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PARANORMAL SPACES UNDER $\diamondsuit^*$

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Abstract. We prove that paranormal spaces of character $\leq \omega_1$ are ω_1-collectionwise Hausdorff assuming the set-theoretic principle $\diamondsuit^*$. This gives an affirmative answer to problem 197 in *Problems I wish I could solve*, by W. S. Watson (Open Problems in Topology (1990), 37–76).

1. Introduction and notation

In the classic paper [F], W. G. Fleissner proved that normal first countable spaces are collectionwise Hausdorff assuming the set-theoretic principle $\diamondsuit_{SS}$, a consequence of Godel's Axiom of Constructibility (V=L). A few years later, Fleissner, Shelah, and Taylor each showed (independently) that assuming the popular set-theoretic principle $\diamondsuit^*$ normal first countable spaces are ω_1-collectionwise Hausdorff. Thus, the following results were established:

Theorem 1.1 (Fleissner, 1974)**.** *Assume $\diamondsuit_{SS}$. Normal first countable spaces are collectionwise Hausdorff.*

Theorem 1.2 (Fleissner, Shelah and Taylor, ca 1979)**.** *Assume $\diamondsuit^*$. Normal first countable spaces are ω_1-collectionwise Hausdorff.*

It is now common to ask if certain separation properties (e.g. collectionwise Hausdorff) of normal spaces also hold for countably paracompact spaces. In fact in [W1], W. S. Watson proved that countably paracompact first countable spaces are collectionwise Hausdorff assuming $\diamondsuit_{SS}$.

Theorem 1.3 (Watson, 1985)**.** *Assume $\diamondsuit_{SS}$. Countably paracompact first countable spaces are collectionwise Hausdorff.*

This result came ten years after Fleissner's original result on normal spaces. However, the corresponding theorem on countably paracompact spaces has resisted a solution until now. In this paper we will show the following:

Theorem 1.4. *Assume $\diamondsuit^*$. Countably paracompact first countable spaces are ω_1-collectionwise Hausdorff.*

This gives an affirmative answer to problem 197 in [W2]. In fact, we prove the more general theorem below:

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Theorem 1.5. *Assume $\diamondsuit^*$. Paranormal spaces of character $\leq \omega_1$ are ω_1-collectionwise Hausdorff.*

Paranormality is a topological property that generalizes both normality and countable paracompactness. Thus Theorem 1.5 extends the earlier result of Fleissner, Shelah, and Taylor.

We recall the definitions. Let X be a topological space and D be a closed discrete subspace of X. Then D is said to be *separated* in X if there is a disjoint collection $\{U(x) : x \in D\}$ of open sets with $x \in U(x)$ for each $x \in D$. A space X is said to be *collectionwise Hausdorff* if all closed discrete subspaces are separated. We use the following definition of paranormality, due to P. Nyikos [N]:

Definition 1.6. A space is paranormal if every countable discrete collection of closed sets $\{D_n : n \in \omega\}$ can be expanded to a locally finite collection of open sets $\{U_n : n \in \omega\}$, i.e. $D_n \subset U_n$ and $U_n \cap D_m \neq \emptyset$ iff $D_m = D_n$.

The collection $\{U_n : n \in \omega\}$ is said to be a locally finite open expansion of the collection $\{D_n : n \in \omega\}$.

The notation ${}^A B$ denotes the set of all functions $f : A \to B$. Let ω_1 be embedded as a closed discrete subspace of a paranormal space X of character $\leq \omega_1$. For each $\alpha \in \omega_1$, let $\{U(\alpha, \beta) : \beta \in \omega_1\}$ enumerate a neighborhood base at α. Then a potential separation of ω_1 would correspond to a function $f \in {}^{\omega_1}\omega_1$. In particular, given a partial function $f \in {}^{\omega_1}\omega_1$, let $U_f = \bigcup\{U(\alpha, f(\alpha)) : \alpha \in \mathrm{dom}(f)\}$ and define

$$S_f = \{\alpha \in \omega_1 : \alpha \in \overline{U_{f|\alpha}}\}.$$

Similarly, for each $f \in {}^{\omega_1}\omega_1$, $g \in {}^{\omega_1}\omega$ and $n \in \omega$, let $U^n_{f,g} = \bigcup\{U(\alpha, f(\alpha)) : \alpha \in \mathrm{dom}(f)$ and $g(\alpha) = n\}$ and define

$$S^n_{f,g} = \{\alpha \in \omega_1 : \alpha \in \overline{U^n_{f|\alpha, g|\alpha}}\}.$$

We think of the function $f \in {}^{\omega_1}\omega_1$ as the neighborhood picker and the function $g \in {}^{\omega_1}\omega$ as the color picker. Finally, for functions $f_0, f_1 \in {}^{\omega_1}\omega_1$, we say that f_1 *refines* f_0 if $U(\alpha, f_1(\alpha)) \subseteq U(\alpha, f_0(\alpha))$ for every $\alpha \in \omega_1$.

