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DISCRETE SPECTRUM OF ELECTROMAGNETIC DIRAC OPERATORS

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Abstract. We consider the Dirac operators with electromagnetic fields on 2-dimensional Euclidean space. We offer the sufficient conditions for electromagnetic fields that the associated Dirac operator has only discrete spectrum.

1. Introduction

In this paper we consider electromagnetic Dirac operators on the 2-dimensional Euclidean space $\mathbf{R}^2$. Let $(L^2(\mathbf{R}^2))^2$ be a Hilbert space

$$(L^2(\mathbf{R}))^2 = L^2(\mathbf{R}^2) \oplus L^2(\mathbf{R}^2) \tag{1.1}$$

with inner product $(f,g) \equiv \int_{\mathbf{R}^2}(f_1\bar{g_1} + f_2\bar{g_2})dx$ for $f = (f_1, f_2)$, $g = (g_1, g_2) \in (L^2(\mathbf{R}^2))^2$. In this Hilbert space we introduce a Dirac operator with electromagnetic field $\mathbf{a} = (a_j(x))_{j=1}^2$ and potential $V(x)$ by

$$\mathbf{H}_V(\mathbf{a}) = \sum_{j=1}^{2} \sigma_j(-i\frac{\partial}{\partial x_j} - a_j) + V, \tag{1.2}$$

where $(\sigma_j)_{j=1}^2$ is a system of 2×2 Hermitian constant matrices with the anti-commutation relations $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{j,k}I$. Here a_j and V should be understood as operators of multiplication by real-valued functions $a_j(x)$ and $V(x)$, respectively. Let us assume $a_j(x), V(x) \in L^\infty_{loc}(\mathbf{R}^2)$. Then it is well-known that $\mathbf{H}_V(\mathbf{a})$ is self-adjoint (cf. Chernoff [1]). In case $\mathbf{a} = (a_j(x))_{j=1}^2 \in (C^1(\mathbf{R}^2))^2$, we define B as the 2×2 skew-symmetric matrix whose (j,k) component is $\partial a_k/\partial x_j - \partial a_j/\partial x_k$. We describe b as $\partial a_2/\partial x_1 - \partial a_1/\partial x_2$. By using this, we have

$$B(x) = \begin{pmatrix} 0 & b(x) \\ -b(x) & 0 \end{pmatrix}. \tag{1.3}$$

The norm of B is given by $|B| \equiv |b|$. First we offer the conditions of electromagnetic fields that the associated Dirac operator has only discrete spectrum. For this end, we introduce a non-negative form on $(L^2(\mathbf{R}^2))^2$ by

$$h_{\mathbf{a},V}(\phi,\psi) = (\mathbf{H}_V(\mathbf{a})\phi, \mathbf{H}_V(\mathbf{a})\psi), \qquad \phi,\psi \in (C_0^\infty(\mathbf{R}^2))^2 \tag{1.4}$$

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and define, for open sets Ω in $\mathbf{R}^2$

$$e_{\mathbf{a},V}(\Omega) = \inf\left\{ h_{\mathbf{a},V}(\phi,\phi)\,;\, \|\phi\| = 1, \operatorname{supp}\phi \subset \Omega, \phi \in (C_0^\infty(\mathbf{R}^2))^2 \right\}. \tag{1.5}$$

Let us state our main theorem.

Main Theorem. *The following four conditions are equivalent.*

(a) $\mathbf{H}_V(\mathbf{a})$ *has only discrete spectrum.*
(b) $e_{\mathbf{a},V}(W_R) \to \infty$ *as* $R \to \infty$, *where* $W_R = \{\, x \,|\, |x| > R \,\}$.
(c) $e_{\mathbf{a},V}(Q_x) \to \infty$ *as* $|x| \to \infty$, *where* $Q_x = \{\, y \,|\, |x-y| < 1 \,\}$.
(d) *There exists a real-valued continuous function* $\lambda(x)$ *on* $\mathbf{R}^2$ *such that* $\lambda(x) \to \infty$ *as* $|x| \to \infty$ *and that*

$$h_{\mathbf{a},V}(\phi,\phi) \geqq \int_{\mathbf{R}^2} \lambda(x)|\phi(x)|^2\,dx$$

for all $\phi(x) \in (C_0^\infty(\mathbf{R}^2))^2$.

As an application, we give a sufficient condition that $\mathbf{H}_V(\mathbf{a})$ has only discrete spectrum.

