QUASI-ISOMORPHISMS OF KOSZUL COMPLEXES

JOSÉ J. M. SOTO

(Communicated by Wolmer V. Vasconcelos)

Abstract. Let $f : A \rightarrow B$ be a surjective homomorphism of noetherian local commutative rings that induces an isomorphism between the first Koszul homology modules and an epimorphism between the second Koszul homology modules. Then $f$ induces isomorphisms between Koszul homology modules in all dimensions.

Let $(A, m, k)$ be a local noetherian (commutative with unit) ring, $I$ an ideal of $A$, $B = A/I$, and $n$ the maximal ideal of $B$. Let $\{x_1, \ldots, x_n\}$ be a minimal set of generators of the ideal $m$ of $A$, and $y_1, \ldots, y_r$ the images of these elements in $B$. Let $\{v_1, \ldots, v_r\}$ be a minimal set of generators of the ideal $n$ of $B$. Fix for each $1 \leq j \leq n$ elements $b_{ja} \in E$ such that $y_j = \sum_{a=1}^r b_{ja} v_a$. Let $E = A[x_1, \ldots, x_n; dx_i = x_i]$ be the Koszul complex associated to the elements $x_1, \ldots, x_n$ of $A$, and $E' = B[v_1, \ldots, v_r; dv_i = v_i]$ be the Koszul complex associated to the elements $v_1, \ldots, v_r$ of $B$. By a little abuse of language we denote $H_*(A) = H_*(E)$, $H_*(B) = H_*(E')$ (they do not depend, up to isomorphism, of the minimal set of generators of the maximal ideal). Let $f : E \rightarrow E'$ be the homomorphism of complexes extending the projection map $A \rightarrow B$ by sending $X_i$ to $\sum_{a=1}^r b_{ja} V_a$.

L. L. Avramov and E. S. Golod [4, Proposition 1] show that if the ideal $I$ is generated by a regular sequence which is part of a minimal system of generators of the ideal $m$, then $H_*(f) : H_*(A) \rightarrow H_*(B)$ is an isomorphism. In [8] (see [9, (2.3.6)] S. S. Strogalov shows that the converse also holds. The following result shows that it is enough to consider the first two homology modules:

**Proposition 1.** If $H_1(f)$ is an isomorphism and $H_2(f)$ is surjective, then the ideal $I$ is generated by a regular sequence which is part of a minimal system of generators of the ideal $m$.

**Proof.** Let $0 \rightarrow U \rightarrow F \xrightarrow{p} m \rightarrow 0$ be an exact sequence of $A$-modules with $F$ free with basis $\{z_1, \ldots, z_n\}$ and $p(z_i) = x_i$, $1 \leq i \leq n$, and let $0 \rightarrow U' \rightarrow F' \xrightarrow{p'} n \rightarrow 0$ be an exact sequence of $B$-modules with $F'$ free with basis $\{z'_1, \ldots, z'_r\}$ and $p'(z'_i) = v_i$, $1 \leq i \leq r$. Let $g : F \wedge F' \rightarrow F$, $g(a \wedge b) = p(a)b - p(b)a$, and $V = \text{Im}(g)$. Define similarly $g' : F' \wedge F' \rightarrow F'$ and $V' = \text{Im}(g')$. We have isomorphisms $U/V = H_1(A)$, $U'/V' = H_1(B)$.
Consider now the André-Quillen homology modules \([1], [6]\). We have a commutative diagram of \(k\)-vector spaces with exact rows \([1, 15.12]\)

\[
\begin{array}{c}
0 \longrightarrow H_2(A, k, k) \overset{\alpha}{\longrightarrow} H_1(A) \longrightarrow F/mF \overset{\beta}{\longrightarrow} m/m^2 \longrightarrow 0 \\
\downarrow \lambda \hspace{1cm} \downarrow H_1(f) \hspace{1cm} \downarrow \mu \\
0 \longrightarrow H_2(B, k, k) \overset{\alpha'}{\longrightarrow} H_1(B) \longrightarrow F'/nF' \overset{\beta'}{\longrightarrow} n/n^2 \longrightarrow 0
\end{array}
\]

where \(\mu\) is the map such that \(\mu(z_j + mF) = \sum_{\alpha=1}^{\gamma} b_{j\alpha}(z_{\alpha} + nF')\), \(1 \leq j \leq n\), and \(\lambda\) is the canonical map.

The maps \(\beta\) and \(\beta'\) are isomorphisms and so \(\alpha\) and \(\alpha'\) are also isomorphisms.

