a sequence of functions from $\mathcal{M}$ such that each subsequence $\{f_{n_m}\}_{m\in N}$ of $\{f_n\}_{n\in N}$ diverges by oscillating. Then there exist a set $Y \subset [0,1]$ of the second category and having the Baire property, a pair of numbers $r \in R$ and $\delta > 0$ and an infinite set $M \subset N$ such that for each infinite set $L \subset^* M$ and for each set $A \subset Y$, which has the Baire property and is of the second category, there exist a pair of points $x, y \in A$ and an infinite set $L_0 \subset L$ such that $f_n(x) > r + \delta$ and $f_n(y) < r$ for each $n \in L_0$.*

Lemma 2. *Let $\{f_n\}_{n\in N}$ be a sequence of functions from $\mathcal{M}$ such that each subsequence $\{f_{n_m}\}_{m\in N}$ of $\{f_n\}_{n\in N}$ diverges by oscillating. Then there exist two increasing sequences $\{n_k\}_{k\in N}$ and $\{m_k\}_{k\in N}$ of natural numbers such that $\{f_{n_k} - f_{m_k}\}_{k\in N}$ does not converge to zero in category.*

Proof. Let $P_n \subset [0,1]$ for $n \in N$ be a set of the first category such that $f_n|[0,1]-P_n$ is continuous (compare [O], th. 8.1).

Let Y, r, δ and M be as in Lemma 1. We can obviously suppose that $Y \subset [0,1]-\bigcup_{n=1}^{\infty} P_n$, so all functions are continuous at each point of Y. Let $(a,b) \subset [0,1]$ be an interval such that $(a,b) - Y \in \mathcal{I}$ (recall that $Y \in \mathcal{S} - \mathcal{I}$). Observe that if $(c,d) \subset (a,b)$, then also $(c,d) - Y \in \mathcal{I}$.

Put $A = (a,b) \cap Y$. From Lemma 1 there exist an infinite set $M_1^1 \subset M$ and a point $x_1^1 \in A$ such that $f_n(x_1^1) > r + \delta$ for $n \in M_1^1$. Choose $n_1 \in M_1^1$. From the continuity of f_{n_1} at x_1^1 on Y it follows that there exists $\eta_1 > 0$ such that $f_{n_1}(x) > r+\delta$ for $x \in (x_1^1-\eta_1, x_1^1+\eta_1)\cap Y$. Now for $A = (x_1^1-\eta_1, x_1^1+\eta_1)\cap Y$ and M_1^1 again by virtue of Lemma 1 there exist an infinite set $M_2^1 \subset M_1^1$ and a point $y_1^1 \in A$ such that $f_n(y_1^1) < r$ for $n \in M_2^1$. Choose $m_1 \in M_2^1$. From the continuity of f_{m_1} at y_1^1 on Y it follows that there exists $\varepsilon_1 > 0$ such that $f_{m_1}(x) < r$ for $x \in (y_1^1 - \varepsilon_1, y_1^1 + \varepsilon_1) \cap Y \subset (x_1^1 - \eta_1, x_1^1 + \eta_1) \cap Y$. Let $G_1 = (y_1^1 - \varepsilon_1, y_1^1 + \varepsilon_1)$. Then we have $f_{n_1}(x) - f_{m_1}(x) > \delta$ on $G_1 \cap Y$.

Suppose that we have chosen increasing finite sequences $(n_1, \dots, n_{k-1})$ and $(m_1, \dots, m_{k-1})$ and a sequence $(G_1, \dots, G_{k-1})$ of open sets such that

$$G_j \cap \left(a + \frac{i-1}{j}(b-a),\, a + \frac{i}{j}(b-a)\right) \neq \emptyset$$

for $j \in \{1, \dots, k-1\}$ and $i \in \{1, \dots, j\}$ and $f_{n_j}(x) - f_{m_j}(x) > \delta$ for $x \in G_j \cap Y$, $j \in \{1, \dots, k-1\}$. We have also chosen descending sequences of infinite sets of natural numbers $M_1^1 \supset M_2^1; M_1^2 \supset \ldots \supset M_4^2; \ldots ; M_1^{k-1} \supset \ldots \supset M_{2(k-1)}^{k-1}$.

