

QUASI-ISOMORPHISMS OF KOSZUL COMPLEXES

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ABSTRACT. Let $f : A \rightarrow B$ be a surjective homomorphism of noetherian local commutative rings that induces an isomorphism between the first Koszul homology modules and an epimorphism between the second Koszul homology modules. Then f induces isomorphisms between Koszul homology modules in all dimensions.

Let (A, \mathfrak{m}, k) be a local noetherian (commutative with unit) ring, I an ideal of A , $B = A/I$, and \mathfrak{n} the maximal ideal of B . Let $\{x_1, \dots, x_n\}$ be a minimal set of generators of the ideal \mathfrak{m} of A , and y_1, \dots, y_n the images of these elements in B . Let $\{v_1, \dots, v_r\}$ be a minimal set of generators of the ideal \mathfrak{n} of B . Fix for each $1 \leq j \leq n$ elements $b_{j\alpha} \in B$ such that $y_j = \sum_{\alpha=1}^r b_{j\alpha} v_\alpha$. Let $E = A\langle X_1, \dots, X_n; dX_i = x_i \rangle$ be the Koszul complex associated to the elements x_1, \dots, x_n of A , and $E' = B\langle V_1, \dots, V_r; dV_i = v_i \rangle$ be the Koszul complex associated to the elements v_1, \dots, v_r of B . By a little abuse of language we denote $H_*(A) = H_*(E)$, $H_*(B) = H_*(E')$ (they do not depend, up to isomorphism, of the minimal set of generators of the maximal ideal). Let $f : E \rightarrow E'$ be the homomorphism of complexes extending the projection map $A \rightarrow B$ by sending X_i to $\sum_{\alpha=1}^r b_{j\alpha} V_\alpha$.

L. L. Avramov and E. S. Golod [4, Proposition 1] show that if the ideal I is generated by a regular sequence which is part of a minimal system of generators of the ideal \mathfrak{m} , then $H_*(f) : H_*(A) \rightarrow H_*(B)$ is an isomorphism. In [8] (see [9, (2.3.6)]) S. S. Strogalov shows that the converse also holds. The following result shows that it is enough to consider the first two homology modules:

Proposition 1. *If $H_1(f)$ is an isomorphism and $H_2(f)$ is surjective, then the ideal I is generated by a regular sequence which is part of a minimal system of generators of the ideal \mathfrak{m} .*

Proof. Let $0 \rightarrow U \rightarrow F \xrightarrow{p} \mathfrak{m} \rightarrow 0$ be an exact sequence of A -modules with F free with basis $\{z_1, \dots, z_n\}$ and $p(z_i) = x_i$, $1 \leq i \leq n$, and let $0 \rightarrow U' \rightarrow F' \xrightarrow{p'} \mathfrak{n} \rightarrow 0$ be an exact sequence of B -modules with F' free with basis $\{z'_1, \dots, z'_r\}$ and $p'(z'_i) = v_i$, $1 \leq i \leq r$. Let $g : F \wedge F \rightarrow F$, $g(a \wedge b) = p(a)b - p(b)a$, and $V = \text{Im}(g)$. Define similarly $g' : F' \wedge F' \rightarrow F'$ and $V' = \text{Im}(g')$. We have isomorphisms $U/V = H_1(A)$, $U'/V' = H_1(B)$.

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Consider now the André-Quillen homology modules [1], [6]. We have a commutative diagram of k -vector spaces with exact rows [1, 15.12]

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & H_2(A, k, k) & \xrightarrow{\alpha} & H_1(A) & \longrightarrow & F/\mathfrak{m}F & \xrightarrow{\beta} & \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & 0 \\
 & & \downarrow \lambda & & \downarrow H_1(f) & & \downarrow \mu & & \downarrow & & \\
 0 & \longrightarrow & H_2(B, k, k) & \xrightarrow{\alpha'} & H_1(B) & \longrightarrow & F'/\mathfrak{n}F' & \xrightarrow{\beta'} & \mathfrak{n}/\mathfrak{n}^2 & \longrightarrow & 0
 \end{array}$$

where μ is the map such that $\mu(z_j + \mathfrak{m}F) = \sum_{\alpha=1}^r b_{j\alpha}(z'_\alpha + \mathfrak{n}F')$, $1 \leq j \leq n$, and λ is the canonical map.

The maps β and β' are isomorphisms and so α and α' are also isomorphisms. Since $H_1(f)$ is an isomorphism by hypothesis, we see that λ is an isomorphism.

