TYPE $II_{\infty}$ FACTORS GENERATED BY PURELY INFINITE SIMPLE $C^*$-ALGEBRAS ASSOCIATED WITH FREE GROUPS

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Abstract. Let $\Gamma = G_1 \ast G_2 \ast \ldots \ast G_n \ast \ldots$ be a free product of at least two but at most countably many cyclic groups. With each such group $\Gamma$ we associate a family of $C^*$-algebras, denoted $C^*_r(\Gamma, P_{\Lambda})$ and generated by the reduced group $C^*$-algebra $C^*_r \Gamma$ and a collection $P_{\Lambda}$ of projections onto the $\ell^2$-spaces over certain subsets of $\Gamma$. We determine $W^*(\Gamma, P_{\Lambda})$, the weak closure of $C^*_r(\Gamma, P_{\Lambda})$ in $L(\ell^2(\Gamma))$, and use this result to show that many of the $C^*$-algebras in question are non-nuclear.

0. Introduction

In recent years a great deal of progress has been made towards the classification of separable, simple $C^*$-algebras. This program, initiated by George Elliott [6] and Mikael Rørdam [14], has been advanced significantly through combined efforts of a number of researchers. The progress has been particularly visible in the subcategory of unital, purely infinite and nuclear $C^*$-algebras; e.g. see [7], [9], [13]. And yet a number of important related problems still remain open. One such outstanding problem is whether there exists a simple $C^*$-algebra containing both finite and infinite projections. Partially motivated by the desire to find an example of such an algebra the second named author initiated investigations of a class of $C^*$-algebras related to free-product groups [17], [18].

Let $\Gamma = G_1 \ast G_2 \ast \ldots \ast G_n \ast \ldots$ be a free product of finitely or countably many cyclic groups. For each $i$ we fix a generator $g_i$ of $G_i$. For any $h \in \Gamma \setminus \{e\}$ let $\Gamma(h)$ be the set of reduced words, in the fixed set $\{g_i\}$ of generators of $\Gamma$, whose initial segments coincide with $h$. Let $P_h$ denote the orthogonal projection from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma(h))$, and let $C^*_r(\Gamma, P_h)$ be the $C^*$-subalgebra of $L(\ell^2(\Gamma))$ generated by the reduced group $C^*$-algebra $C^*_r \Gamma$ and the projection $P_h$ [17], [18]. This construction is somewhat similar, and actually related to those of [2], [12], [15].

Against the hope of finding a counterexample to the outstanding problem mentioned above, the second named author showed [17], [18] that all $C^*$-algebras of the form $C^*_r(\Gamma, P_h)$ are either purely infinite (see [5] for the definition) and simple, or extensions of purely infinite, simple $C^*$-algebras by the compacts. This result brought several natural questions about the finer structure of these algebras. The
most important one is whether the algebras are nuclear, and thus fit into the classifiable category, or not. In this article we provide a partial answer to this question. Our investigations are carried out for a larger class of C*-algebras, containing those considered in [17], [18].

Let Λ be a subset of Γ and denote \( \mathcal{P}_\Lambda = \{ P_h : h \in \Lambda \} \). We define \( C^*_\Lambda(\Gamma, \mathcal{P}_\Lambda) \) as the C*-subalgebra of \( \mathcal{L}(\ell^2(\Gamma)) \) generated by \( C^*_\Gamma \) and \( \mathcal{P}_\Lambda \). Let \( \Gamma_\Lambda \) be the subgroup of \( \Gamma \) generated by those of \( \{ g_i^{±1} \} \) which do not occur as final letters of reduced forms of elements in \( \Lambda \). We denote by \( W^*(\Gamma, \mathcal{P}_\Lambda) \) the weak closure of \( C^*_\Lambda(\Gamma, \mathcal{P}_\Lambda) \) in \( \mathcal{L}(\ell^2(\Gamma)) \).