We will prove Theorem 1.5 by contradiction. So throughout this paper let us assume that X is a paranormal space of character $\leq \omega_1$ and that ω_1 is a *non*-separated closed discrete subset of X.

2. The ideal

In this section, we define an ideal I of subsets of ω_1 and use paranormality of X to show that this ideal is countably complete.

We define a subset $A \subset \omega_1$ to be in I if there are functions $f \in {}^{\omega_1}\omega_1$ and $g \in {}^{\omega_1}\omega$ such that $S^n_{f,g} \cap A = \emptyset$ for every $n \in \omega$. It is a straightforward argument to show that I is an ideal. The remainder of this section is devoted to showing that I is countably complete and proper.

Before we begin, note that we could have defined an ideal by requiring that a set A is in the ideal if and only if there is a function $f \in {}^{\omega_1}\omega_1$ with $S_f \cap A = \emptyset$. Actually, in the class of paranormal spaces, these ideals are the same! Clearly a set in the latter ideal is in the former. Let us prove that the inclusion is reversed:

Lemma 2.1. *If $A \subset \omega_1$, $f \in {}^{\omega_1}\omega_1$, and $g \in {}^{\omega_1}\omega$ are such that $S^n_{f,g} \cap A = \emptyset$ for every $n \in \omega$, then there is a function $h \in {}^{\omega_1}\omega_1$ such that $S_h \cap A = \emptyset$.*

Proof. Let A, f, and g be as in the hypothesis. For every $n \in \omega$, define

$$B_n = \{\beta \in \omega_1 : g(\beta) = n\}$$

Let $\{U_n : n \in \omega\}$ be a locally finite open expansion of $\{B_n : n \in \omega\}$ (apply paranormality). Let $h \in {}^{\omega_1}\omega_1$ be such that, for every $\beta \in \omega_1$,

$$U(\beta, h(\beta)) \subseteq U(\beta, f(\beta)) \cap U_{g(\beta)}.$$

We claim that h satisfies the conclusion of the lemma. For every $n \in \omega$, let h_n be the partial function defined by $h_n = h|B_n$ and let $\alpha \in A$. By assumption, $\alpha \notin \overline{U^n_{f|\alpha,g|\alpha}}$ and since h refines f, we have $\alpha \notin \overline{U_{h_n|\alpha}}$ for every $n \in \omega$. Then, since $U_{h_n|\alpha} \subset U_n$ and $\{U_n : n \in \omega\}$ is locally finite, it follows that $\alpha \notin \overline{U_{h|\alpha}}$. Thus, $\alpha \notin S_h$ and hence, $S_h \cap A = \emptyset$. □

By Lemma 2.1, for any set A in our ideal I, there is a corresponding function f such that $S_f \cap A = \emptyset$. In such a case, we say that f *witnesses* that $A \in I$.

Lemma 2.2. *The ideal I is countably complete.*

Proof. Let $\{A_n : n \in \omega\} \subseteq I$ and set $A = \bigcup_{n\in\omega} A_n$. Without loss of generality, we may assume that $A_n \cap A_m = \emptyset$ for $n \neq m$. For each $n \in \omega$, fix a function $g_n : \omega_1 \to \omega_1$ such that $A_n \cap S_{g_n} = \emptyset$. Let U_n be open such that $A_n \subseteq U_n$ and so that $\{U_n : n \in \omega\}$ is locally finite. Now let $g : \omega_1 \to \omega_1$ so that if $U(\beta, g(\beta)) \cap U_n \neq \emptyset$, then $U(\beta, g(\beta)) \subseteq U(\beta, g_n(\beta))$ for each $\beta \in \omega_1$ and each $n \in \omega$. Such a g exists since the U_n's are locally finite.