Consider the commutative diagram of exact rows [6, 10.4]:

$$\begin{array}{ccccccc}
 \bigwedge^2 H_1(A) & \longrightarrow & H_2(A) & \longrightarrow & H_3(A, k, k) & \longrightarrow & 0 \\
 \downarrow & & \downarrow H_2(f) & & \downarrow \varepsilon & & \\
 \bigwedge^2 H_1(B) & \longrightarrow & H_2(B) & \longrightarrow & H_3(B, k, k) & \longrightarrow & 0
 \end{array}$$

Since $H_2(f)$ is surjective we have that ε is surjective and so from the Jacobi-Zariski exact sequence [1, 5.1] associated to $A \rightarrow B \rightarrow k$

$$H_3(A, k, k) \xrightarrow{\varepsilon} H_3(B, k, k) \rightarrow H_2(A, B, k) \rightarrow H_2(A, k, k) \xrightarrow{\sim} H_2(B, k, k)$$

we have $H_2(A, B, k) = 0$, i.e., the ideal I is generated by a regular sequence [1, 6.25].

Finally, the same Jacobi-Zariski exact sequence

$$H_2(A, k, k) \xrightarrow{\sim} H_2(B, k, k) \rightarrow H_1(A, B, k) = I/\mathfrak{m}I \rightarrow H_1(A, k, k) = \mathfrak{m}/\mathfrak{m}^2$$

shows that $I/\mathfrak{m}I \rightarrow \mathfrak{m}/\mathfrak{m}^2$ is injective.

Remarks. i) We can see in the proof that in order to assure that I is generated by a regular sequence, it is enough to assume that $H_1(f)$ is injective and $H_2(f)$ surjective.

ii) If the ideal I is of finite projective dimension and $H_2(f)$ is surjective, then I is generated by a regular sequence. The proof is the same, having in mind that by the assumption on the projective dimension of I , a result of L. L. Avramov [3] (see [7, Lemma 1]) gives us that $H_2(A, k, k) \rightarrow H_2(B, k, k)$ is injective.

Corollary 2. *If $H_1(f)$ is an isomorphism and B is a complete intersection ring, then A is complete intersection.*

Proof. It suffices to show that $H_2(f)$ is surjective. In the commutative diagram

$$\begin{array}{ccc}
 \bigwedge^2 H_1(A) & \xrightarrow{\gamma_2} & H_2(A) \\
 \downarrow \simeq & & \downarrow H_2(f) \\
 \bigwedge^2 H_1(B) & \xrightarrow{\gamma'_2} & H_2(B)
 \end{array}$$

we know that γ'_2 is an isomorphism [10, Theorem 6], and so $H_2(f)$ is surjective.

Remark. If A contains a field, B is complete intersection and $H_1(f)$ is injective, then A is complete intersection: from the diagram

$$\begin{array}{ccc} \bigwedge^* H_1(A) & \longrightarrow & H_*(A) \\ \downarrow \bigwedge^* H_1(f) & & \downarrow \\ \bigwedge^* H_1(B) & \xrightarrow{\sim} & H_*(B) \end{array}$$

we deduce that $\bigwedge^* H_1(A) \rightarrow H_*(A)$ is injective. Then it follows from a result of W. Bruns [5], based on deep results on commutative algebra, that A is complete intersection.

We end this paper with a slight generalization of a result of E. F. Assmus. Let (A, \mathfrak{m}, k) be a local noetherian ring. We know that A is complete intersection if and only if the canonical homomorphism $\bigwedge^* H_1(A) \rightarrow H_*(A)$ is an isomorphism, i.e., $H_*(A)$ is a free graded (anticommutative) k -algebra generated by the degree 1 elements. The “only if” part is [10, Theorem 6], and the “if” part is [2, Theorem 2.7] (see also [2] and [5] for related characterizations). This result by Assmus can be stated in a slightly stronger form:

Proposition 3. *If $H_*(A)$ is a free graded k -algebra, then A is complete intersection.*

Proof. Let $H_*(A) = \mathbf{S}^*M$, where M is a graded k -module and \mathbf{S}^* denotes the graded symmetric k -algebra functor (i.e., symmetric k -algebra S on the even degree part, and exterior k -algebra \wedge on the odd degree part). By [2, Theorem 2.7] it suffices to show that $M_2 = 0$. We have $S^n M_2 \subset H_{2n}(A)$ for all $n \geq 0$, and so if $M_2 \neq 0$ we would have $H_{2n}(A) \neq 0$ for all n .

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