The main result of this article, Theorem I, states that there is an isomorphism

\[
W^*(\Gamma, \mathcal{P}_\Lambda) \cong W^*(\Gamma_\Lambda) \otimes \mathcal{L}(\mathcal{H}),
\]

where \( \mathcal{H} \) is an infinite dimensional, separable Hilbert space, and \( W^*(\Gamma_\Lambda) \) denotes the standard group von Neumann algebra. Since the subgroups \( \Gamma_\Lambda \) themselves are isomorphic to free-products of cyclic groups, all of them, with the exceptions of \( \Gamma_\Lambda \cong \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z} \ast \mathbb{Z}_2 \), are non-amenable. We are then able to conclude that for non-amenable \( \Gamma_\Lambda \) the von Neumann algebras \( W^*(\Gamma, \mathcal{P}_\Lambda) \) are not injective and, consequently, the corresponding C*-algebras \( C^*_\Lambda(\Gamma, \mathcal{P}_\Lambda) \) are non-nuclear (Theorem II). Thus, our construction yields a large class of non-nuclear, purely infinite, simple C*-algebras, which are not classifiable by the existing K-theoretic invariants. It seems plausible that these algebras may become helpful examples when the classification program extends beyond the class of nuclear C*-algebras. Having this ultimate objective in mind we carry out a detailed investigation, in a separate paper [16], of the structure of the algebras \( C^*_\Lambda(\Gamma, \mathcal{P}_\Lambda) \).

1. Main results

1.0. Throughout this article \( \Gamma \) denotes a free product of finitely or countably many cyclic groups \( G_i \). Let \( e \) denote the unit of \( \Gamma \) and let

\[
\mathcal{G} = \{ g_1, g_2, ..., g_n, ... \}
\]

be a fixed set of generators of \( \Gamma \), where \( g_i \) is a generator of \( G_i \).

If \( g_i \) has finite order \( m \), then we make a convention that elements of \( G_i \) be written as \( e, g_i, g_i^2, ..., g_i^{m−1} \). Then each element of \( \Gamma \) can be uniquely written as a reduced word

\[
g_{i_1}^{\epsilon_{i_1}}g_{i_2}^{\epsilon_{i_2}}...g_{i_k}^{\epsilon_{i_k}}, \quad \epsilon_j = +1 \text{ or } -1.
\]

A word is called reduced if it does not contain a subword of the form \( hh^{-1}, h \in \Gamma \).

Let \( \{ \xi_h : h \in \Gamma \} \) be the standard basis of the Hilbert space \( \ell^2(\Gamma) \), where \( \xi_h(s) = 1 \) if \( h = s \) and \( \xi_h(s) = 0 \) if \( h \neq s \). Let \( L : \Gamma \rightarrow \mathcal{L}(\ell^2(\Gamma)) \) be the left regular representation, i.e. \( L_h(\xi_s) = \xi_{hs} \) for \( h, s \in \Gamma \). The reduced group C*-algebra \( C^*_\Gamma \) is generated by \( \{ L_h : h \in \Gamma \} \), and the von Neumann algebra \( W^*(\Gamma) \) is the weak closure of \( C^*_\Gamma \) [8, Vol. II, 6.7].