We aim to show that g witnesses that $A \in I$. By way of contradiction, suppose that $A \cap S_g \neq \emptyset$. Then there are an $n \in \omega$ and an $\alpha \in A_n \cap S_g$. Let V be open such that $\alpha \in V \subseteq U_n$ and so that $V \cap U_{g_n|\alpha} = \emptyset$ (since $A_n \cap S_{g_n} = \emptyset$). But $\alpha \in S_g$ implies that there is a $\beta < \alpha$ such that $U(\beta, g(\beta)) \cap V \neq \emptyset$. Since $V \subseteq U_n$, we have $U(\beta, g(\beta)) \cap U_n \neq \emptyset$. But this implies that $U(\beta, g(\beta)) \subseteq U(\beta, g_n(\beta))$. Therefore $V \cap U(\beta, g_n(\beta)) \neq \emptyset$, which is a contradiction. □

Lemma 2.3. *The set ω_1 is not in I.*

Proof. Suppose that $\omega_1 \in I$. Let f witness that $\omega_1 \in I$. For every $\alpha \in \omega_1$, let $g(\alpha) \in \omega_1$ such that $U(\alpha, g(\alpha)) \subseteq U(\alpha, f(\alpha))$ and $U(\alpha, g(\alpha)) \cap U_{f|\alpha} = \emptyset$. Then $\{U(\alpha, g(\alpha)) : \alpha \in \omega_1\}$ separates the points of ω_1 contradicting our assumption. □

By Lemmas 2.2 and 2.3, we see that I is a countably complete proper ideal on ω_1. By Theorem II.3.6(b) in [BTW], there is a rearrangement $r : \omega_1 \to \omega_1$ of ω_1 such that the corresponding ideal I_r contains no cub sets. For notational simplicity, we assume that the rearrangement r is the identity map. Thus, we may assume that our ideal I does not contain any cub sets.

3. Predicting the destruction of paranormality

We are now ready to see how to use the principle $\diamondsuit^*$ to get a countable partition of the closed discrete set ω_1 which does not have a locally finite open expansion. Actually, we will follow Taylor's approach in [Ta] and use a diamond principle that is a consequence of $\diamondsuit^*$.

If I is a countably complete proper ideal of subsets of ω_1, then $\diamondsuit(I)$ is the assertion that there is a sequence $\{(f_\alpha, g_\alpha) : \alpha \in \omega_1\}$ such that $f_\alpha : \alpha \to \alpha$,

$g_\alpha : \alpha \to \omega$ and for every pair $f : \omega_1 \to \omega_1$ and $g : \omega_1 \to \omega$,

$$T_{f,g} = \{\alpha \in \omega_1 : f|\alpha = f_\alpha \text{ and } g|\alpha = g_\alpha\} \notin I.$$

While this is different than the definition of $\diamondsuit(I)$ given in [Ta], the above definition follows by a standard coding argument (see, for example, Exercise 51 Chapter 2 in [K]).

By Theorem 4.5(ii) in [Ta], the principle $\diamondsuit^*$ implies that $\diamondsuit(I)$ holds for every countably complete proper ideal on ω_1 which does not contain any cub sets. Our work in section 2 guarantees that these assumptions have been satisfied. Therefore, we assume that $\diamondsuit(I)$ holds for our ideal I.

For each $f \in {}^{\omega_1}\omega_1$, $g \in {}^{\omega_1}\omega$ and $n \in \omega$, we recall the definition,

$$S^n_{f,g} = \{\alpha \in \omega_1 : \alpha \in \overline{\bigcup\{U(\beta, f(\beta)) : \beta < \alpha \text{ and } g(\beta) = n\}}\}.$$

Fix $\alpha \in \omega_1$. Let n_α be the least $n \in \omega$ such that

$$\alpha \in \overline{\bigcup\{U(\beta, f_\alpha(\beta)) : g_\alpha(\beta) = n\}} = \overline{U^n_{f_\alpha, g_\alpha}}$$

if such an n exists; otherwise, let $n_\alpha = 0$. This defines a partition $\{A_n : n \in \omega\}$ of ω_1 where $A_n = \{\alpha \in \omega_1 : n_\alpha = n\}$. By paranormality, there is a locally finite open expansion $\{U_n : n \in \omega\}$ of $\{A_n : n \in \omega\}$.

Pick $f \in {}^{\omega_1}\omega_1$ and $g \in {}^{\omega_1}\omega$ such that, for every $\alpha \in \omega_1$,

(1) $U(\alpha, f(\alpha)) \subset U_{n_\alpha}$ and
(2) if $U(\alpha, f(\alpha)) \cap U_n \neq \emptyset$, then $n < g(\alpha)$