To make an inductive step we divide (a,b) into k parts $\{(a_i^k, b_i^k)\}_{i=1,\dots,k}$ of equal length, i.e. such that $a_i^k = a + \frac{i-1}{k}(b-a)$, $b_i^k = a + \frac{i}{k}(b-a)$. For $A = (a_1^k, b_1^k) \cap Y$ and M we use Lemma 1 to find an infinite set $M_1^k \subset M$ and a point $x_1^k \in A$ such that $f_n(x_1^k) > r+\delta$ for $n \in M_1^k$. Then we repeat the procedure for $A = (a_2^k, b_2^k) \cap Y$ and M_1^k obtaining $x_2^k \in A$ and an infinite set $M_2^k \subset M_1^k$ such that $f_n(x_2^k) > r + \delta$

for $n \in M_2^k$ (observe that also $f_n(x_1^k) > r + \delta$ for $n \in M_2^k$). After k choices we obtain an infinite set M_k^k of natural numbers and a set $\{x_1^k, \dots, x_k^k\} \subset Y$ such that $x_1^k \in (a_i^k, b_i^k)$ and $f_n(x_i^k) > r + \delta$ for $i \in \{1, \dots, k\}$ and $n \in M_k^k$.

Choose $n_k \in M_k^k$ such that $n_k > n_{k-1}$. From the continuity of f_{n_k} at each x_i^k, $i \in \{1, \dots, k\}$, on Y it follows that there exists $\eta_k > 0$ such that $f_{n_k}(x) > r + \delta$ for $x \in \bigcup_{i=1}^k (x_i^k - \eta_k, x_i^k + \eta_k) \cap Y$.

To find m_k and the open set G_k we proceed as follows: for $A = (x_1^k - \eta_1, x_1^k + \eta_1) \cap (a_1^k, b_1^k) \cap Y$ and M_k^k applying Lemma 1 we choose $y_1^k \in A$ and an infinite set $M_{k+1}^k \subset M_k^k$ such that $f_n(y_1^k) < r$ for $n \in M_{k+1}^k$. Then we choose successively infinite sets of natural numbers $M_{k+2}^k \supset M_{k+3}^k \supset \dots \supset M_{2k}^k$ (of course, $M_{k+2}^k \subset M_{k+1}^k$) and points $y_2^k, \dots, y_k^k$ such that $y_i^k \in (x_i^k - \eta_k, x_i^k + \eta) \cap (a_i^k, b_i^k) \cap Y$ for $i \in \{2, \dots, k\}$ and $f_n(y_i^k) < r$ for $n \in M_{k+i}^k$. At last, we choose $m_k > m_{k-1}$ from the set M_{2k}^k. Then we have $f_{m_k}(y_i^k) < r$ for $i \in \{1, \dots, k\}$. From the continuity of f_{m_k} at each y_i^k, $i \in \{1, \dots, k\}$, it follows that there exists $\varepsilon_k > 0$ such that $(y_i^k - \varepsilon_k, y_i^k + \varepsilon_k) \subset (x_i^k - \eta_k, x_i^k + \eta)$ and if we put $G_k = \bigcup_{i=1}^k (y_i^k - \varepsilon_k, y_i^k + \varepsilon_k)$, then $f_{m_k}(x) < r$ for $x \in G_k \cap Y$. Then we have $f_{n_k}(x) - f_{m_k}(x) > \delta$ for $x \in G_k \cap Y$ and $G_k \cap \left(a + \frac{i-1}{k}(b-a),\, a + \frac{i}{k}(b-a)\right) \neq \emptyset$ for $i \in \{1, \dots, k\}$.