For a reduced word \( h \in \Gamma \setminus \{ e \} \) let \( \Gamma(h) \) be the set of all reduced words in \( \Gamma \) whose initial segments coincide with \( h \). Let \( P_h \) be the orthogonal projection from \( \ell^2(\Gamma) \) onto the subspace \( \ell^2(\Gamma(h)) \). For a subset \( \Lambda \) of \( \Gamma \) let \( P_\Lambda \) denote the set of projections \( \{ P_h : h \in \Lambda \} \). We define \( C^*_\Lambda(\Gamma, \mathcal{P}_\Lambda) \) as the C*-subalgebra of \( \mathcal{L}(\ell^2(\Gamma)) \) generated by \( C^*_\Gamma \) and \( \mathcal{P}_\Lambda \). We denote by \( W^*(\Gamma, \mathcal{P}_\Lambda) \) the weak closure of \( C^*_\Lambda(\Gamma, \mathcal{P}_\Lambda) \).
It is shown in [17, 1.1] that if $h$ is a reduced word ending with $g_i^{±1}$, then
\[ C_r^*(\Gamma, P_h) = C_r^*(\Gamma, P_{g_i}) = C_r^*(\Gamma, P_{g_i^{−1}}). \]
Similarly, $P_h \in C_r^*(\Gamma, \mathcal{P})$ if and only if $P_{g_i} \in C_r^*(\Gamma, \mathcal{P})$. Thus, it suffices to consider the case when $\Lambda$ is a subset of $\mathcal{G}$, and from now on we assume that this is indeed the case.

Let $\Lambda \subseteq \mathcal{G}$. We define $\Gamma_{\Lambda}$ as the subgroup of $\Gamma$ generated by $\mathcal{G} \setminus \Lambda$. It turns out that the subgroup $\Gamma_{\Lambda}$ plays an important role in determining the structures of $C_r^*(\Gamma, \mathcal{P}_{\Lambda})$ and $W^*(\Gamma, \mathcal{P}_{\Lambda})$. We denote by $\mathcal{H}$ an infinite dimensional, separable Hilbert space.

1.1. Theorem I. There exists a $*$-isomorphism
\[ W^*(\Gamma, \mathcal{P}_{\Lambda}) \cong W^*(\Gamma_{\Lambda}) \otimes \mathcal{L}(\mathcal{H}). \]
Furthermore, the commutant $W^*(\Gamma, \mathcal{P}_{\Lambda})'$ is $*$-isomorphic to $W^*(\Gamma_{\Lambda})$.

In order to prove Theorem I we need two lemmas. Let $R : \Gamma \rightarrow \mathcal{L}(\ell^2(\Gamma))$ be the right regular representation, i.e. $R_h(\xi_s) = \xi_{sh−1}$ for $h, s \in \Gamma$. If $f \in \ell^2(\Gamma)$ and $R_f \in \mathcal{L}(\ell^2(\Gamma))$, then
\[ R_f(\xi_s) = \sum_{h \in \Gamma} f(h)\xi_{sh−1} \quad \forall s \in \Gamma. \]

1.2. Lemma. Let $f \in \ell^2(\Gamma)$ and $R_f \in \mathcal{L}(\ell^2(\Gamma))$ be self-adjoint. Let $g \in \mathcal{G}$ (i.e. there exists $i$ such that $g = g_i$). Then $R_f P_g = P_g R_f$ if and only if $\{s \in \Gamma : f(s) \neq 0\} \subseteq \Gamma(g)$.

Proof. Clearly, $R_f = R_f^*$ if and only if $f(s) = \overline{f(s−1)}$ for all $s \in \Gamma$. Also, $R_f P_g = P_g R_f$ if and only if the subspace $\ell^2(\Gamma(g))$ is $R_f$-invariant (i.e. $R_f(\ell^2(\Gamma(g))) \subseteq \ell^2(\Gamma(g))$), i.e. $R_f(\xi_h) \in \ell^2(\Gamma(g))$ for all $h \in \Gamma(g)$. This is equivalent to
\[ \{\Gamma(g)s^{-1}\} \cup \{\Gamma(g)s\} \subseteq \Gamma(g) \quad \text{whenever} \quad f(s) \neq 0. \]
Notice that $f(s) \neq 0$ if and only if $f(s−1) \neq 0$. Thus, if $R_f P_g = P_g R_f$, then the function $f$ must satisfy
\[ \{s \in \Gamma : f(s) \neq 0\} \subseteq \{s \in \Gamma : \{\Gamma(h)s^{-1}\} \cup \{\Gamma(g)s\} \subseteq \Gamma(g)\}. \]
We claim that $\{\Gamma(g)s^{-1}\} \cup \{\Gamma(g)s\} \subseteq \Gamma(g)$ if and only if neither $g$ nor $g^{-1}$ is contained as a factor in the reduced word $s$. In order to prove this claim we consider separately two cases:
1. $g$ has infinite order,
2. $g$ has finite order $m$.