Theorem 1.1. *On the 2-dimensional Euclidean space, we assume that* $\mathbf{a} = (a_j(x))_{j=1}^2 \in (C^2(\mathbf{R}^2))^2$ *and* $V(x) \in C^1(\mathbf{R}^2)$ *and that the following three conditions* (1.6) , (1.7) *and* (1.8) *are fulfilled. Then* $\mathbf{H}_V(\mathbf{a})$ *has only discrete spectrum.*

$$\lim_{|x|\to\infty} \frac{|b|}{V^2} > 2, \tag{1.6}$$

$$\lim_{|x|\to\infty} |V| = \infty, \tag{1.7}$$

$$\lim_{|x|\to\infty} \frac{\nabla V}{V^2} = 0. \tag{1.8}$$

Iwatsuka got a similar result to the Main Theorem in the case where $\mathbf{H}_V(\mathbf{a})$ is a Schrödinger operator. However, the conditions for Schrödinger operators have only discrete spectrum are quite different from those for Dirac operators. In fact, under the conditions $V = 0$ and $\lim_{|x|\to\infty} |b| = \infty$, Schrödinger operators have only discrete spectrum, but Dirac operators may have an essential one. We know a similar fact as above when $b = 0$ and $\lim_{|x|\to\infty} V = \infty$ (cf. Thaller [6]).

2. Proof of the Main Theorem

Let E be the spectral measure associated with $\mathbf{H}_V(\mathbf{a})$. $\sigma_{\text{ess}}(\mathbf{H}_V(\mathbf{a}))$ is defined by

$$\sigma_{\text{ess}}(\mathbf{H}_V(\mathbf{a})) = \{\, \mu \in \mathbf{R} \,|\, \dim \operatorname{Ran}(E(\mu-\varepsilon, \mu+\varepsilon)) \,=\, \infty \quad \text{for all } \varepsilon > 0 \,\},$$

where Ran(**.**) denotes the range of an operator. Note that

$$\sigma_{\text{ess}}(\mathbf{H}_V(\mathbf{a})) = \phi \quad \text{if and only if} \quad \mathbf{H}_V(\mathbf{a}) \text{ has only discrete spectrum.} \tag{2.1}$$

We need the following lemma to prove the Main Theorem.

Lemma 2.1. *Let* $s > 0$. *Then the following conditions are equivalent.*

(a) $\inf\{\, |\lambda| \,|\, \lambda \in \sigma_{\text{ess}}(\mathbf{H}_V(\mathbf{a}))\,\} \geqq s$.
(b) $\lim_{R\to\infty} e_{\mathbf{a},V}(W_R) \geqq s^2$, *where* $W_R = \{x \| x| > R\}$.

(c) *There exists a real-valued continuous function* $\lambda(x)$ *on* $\mathbf{R}^2$ *such that*

$$\liminf_{|x|\to\infty} \lambda(x) \geqq s^2 \quad \text{and} \quad h_{\mathbf{a},V}(\phi,\phi) \geqq \int_{\mathbf{R}^d} \lambda(x)|\phi(x)|^2 dx$$

for all $\phi(x) \in (C_0^\infty(\mathbf{R}^2))^2$.

Proof. (a) $\Rightarrow$ (b): Suppose that (b) does not hold. There exists some $s' > 0$ such that $\lim_{R\to\infty} e_{\mathbf{a},V}(W_R) < s'^2 < s^2$. Since $e_{\mathbf{a},V}(W_R)$ is increasing in R by (1.5), we notice that $e_{\mathbf{a},V}(W_R) < s'^2$ for all $R > 0$. Hence, one can choose successively a sequence $\{\phi_k\}_{k=1}^\infty \subset (C_0^\infty(\mathbf{R}^2))^2$ such that

$$\begin{cases} \|\phi_k\| = 1\,, \\ h_{\mathbf{a},V}(\phi_k,\phi_k) < s'^2, \\ \operatorname{supp}\phi_k \subset \{\, x \,|\, a_k < |x| < a_{k+1}\}\,, \end{cases} \tag{2.2}$$

for some increasing sequence $\{a_k\}_{k=1}^\infty$ such that $a_k \to \infty$ as $k \to \infty$. Let $E(t) = E((\infty\,,\,t])$. If $s' < t < s$, (a) implies that $E(t) - E(-t)$ is compact; hence

$$\lim_{k\to\infty} \|E(t)\phi_k\|^2 = \lim_{k\to\infty} \|E(-t)\phi_k\|^2.$$

Therefore we have

$$\begin{aligned} h_{\mathbf{a}V}(\phi_k,\phi_k) &= \int_{(-\infty\,,\,\infty)} \mu^2 d\|E(\mu)\phi_k\|^2 dx \\ &\geqq \int_{(t\,,\,\infty)} t^2 d\|E(\mu)\phi_k\|^2 dx + \int_{(-\infty\,,\,-t)} t^2 d\|E(\mu)\phi_k\|^2 dx \\ &= t^2(\|\phi_k\|^2 - \|E(t)\phi_k\|^2 + \|E(-t)\phi_k\|^2) \to t^2 \quad \text{as } k \to \infty. \end{aligned}$$