Let $A = \{\alpha \in \omega_1 : \forall n \in \omega(\alpha \notin S^n_{f,g})\}$. Then $A \in I$ so there is an $\alpha \in T_{f,g} \setminus A$. That is, $f|\alpha = f_\alpha$, $g|\alpha = g_\alpha$, and there is an n such that $\alpha \in S^n_{f,g}$. By above n_α was defined as the minimum such n. Thus we have $\alpha \in S^{n_\alpha}_{f_\alpha, g_\alpha}$. Let $\beta < \alpha$ be such that $g_\alpha(\beta) = n_\alpha$ and such that $U(\alpha, f(\alpha)) \cap U(\beta, f_\alpha(\beta)) \neq \emptyset$. Then $U(\alpha, f(\alpha)) \cap U(\beta, f(\beta)) \neq \emptyset$ and by definition of $f(\alpha)$ we have $U(\alpha, f(\alpha)) \subseteq U_{n_\alpha}$. Thus, since $U(\beta, f(\beta)) \cap U_{n_\alpha} \neq \emptyset$, by definition of $g(\beta)$ we have $g(\beta) > n_\alpha$. But $n_\alpha = g_\alpha(\beta) = g(\beta)$, which is a contradiction. This finishes the proof of Theorem 1.5.

4. Locally compact spaces

It now follows fairly easily from results of Z. Balogh [B] that locally compact paranormal spaces are ω_1-collectionwise Hausdorff assuming $\diamondsuit^*$. In fact, we can prove they are ω_1-collectionwise normal with respect to compact sets. To see this, suppose that X is locally compact and that $\mathcal{C} = \{C_\alpha : \alpha \in \omega_1\}$ is a discrete family of compact subsets of X. We first note that the proof of Theorem 1.5 easily generalizes to establish that $\diamondsuit^*$ implies that countably paracompact spaces are collectionwise normal with respect to compact sets of character $\leq \omega_1$. By Balogh's character reduction theorem (Lemma 2.1 in [B]), $\mathcal{C}$ has a discrete expansion $\mathcal{C}' = \{C'_\alpha : \alpha \in \omega_1\}$ by compact sets each of character $\leq \omega_1$. Therefore, by the generalized version of Theorem 1.5, $\mathcal{C}'$ can be separated and hence $\mathcal{C}$ can be separated.

5. Questions

The work presented in this paper leads us in two directions for further investigation. The first concerns separation of discrete families in normal, first countable spaces versus separation of discrete families in countably paracompact, first countable spaces. As mentioned in the introduction, several researchers have investigated this problem. The general trend has been that any theorem proved for normal spaces was eventually established for countably paracompact spaces. A positive answer to the following question, due to D.K. Burke, would unify a great body of work. A negative answer would help clarify the subtle distinction between normality and countable paracompactness.

Question 5.1 ([Bu]). If every first countable normal space is collectionwise Hausdorff, is every countably paracompact, first countable space collectionwise Hausdorff?

The second line of investigation, also around the relationship between normality and countably paracompactness, concerns *strong* separation of discrete families in first countable spaces. Probably the most important open problem is Nyikos's question whether V=L implies that first countable, countably paracompact spaces are strongly collectionwise Hausdorff. A space is said to be strongly collectionwise Hausdorff if each closed discrete set can be separated by a *discrete* family of open sets. This is certainly true for normal spaces since collectionwise Hausdorff is equivalent to strongly collectionwise Hausdorff for the class of normal spaces. In addition, D.K. Burke has shown that PMEA implies that first countable, countably paracompact spaces are strongly collectionwise Hausdorff [Bu], and F.D. Tall has shown the same holds after adding supercompact many Cohen reals [T]. Recently in [KSS] it was shown that countably paracompact subspaces of ω_1^2 are strongly collectionwise Hausdorff and the techniques developed to prove this result were extended to establish the main result of this paper. As such we conjecture that this technique may be applicable at least to the following special cases of the original question of Nyikos.

Question 5.2 ([N]). Assume V=L. Are countably paracompact, locally countable, first countable spaces strongly collectionwise Hausdorff?

Question 5.3 ([N]). Assume V=L. Are countably paracompact ω_1-trees strongly collectionwise Hausdorff?

Probably the nicest positive answer to Nyikos's questions would be provided by a positive answer to the following question.

Question 5.4. Are first countable, countably paracompact, collectionwise Hausdorff spaces strongly collectionwise Hausdorff?

While the results of [Bu] and [T] provide consistent positive answers to this question (relative to the consistency of large cardinals), this question is otherwise open. There are only two related counterexamples in the literature. One is a ZFC example of a countably paracompact, collectionwise Hausdorff but not strongly collectionwise Hausdorff space of character 2^{ω_1} (Theorem 2 of [W3]). The other example is a paranormal, first countable, collectionwise Hausdorff not strongly collectionwise Hausdorff space constructed assuming V=L [S].

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