Case (1). Suppose that $s$ contains a factor $g$ or $g^{-1}$, i.e. in reduced form
\[ s = g_{i_1}^{\epsilon_1} \cdots g_{i_r}^{\epsilon_r} g_{r+1}^{\epsilon_{r+1}} \cdots g_{i_k}^{\epsilon_k}, \]
where $\epsilon, \epsilon_i = 1$ or $−1$. If $\epsilon = 1$, set $h = gg_{i_{r+1}}^{−1} \cdots g_{i_k}^{−1}$. Since $g_{r+1}^{−1} \neq g^{-1}$, we have $h \in \Gamma(g)$. Then $hs^{-1} \notin \Gamma(g)$, because $g_{i_r}^{\epsilon_r} \neq g^{-1}$. If $\epsilon = −1$, set $h = gg_{i_{r+1}}^{−1}g_{r+1}^{−1} \cdots g_{i_k}^{−1}$, again an element of $\Gamma(g)$. Also $hs \notin \Gamma(g)$, due to reasoning as above. Thus, $\{\Gamma(g)s^{-1}\} \cup \{\Gamma(g)s\} \subseteq \Gamma(g)$.

Conversely, suppose that the reduced form of $s$ contains neither $g$ nor $g^{-1}$ as a factor, and let $h \in \Gamma(g)$. Then the initial letter $g$ of $h$ survives all cancellations in $hs$ (or $hs^{-1}$) and, thus, $\{\Gamma(g)s\} \cup \{\Gamma(g)s^{-1}\} \subseteq \Gamma(g)$.
Case (2). By our convention, all elements of the subgroup \( G_i \) \((g = g_i)\) are written as \( e, g, g^2, ..., g^{m-1} \). Suppose first that the reduced form of \( s \) contains \( g \) as a factor, i.e.

\[
s = g_i^{e_1} ... g_i^{e_r} g_i^{n-1} g_{i+1}^{e_r+1} ... g_{i_k}^{e_k},
\]

where \( 1 \leq n \leq m-1 \), and \( g_{i_r} \) and \( g_{i_r+1} \) are different from \( g \). Set

\[
h = g^{m-n}(g_{i_r}^{e_r})^{-1}(g_{i_r+1}^{e_r+1})^{-1} ... (g_{i_k}^{e_k})^{-1}.
\]

Then \( hs \in \Gamma(g_{i_r+1}) \) and, hence, \( hs \notin \Gamma(g) \).

Conversely, if \( s \) does not contain a factor \( g \), then \( hs, hs^{-1} \) are in \( \Gamma(g) \) for any \( h \in \Gamma(g) \), i.e. \( \{\Gamma(g)s\} \cup \{\Gamma(g)s^{-1}\} \subseteq \Gamma(g) \).

Consequently, \( R_fP_g = P_gR_f \) if and only if \( \{s \in \Gamma : f(s) \neq 0\} \subseteq \Gamma(g) \), as desired.

\[\Box\]

1.3. Lemma. \( W^*(\Gamma, \mathcal{P}_\Lambda)' \cong W^*(\Gamma_\Lambda) \).

Proof. It is obvious that \( W^*(\Gamma, \mathcal{P}_\Lambda)' = W^*(\Gamma)' \cap ( \bigcap_{h \in \Lambda} \{P_h\}' \). By the classical theory [8, Vol. II, 6.7.2], the commutant \( W^*(\Gamma)' \) in \( \mathcal{L}(\ell^2(\Gamma)) \) coincides with the von Neumann algebra generated by \( \{R_s : s \in \Gamma\} \). Also, each element of \( W^*(\Gamma)' \) can be written in the form \( R_f \), for some \( f \in \ell^2(\Gamma) \). It is immediate that

\[
W^*(\Gamma, \mathcal{P}_\Lambda)' = \{R_f \in \mathcal{L}(\ell^2(\Gamma)) : R_fP_h = P_hR_f \ \forall h \in \Lambda\}.
\]