But this contradicts (2.2).

(b) $\Rightarrow$ (c): Let $\{\xi_k\}_{k=0,1,2,\cdots}$ be a sequence of real-valued functions of class $C_0^\infty(\mathbf{R}^2)$ such that

$$\begin{cases} \displaystyle\sum_{k=0}^\infty \xi_k^2(x) = 1, \\ \xi_k(x) = \xi_1(\dfrac{x}{2^{k-1}}) \quad (k \geqq 1), \\ \operatorname{supp}\xi_0 \subset \{\, x \,|\, |x| < 2\,\}\,, \\ \operatorname{supp}\xi_k \subset \{\, x \,|\, 2^{k-1} < |x| < 2^{k+1}\,\} \quad (k \geqq 1). \end{cases} \tag{2.3}$$

See Iwatsuka [3] for the existence of such a sequence. By direct computation,

$$\begin{cases} \mathbf{H}_V(\mathbf{a})(\xi^2\phi) = \xi^2\mathbf{H}_V(\mathbf{a})\phi - \displaystyle\sum_{j=1}^2 2i\sigma_j\xi\frac{\partial\xi}{\partial x_j}\phi, \\ \mathbf{H}_V(\mathbf{a})(\xi\phi) = \xi\mathbf{H}_V(\mathbf{a})\phi - \displaystyle\sum_{j=1}^2 i\sigma_j\frac{\partial\xi}{\partial x_j}\phi \end{cases} \tag{2.4}$$

for $\phi \in (C_0^\infty(\mathbf{R}^2))^2$, and a real-valued function $\xi \in C^\infty(\mathbf{R}^2)$. By simple calculation, we have

$$\begin{aligned} h_{\mathbf{a},V}(\phi, \xi_k^2\phi) - h_{\mathbf{a},V}(\xi_k\phi, \xi_k\phi) &= (\sum_{j=1}^{2} i\sigma_j \frac{\partial \xi_k}{\partial x_j}\phi, i\sigma_j \frac{\partial \xi_k}{\partial x_j}\phi) \\ =(\phi, (-\sum_{j=1}^{2} i\sigma_j \frac{\partial \xi_k}{\partial x_j})(\sum_{j=1}^{2} i\sigma_j \frac{\partial \xi_k}{\partial x_j})\phi) &= \||\nabla \xi_k|\phi\|^2. \end{aligned} \tag{2.5}$$

We obtain

$$\begin{aligned} h_{\mathbf{a},V}(\phi, \phi) &= \mathbf{R}e(\sum_{k=0}^{\infty} h_{\mathbf{a},V}(\phi, \xi_k^2\phi)) \\ &= \sum_{k=0}^{\infty} h_{\mathbf{a},V}(\xi_k\phi, \xi_k\phi) - \sum_{k=0}^{\infty} \||\nabla \xi_k|\phi\|^2, \end{aligned} \tag{2.6}$$

where $|\nabla \xi| = \sqrt{\sum_{j=1}^{2}(\partial \xi/\partial x_j)^2}$. Let $\lambda(x) = \sum_{k=0}^{\infty}(e_k\xi_k(x)^2 - |\nabla \xi_k|^2)$, where $e_0 = e_{\mathbf{a},V}(\{\, x \,|\, |x| < 2 \,\})$ and $e_k = e_{\mathbf{a},V}(W_{2^{k-1}})$, $k = 1, 2, \cdots$, now one notices that (c) holds.