Since for each increasing sequence $\{k_p\}_{p \in N}$ of natural numbers we have $E = \{x \in [0,1] : \limsup_p (f_{n_{k_p}}(x) - f_{m_{k_p}}(x)) \geq \delta\} \supset \bigcap_{j=1}^\infty \bigcup_{p=j}^\infty \{x \in [0,1] : f_{n_{k_p}}(x) - f_{m_{k_p}}(x) > \delta\} \supset Y \cap \bigcap_{j=1}^\infty \bigcup_{p=j}^\infty G_{k_p}$ and, as is not difficult to see, the set $\bigcup_{p=j}^\infty G_{k_p}$ is open and dense in (a,b) for each $j \in N$, so E is residual in (a,b) and hence $\{f_{n_{k_p}} - f_{m_{k_p}}\}_{p \in N}$ does not converge $\mathcal{I}$-a.e. Finally, $\{f_{n_k} - f_{m_k}\}_{k \in N}$ does not converge in category. □

Theorem 2. *If a sequence $\{f_n\}_{n \in N}$ of functions from $\mathcal{M}$ is fundamental in category, then it converges in category to some function $f \in \mathcal{M}$.*

Proof. Suppose that $\{f_n\}_{n \in N}$ does not converge in category to any function from $\mathcal{M}$. Then, according to remarks preceding the series of lemmas, either (1) there exists a pair of functions $g_1, g_2 \in \mathcal{M}$ which are not equivalent and a pair of increasing sequences $\{n_k\}_{k \in N}$ and $\{m_k\}_{k \in N}$ of natural numbers such that $f_{n_k} \underset{k \to \infty}{\to} g_1$ $\mathcal{I}$-a.e. and $f_{m_k} \underset{k \to \infty}{\to} g_2$ $\mathcal{I}$-a.e., or (2) there exists an increasing sequence $\{n_m\}_{m \in N}$ of natural numbers such that for each subsequence $\{n_{m_p}\}_{p \in N}$ of $\{n_m\}_{m \in N}$ the sequence $\{f_{n_{m_p}}\}_{p \in N}$ diverges by oscillating, or (3) there exists an increasing sequence $\{n_m\}_{m \in N}$ of natural numbers such that $\{x \in [0,1] : \lim_{m \to \infty} f_{n_m}(x) = +\infty\} \notin \mathcal{I}$ or $\{x \in [0,1] : \lim_{m \to \infty} f_{n_m}(x) = -\infty\} \notin \mathcal{I}$. In case (1) the sequence $\{f_n\}_{n \in N}$ is not fundamental in category because $\{f_{n_k} - f_{m_k}\}_{k \in N}$ does not converge to zero in category. In case (2) from Lemma 2 we obtain immediately that $\{f_n\}_{n \in N}$ is not fundamental in category. Suppose now that in case (3) the first possibility holds. To simplify the description assume that $\{n_m\}_{m \in N}$ is the sequence $\{n\}_{n \in N}$ of all natural numbers, i.e. that $E = \{x \in [0,1] : \lim_{n \to \infty} f_n(x) = +\infty\} \notin \mathcal{I}$. Since $E = \bigcap_{k=1}^\infty \bigcup_{m=1}^\infty \bigcap_{n=m}^\infty \{x \in [0,1] : f_n(x) \geq k\}$, the set E has the Baire property and there exists an interval (a,b) such that $(a,b) - E \in \mathcal{I}$. Take $Y = (a,b) \cap E - \bigcup_{n=1}^\infty P_n$, where, as before, $P_n \in \mathcal{I}$ is a set such that $f_n | [0,1] - P_n$ is continuous. We shall find an increasing sequence $\{j_n\}_{n \in N}$ of natural numbers such that $\{f_{j_n} - f_n\}_{n \in N}$ does not converge to zero in category. Put $j_1 = 1$. Suppose that we have constructed j_{n-1}. To find j_n choose an increasing sequence $x_1^n < x_2^n < \dots < x_{k_n}^n$ such that

$x_i^n \in Y$ for $i \in \{1, \dots, k_n\}$, $x_1^n - a < \frac{1}{n}$, $b - x_{k_n}^n < \frac{1}{n}$ and $x_i^n - x_{i-1}^n < \frac{1}{n}$ for $i \in \{2, \dots, k_n\}$.