Let us consider the self-adjoint part of \( W^*(\Gamma, \mathcal{P}_\Lambda)' \), which of course generates the whole \( W^*(\Gamma, \mathcal{P}_\Lambda)' \). If \( R_f = R_f' \) and \( R_f \in W^*(\Gamma, \mathcal{P}_\Lambda)' \), then by Lemma 1.2

\[
\{s \in \Gamma : f(s) \neq 0\} \subseteq \Gamma_\Lambda.
\]

It follows that \( W^*(\Gamma, \mathcal{P}_\Lambda)' \) equals the von Neumann algebra generated by the unitaries \( \{R_s : s \in \Gamma_\Lambda\} \). Therefore, \( W^*(\Gamma, \mathcal{P}_\Lambda)' \) is *-isomorphic to \( W^*(\Gamma_\Lambda) \) [8, Vol. II, 6.7].

\[\Box\]

1.4. Proof of Theorem I. We write \( \tilde{\Gamma} \) for a fixed set of representatives of cosets \( \Gamma/\Gamma_\Lambda \). Then the disjoint union \( \Gamma = \bigcup_{h \in \Gamma} h\Gamma_\Lambda \) induces a decomposition

\[
\ell^2(\Gamma) = \bigoplus_{h \in \Gamma} \ell^2(h\Gamma_\Lambda).
\]

For any pair \( h_1, h_2 \in \tilde{\Gamma} \) we define a partial isometry \( V_{h_1h_2} \) from \( \ell^2(h_1\Gamma_\Lambda) \) onto \( \ell^2(h_2\Gamma_\Lambda) \) by

\[
V_{h_1h_2}(\xi_{h_1s}) = \xi_{h_2s}, \ \forall s \in \Gamma_\Lambda.
\]

It is clear that \( \{V_{h_1h_2} : h_1, h_2 \in \Gamma\} \) form a system of matrix units which generates a type I factor, which is *-isomorphic to \( \mathcal{L}(\mathcal{H}) \). Furthermore, all \( V_{h_1h_2} \) are in the double commutant \( W^*(\Gamma, \mathcal{P}_\Lambda)' \). Let \( P_0 \) be the projection onto \( \ell^2(\Gamma_\Lambda) \). Then it is clear from [8, Vol. II, 6.7] that \( P_0W^*(\Gamma, \mathcal{P}_\Lambda)'P_0 \) is the von Neumann algebra generated by the unitaries \( \{L_h : h \in \Gamma_\Lambda\} \). Thus, \( P_0W^*(\Gamma, \mathcal{P}_\Lambda)'P_0 \cong W^*(\Gamma_\Lambda) \). Consequently, the Double Commutant Theorem [8, Vol. I, 5.3.1] implies

\[
W^*(\Gamma, \mathcal{P}_\Lambda) \cong W^*(\Gamma, \mathcal{P}_\Lambda)' \cong W^*(\Gamma_\Lambda) \otimes \mathcal{L}(\mathcal{H})
\]

\[\Box\]
1.5. Corollary. (i) If $\Gamma_A$ contains at least two generating cyclic groups and $\Gamma_A \neq \mathbb{Z}_2 \ast \mathbb{Z}_2$, then $W^*(\Gamma_A)$ is a factor of type $II_1$ and, hence, $W^*(\Gamma, P_A)$ is a factor of type $II_\infty$. In particular, if $\Gamma$ is a free group $F_n$ on $n$ generators and $\Lambda = \{g_{n-m}, g_{n-m+1}, \ldots, g_n\}$, where $3 \leq n \leq +\infty$ and $n - m \geq 1$, then

$$W^*(\Gamma, P_A) \cong \begin{cases} \mathcal{L}^\infty(S^1) \otimes \mathcal{L}(\mathcal{H}) & \text{if } n - m = 1, \\
\mathcal{L}(\mathcal{F}_{n-m}) \otimes \mathcal{L}(\mathcal{H}) & \text{if } n - m > 1,
\end{cases}$$

where $+\infty - 1$ is understood as $+\infty$.