(c) $\Rightarrow$ (a): Suppose that (a) does not hold. Choose $\sigma \in \sigma_{\mathrm{ess}}(\mathbf{H}_V(\mathbf{a}))$ such that $|\sigma| < s$, and a sequence $\{u_k\}_{k=1}^{\infty} \in \mathbf{D}(\mathbf{H}_V(\mathbf{a}))$ such that

$$\begin{cases} \|u_k\| = 1, \\ u_k \to 0 \quad \text{weakly as} \quad k \to \infty, \\ \|\mathbf{H}_V(\mathbf{a})u_k - \sigma u_k\| \to 0 \quad \text{as} \quad k \to \infty. \end{cases} \tag{2.7}$$

From (c) for any $0 < \varepsilon < s^2 - \sigma^2$, there exists $M > 0$ such that $\lambda(x) > s^2 - \varepsilon$ for x : $|x| \geqq M$. Then we have

$$\begin{aligned} h_{\mathbf{a},V}(u_k, u_k) &\geqq \int_{|x|>M} \lambda(x)|u_k(x)|^2 dx + \int_{|x|\leqq M} \lambda(x)|u_k(x)|^2 dx \\ &\geqq (s^2 - \varepsilon) \int_{|x|>M} |u_k(x)|^2 dx - M_\lambda \int_{|x|\leqq M} |u_k(x)|^2 dx, \end{aligned}$$

where $M_\lambda = \max_{|x|\leqq M} |\lambda(x)|$.

It suffices to find a subsequence $\{u_{k'}\}$ of $\{u_k\}$ such that $\lim_{k'\to\infty} \int_{|x|\leqq M} |u_{k'}|^2 dx = 0$. In fact, it implies that $\sigma^2 = \lim_{k'\to\infty} h_{\mathbf{a}V}(u_{k'}, u_{k'}) \geqq s^2 - \varepsilon$, which contradicts $\varepsilon < s^2 - \sigma^2$. We can choose the following real-valued functions of class $C^\infty(\mathbf{R}^2)$:

$$\begin{cases} \varphi_0^2(x) + \varphi_1^2(x) = 1, & \\ \varphi_0(x) = 1 & \text{if } |x| \leq M, \\ \varphi_0(x) = 0 & \text{if } |x| \geq M + 1. \end{cases}$$

In similar fashion to (2.4), we get

$$h_{\mathbf{a},V}(u_k, u_k) = \sum_{l=0}^{1} h_{\mathbf{a},V}(\varphi_l u_k, \varphi_l u_k) - \sum_{l=0}^{1} \||\nabla \varphi_l|u_k\|^2.$$

Hence, we obtain

$$h_{\mathbf{a},V}(\varphi_0 u_k, \varphi_0 u_k) \leq h_{\mathbf{a},V}(u_k, u_k) + \sum_{l=0}^{1} \||\nabla \varphi_l| u_k\|^2,$$

which proves that $h_{\mathbf{a},V}(\varphi_0 u_k, \varphi_0 u_k)$ is bounded. We define that $\Omega_{M+1} = \{x \| x| \leq M+1\}$, which includes the support of φ_0. By the assumption that $\mathbf{a}_j, V \in L^\infty_{loc}$, we notice that $\|\mathbf{H}_V(\mathbf{a})(\varphi_0 \cdot)\|$ is equivalent to $\|\varphi_0 \cdot \|_{H_1}$. Because the embedding map from $(H_1(\Omega_{M+1}))^2$ to $(L^2(\Omega_{M+1}))^2$ is compact, there exists a subsequence $\{u_{k'}\}$ of $\{u_k\}$ such that $\lim_{k' \to \infty} \int_{\mathbf{R}^2} (\varphi_0(x)|u_{k'}|)^2 dx = 0$. We have

$$0 \leq \lim_{k' \to \infty} \int_M |u_{k'}|^2 dx \leq \lim_{k' \to \infty} \int_{\mathbf{R}^2} |\varphi_0(x) u_{k'}|^2 dx = 0,$$

which completes the proof. □

Owing to this lemma, the proof of the Main Theorem is analogous to that by Iwatsuka [3, Main Theorem].

3. Sufficient conditions for $\mathbf{H}_V(\mathbf{a})$ to have only discrete spectrum

Through this section, we assume that a_j and V are smooth, and we shall offer sufficient conditions for $\mathbf{H}_V(\mathbf{a})$ to have only discrete spectrum. We have by self-adjointness of $\mathbf{H}_V(\mathbf{a})$.

$$h_{\mathbf{a},V}(\phi, \phi) = (\mathbf{H}_V(\mathbf{a})\phi, \mathbf{H}_V(\mathbf{a})\phi) = (\mathbf{H}_V(\mathbf{a})^2 \phi, \phi). \tag{3.1}$$