Let $c_n = \max\{f_n(x_1^n), \dots, f_n(x_{k_n}^n)\}$. Since $\{f_n(x)\}_{n \in N}$ diverges to $+\infty$ at each point $x \in Y$, there exists a natural number $j_n > j_{n-1}$ such that

$$\min\{f_{j_n}(x_1^n), \dots, f_{j_n}(x_{k_n}^n)\} > c_n + 1.$$

From the continuity of all f_n's on Y we conclude that there exists an open set $G_n \supset \{x_1^n, \dots, x_{k_n}^n\}$ such that $f_{j_n}(x) - f_n(x) > 1$ for all $x \in G_n \cap Y$. Hence for each increasing sequence $\{n_p\}_{p \in N}$ of natural numbers we have $\{x \in (a, b) : \{f_{j_{n_p}}(x) - f_{n_p}(x)\}_{p \in N}$ does not converge to zero$\} \supset Y \cap \limsup_p G_{n_p}$. It is not difficult to see that the last set is residual in (a, b). Hence $\{f_{j_n} - f_n\}_{n \in N}$ does not converge to zero in category, so $\{f_n\}_{n \in N}$ is not fundamental in category. □

3

Suppose now that $X = [0, 1]$, $\mathcal{S}_1$ is the σ-algebra of measurable sets having the Baire property, $\mathcal{I}_1$ is the σ-ideal of null sets of the first category and $\mathcal{M}_1$ is the family of all real, $\mathcal{S}_1$-measurable, $\mathcal{I}_1$-a.e. finite and $\mathcal{I}_1$-a.e. defined on $[0, 1]$ functions. The notion of the convergence with respect to $\mathcal{I}_1$ as well as the notion of fundamentality in $\mathcal{I}_1$ are defined in the natural way. The proof of the following theorem is exactly the same as the proof of Theorem 1:

Theorem 1a. *If a sequence $\{f_n\}_{n \in N}$ of functions from $\mathcal{M}_1$ converges to a function $f \in \mathcal{M}_1$ with respect to $\mathcal{S}_1$, then $\{f_n\}_{n \in N}$ is fundamental in $\mathcal{S}_1$.*

To prove the inverse theorem observe that if $[0, 1] = A \cup B$, where A is a null set and B is a set of the first category, then a set $E \subset [0, 1]$ is a null set of the first category if and only if $E \cap A$ is of the first category and $E \cap B$ is a null set.

Theorem 2a. *If a sequence $\{f_n\}_{n \in N}$ of functions from $\mathcal{M}_1$ is fundamental in $\mathcal{I}_1$, then it converges with respect to $\mathcal{I}_1$ to some function $f \in \mathcal{M}_1$.*

Proof. Let $[0, 1] = A \cup B$, where A is a null set, B is a set of the first category and $A \cap B = \emptyset$. Put for each $n \in N$

$$g_n(x) = \begin{cases} f_n(x) & \text{for } x \in A, \\ 0 & \text{for } x \in B, \end{cases}$$

and

$$h_n(x) = \begin{cases} f_n(x) & \text{for } x \in B, \\ 0 & \text{for } x \in A. \end{cases}$$

Then a sequence $\{g_n\}_{n \in N}$ consists of functions from $\mathcal{M}$ and is fundamental in category, so by virtue of Theorem 2 it converges to some function $g \in \mathcal{M}$. Also $\{h_n\}_{n \in N}$ is a sequence of measurable functions fundamental in measure, so it converges to some measurable function h.

Taking into account the above observation, it is not difficult to see that a sequence $\{f_n\}_{n \in N}$ converges in $\mathcal{I}_1$ to a function $f = (g|A) \cup (h|B)$. □

4

In [V] one can find an example of partially ordered space, in which (o)-fundamental sequences need not converge. However, the pair $(\mathcal{S}, \mathcal{I})$ in [V] does not fulfil ccc.

Question. Is it possible to find a σ-algebra $\mathcal{S}$ and a σ-ideal $\mathcal{I} \subset \mathcal{S}$ of subsets of $[0,1]$ such that the pair $(\mathcal{S}, \mathcal{I})$ fulfils ccc and there exists a sequence $\{f_n\}_{n\in N}$ of real $\mathcal{S}$-measurable $\mathcal{I}$-a.e. finite and $\mathcal{I}$-a.e. defined on $[0,1]$ functions which is fundamental with respect to $\mathcal{I}$ but does not converge with respect to $\mathcal{I}$?

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