(ii) If $\Gamma = \mathbb{Z} \ast \mathbb{Z}_m$, where $2 \leq m < +\infty$, and $\Lambda = \{g_i\}$, then

$$W^*(\Gamma, P_A) \cong \left\{ \begin{aligned}
\mathcal{L}^\infty(S^1) \otimes \mathcal{L}(\mathcal{H}) & \text{if } G_i \cong \mathbb{Z}_m, \\
\mathcal{L}(\mathcal{H}) \oplus \mathcal{L}(\mathcal{H}) \oplus \cdots \oplus \mathcal{L}(\mathcal{H}) & \text{if } G_i \cong \mathbb{Z}.
\end{aligned} \right\} \quad \text{m times}$$

(iii) If $\Gamma = \mathbb{Z}_m \ast \mathbb{Z}_m$, where $2 \leq m, n < +\infty$, and $\Lambda = \{g_i\}$, then

$$W^*(\Gamma, P_A) \cong \mathcal{L}(\mathcal{H}) \oplus \mathcal{L}(\mathcal{H}) \oplus \cdots \oplus \mathcal{L}(\mathcal{H}) \quad \text{m times}$$

As the most important application of Theorem I we determine non-nuclearity of certain C*-algebras of the form $C^*_r(\Gamma, P_A)$. At first we recall some relevant known results.

1.6. (a) Assume that $G$ is a discrete group. Then $C^*_rG$ is amenable (nuclear) if and only if the group $G$ is amenable ([1], [11]).

(b) Assume that $G$ is a countable discrete i.c.c. group. Then $W^*(G)$ is $\ast$-isomorphic to the unique hyperfinite type $II_1$ factor $\mathcal{R}$ if and only if $G$ is amenable [4]. Also, $W^*(G) \otimes \mathcal{L}(\mathcal{H})$ is $\ast$-isomorphic to the unique injective type $II_\infty$ factor $\mathcal{R} \otimes \mathcal{L}(\mathcal{H})$ if and only if $G$ is amenable [4].

(c) A C*-algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is nuclear, and again if and only if the enveloping von Neumann algebra $\mathcal{A}^{\ast\ast}$ is injective [3], [4], [10].

1.7. Theorem II. If $\Gamma_A$ contains at least two different generators of $\Gamma$ but $\Gamma_A \neq \mathbb{Z}_2 \ast \mathbb{Z}_2$, then the C*-algebra $C^*_r(\Gamma, P_A)$ is non-nuclear.

Proof. It is well known that our assumptions imply that the group $\Gamma_A$ is non-amenable (e.g. cf. [12, Theorem 1.1]). Consequently, Theorem I and (b) above imply that $W^*(\Gamma, P_A)$, the weak closure of $C^*_r(\Gamma, P_A)$, is not injective. Thus, the enveloping von Neumann algebra $C^*_r(\Gamma, P_A)^{\ast\ast}$ is not injective either, and (c) above implies that the C*-algebra $C^*_r(\Gamma, P_A)$ is non-nuclear, as desired.

1.8. Remark. The problem of nuclearity of those algebras $C^*_r(\Gamma, P_A)$ for which the group $\Gamma_A$ is amenable (i.e. $\Gamma_A$ is either cyclic or $\mathbb{Z}_2 \ast \mathbb{Z}_2$) has already been answered in the affirmative in [16].

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