We have by direct calculation,

$$\begin{aligned} \mathbf{H}_V(\mathbf{a})^2 &= -\sum_{j=1}^{2} (\frac{\partial}{\partial x_j} - i a_j)^2 + V^2 + ib\sigma_1\sigma_2 \\ &\quad + V \sum_{j=1}^{2} \sigma_j (-i \frac{\partial}{\partial x_j} - a_j) + \sum_{j=1}^{2} \sigma_j (-i \frac{\partial}{\partial x_j} - a_j) V \\ &= -\sum_{j=1}^{2} (\frac{\partial}{\partial x_j} - i a_j + i\sigma_j V)^2 - V^2 + ib\sigma_1\sigma_2. \end{aligned} \tag{3.2}$$

Hence, we obtain

$$h_{\mathbf{a},V}(\phi, \phi) = \sum_{j=1}^{2} \|\Pi_j \phi\|^2 - \|V\phi\|^2 + (ib\sigma_1\sigma_2\phi, \phi), \tag{3.3}$$

where

$$\Pi_j = \frac{\partial}{\partial x_j} - i a_j + i\sigma_j V. \tag{3.4}$$

We get by simple calculation,

$$\Pi_2\Pi_1 - \Pi_1\Pi_2 = ib + 2\sigma_1\sigma_2 V^2 + i\sigma_1 \frac{\partial V}{\partial x_2} - i\sigma_2 \frac{\partial V}{\partial x_1}, \tag{3.5}$$

which plays an important role in this section.

Proof of Theorem 1.1. There exists $N > 0$ such that if $|x| \geq N$, then

$$V \neq 0, \tag{3.6}$$

$$|b| > 2V^2, \tag{3.7}$$

$$|\frac{\nabla V}{V^2}| < \frac{1}{4}. \tag{3.8}$$

One can choose a function $\phi \in (C_0^\infty(\mathbf{R}^2))^2$. We assume that $\operatorname{supp}\phi \subset W_R$ and $R > N$. We have by (3.5), (3.6) and (3.8).

$$\begin{aligned} 2\|\Pi_1\phi\|\|\Pi_2\phi\| &\geq |(\Pi_2\phi, \Pi_1\phi) - (\Pi_1\phi, \Pi_2\phi)| \\ &\geq |((\Pi_2\Pi_1 - \Pi_1\Pi_2)\phi, \phi)| \\ &\geq |((ib + 2\sigma_1\sigma_2 V^2)\phi, \phi)| - \frac{1}{2}\|V\phi\|^2. \end{aligned} \tag{3.9}$$

We can regard σ_1, σ_2 as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.10}$$

Now we can choose such a coordinate on $\mathbf{R}^2$ that $b > 0$ on W_R. Then we obtain by (3.7) and (3.9),

$$\begin{aligned} &\|\Pi_1\phi\|^2 + \|\Pi_2\phi\|^2 + (ib\sigma_1\sigma_2\phi, \phi) \\ &\geq \left| \int_{\mathbf{R}^2} i(\phi_1, \phi_2) \begin{pmatrix} b + 2V^2 & 0 \\ 0 & b - 2V^2 \end{pmatrix} \begin{pmatrix} \overline{\phi}_1 \\ \overline{\phi}_2 \end{pmatrix} dx \right| \\ &+ \int_{\mathbf{R}^2} (\phi_1, \phi_2) \begin{pmatrix} -b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \overline{\phi}_1 \\ \overline{\phi}_2 \end{pmatrix} dx - \frac{1}{2}\|V\phi\|^2 \\ &\geq \int_{\mathbf{R}^2} (\phi_1, \phi_2) \begin{pmatrix} b + 2V^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{\phi}_1 \\ \overline{\phi}_2 \end{pmatrix} dx \\ &+ \int_{\mathbf{R}^2} (\phi_1, \phi_2) \begin{pmatrix} -b & 0 \\ 0 & 2V^2 \end{pmatrix} \begin{pmatrix} \overline{\phi}_1 \\ \overline{\phi}_2 \end{pmatrix} dx - \frac{1}{2}\|V\phi\|^2 \\ &\geq \frac{3}{2}\|V\phi\|. \end{aligned} \tag{3.11}$$

Hence, by (3.3), we have

$$h_{\mathbf{a},V}(\phi, \phi) \geq \frac{1}{2} \inf_{x \in W_R} V^2 \|\phi\|^2. \tag{3.12}$$

Therefore, $e_{\mathbf{a},V}(W_R) \to \infty$ as $R \to \infty$, which completes the proof. □

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