Since \(H_1(f)\) is an isomorphism by hypothesis, we see that \(\lambda\) is an isomorphism.

Consider the commutative diagram of exact rows \([6, 10.4]\):

\[
\begin{array}{c}
\bigwedge^2 H_1(A) \longrightarrow H_2(A) \longrightarrow H_3(A, k, k) \longrightarrow 0 \\
\bigwedge^2 H_1(B) \longrightarrow H_2(B) \longrightarrow H_3(B, k, k) \longrightarrow 0
\end{array}
\]

Since \(H_2(f)\) is surjective we have that \(\varepsilon\) is surjective and so from the Jacobi-Zariski exact sequence \([1, 5.1]\) associated to \(A \to B \to k\)

\[
H_3(A, k, k) \xrightarrow{\sim} H_3(B, k, k) \to H_2(A, B, k) \to H_2(A, k, k) \xrightarrow{\sim} H_2(B, k, k)
\]

we have \(H_2(A, B, k) = 0\), i.e., the ideal \(I\) is generated by a regular sequence \([1, 6.25]\).

Finally, the same Jacobi-Zariski exact sequence

\[
H_3(A, k, k) \xrightarrow{\sim} H_2(B, k, k) \to H_1(A, B, k) = I/mI \to H_1(A, k, k) = m/m^2
\]

shows that \(I/mI \to m/m^2\) is injective.

**Remarks.** i) We can see in the proof that in order to assure that \(I\) is generated by a regular sequence, it is enough to assume that \(H_1(f)\) is injective and \(H_2(f)\) surjective.

ii) If the ideal \(I\) is of finite projective dimension and \(H_2(f)\) is surjective, then \(I\) is generated by a regular sequence. The proof is the same, having in mind that by the assumption on the projective dimension of \(I\), a result of L. L. Avramov \([3]\) (see \([7, \text{Lemma 1}]\)) gives us that \(H_2(A, k, k) \to H_2(B, k, k)\) is injective.

**Corollary 2.** If \(H_1(f)\) is an isomorphism and \(B\) is a complete intersection ring, then \(A\) is complete intersection.

**Proof.** It suffices to show that \(H_2(f)\) is surjective. In the commutative diagram

\[
\begin{array}{c}
\bigwedge^2 H_1(A) \xrightarrow{\gamma_2} H_2(A) \\
\bigwedge^2 H_1(B) \xrightarrow{\gamma_2'} H_2(B)
\end{array}
\]

we know that \(\gamma_2\) is an isomorphism \([10, \text{Theorem 6}]\), and so \(H_2(f)\) is surjective.
Remark. If $A$ contains a field, $B$ is complete intersection and $H_1(f)$ is injective, then $A$ is complete intersection: from the diagram

$$
\begin{array}{c}
\bigwedge^* H_1(A) \
\bigwedge^* H_1(f) \downarrow \\
\bigwedge^* H_1(B) \sim \\
\bigwedge^* H_*(A)
\end{array}
$$

we deduce that $\bigwedge^* H_1(A) \to H_*(A)$ is injective. Then it follows from a result of W. Bruns [5], based on deep results on commutative algebra, that $A$ is complete intersection.

We end this paper with a slight generalization of a result of E. F. Assmus. Let $(A, m, k)$ be a local noetherian ring. We know that $A$ is complete intersection if and only if the canonical homomorphism $\bigwedge^* H_1(A) \to H_*(A)$ is an isomorphism, i.e., $H_*(A)$ is a free graded (anticommutative) $k$-algebra generated by the degree 1 elements. The “only if” part is [10, Theorem 6], and the “if” part is [2, Theorem 2.7] (see also [2] and [5] for related characterizations). This result by Assmus can be stated in a slightly stronger form:

**Proposition 3.** If $H_*(A)$ is a free graded $k$-algebra, then $A$ is complete intersection.

**Proof.** Let $H_*(A) = S^*M$, where $M$ is a graded $k$-module and $S^*$ denotes the graded symmetric $k$-algebra functor (i.e., symmetric $k$-algebra $S$ on the even degree part, and exterior $k$-algebra $\wedge$ on the odd degree part). By [2, Theorem 2.7] it suffices to show that $M_2 = 0$. We have $S^2M_2 \subset H_{2n}(A)$ for all $n \geq 0$, and so if $M_2 \neq 0$ we would have $H_{2n}(A) \neq 0$ for all $n$.

**References**


**Departamento de Álgebra, Facultad de Matemáticas, Universidad de Santiago de Compostela, E-15771 Santiago de Compostela, Spain**