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$B_h[g]$-SEQUENCES FROM B_h-SEQUENCES

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ABSTRACT. A sequence A of positive integers is called a $B_h[g]$-sequence if every integer n has at most g representations $n = a_1 + a_2 + \cdots + a_{h'}$ with all a_i in A and $a_1 \le a_2 \le \cdots \le a_h$. A $B_h[1]$-sequence is also called a B_h-sequence or Sidon sequence.

The main result is the following

Theorem. *Let A be a B_h-sequence and $g = m^{h-1}$ for an integer $m \ge 2$. Then there is a $B_h[g]$-sequence B of size $|B| = m|A|$, where $B = \bigcup_{i=0}^{m-1}\{ma + i| a \in A\}$.*

Corollary. *Let $g = m^{h-1}$. The interval $[1, n]$ then contains a $B_h[g]$-sequence of size $(gn)^{1/h}(1 + o(1))$, when $n \to \infty$.*

1. Introduction

A sequence A of positive integers is called a $B_h[g]$-sequence if every integer n has at most g different representations $n = a_1 + \cdots + a_h$, where $a_1 \le a_2 \le \cdots \le a_h$ belong to A. $B_h[1]$-sequences are also called B_h-sequences or Sidon sequences, when $h = 2$.

Sidon sequences were studied by R. C. Bose, S. Chowla, P. Erdös, J. Singer and P. Turán. The book [3] by Halberstam and Roth has a fine survey of their classical results.

Non-trivial upper bounds for the size of B_h-sequences with $h > 2$ have recently been found by S. Chen, X.-D. Jia, M. N. Kolountzakis and A. Lee, which generalize an old bound of mine for B_4-sequences. References to these recent works can be found in [4] and [6].

For the size of $B_h[g]$-sequences with $g \ge 2$ in $[1, n]$ there is only a trivial bound $(ghh!\,n)^{1/h}$ (for a proof see [1]). In [2] (Theorem 3.4) there is a bound which looks non-trivial. But T. Kløve has proved [5] that it is weaker than the trivial bound!

Constructions of $B_h[g]$-sequences have been given by X.-D. Jia [4], T. Kløve [5], and M. N. Kolountzakis [6]. Jia tries to improve a result of Kolountzakis', but there is a flaw in his proof and his $m > g$ violates the restriction (1.3) below. I will generalize Theorem 3 of Kolountzakis by a similar, but correct, construction.

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Let A be a B_h-sequence and choose an integer $m \geq 2$. Define

$$A_i = \{ma + i | a \in A\} \quad \text{for } i = 0, 1, \ldots, m-1, \tag{1.1}$$

$$B = \bigcup_{i=0}^{m-1} A_i. \tag{1.2}$$

We want B to be a $B_h[g]$-sequence. Then there are restrictions on m and g. One is

$$m \leq g \quad \text{when } |A| \geq 2. \tag{1.3}$$

If $a_1 < a_2$ are elements of A, we have at least the following elements in B : $ma_1 < ma_1 + 1 < \cdots < ma_1 + m - 1 < ma_2 < ma_2 + 1 < \cdots < ma_2 + m - 1$. Since $ma_1 + (ma_2 + m - 1) = (ma_1 + 1) + (ma_2 + m - 2) = \cdots = (ma_1 + m - 1) + ma_2$ it follows that $g \geq m$ for $h = 2$. When $h > 2$ add $(h-2)ma_1$ everywhere. Hence, (1.3) is necessary and $m = 2g - 1$ (in [4]) is impossible.

Assume that $a_1 < a_2 < \cdots < a_h$ are elements of A. If $(x_1, x_2, \ldots, x_h)$ satisfies a diophantine equation

$$x_1 + x_2 + \cdots + x_h = c, \qquad 0 \leq x_1, x_2, \ldots, x_h \leq m - 1, \tag{1.4}$$

then the elements of B

$$ma_1 + x_1 < ma_2 + x_2 < \cdots < ma_h + x_h \tag{1.5}$$

have the sum $m(a_1 + a_2 + \cdots + a_h) + c$. Different solutions to (1.4) give different h-tuples (1.5) of elements of B with the same sum. For a suitable integer c (1.4) has at least m^{h-1}/h solutions. Therefore

$$m^{h-1}/h \leq g \quad \text{when } |A| \geq h \tag{1.6}$$

is also necessary.

2. The main result

Our main result is the following theorem.

Theorem. *Assume that A is a B_h-sequence. Let $g = m^{h-1}$. If B is defined by (1.1) and (1.2), then B will be a $B_h[g]$-sequence of size $|B| = m|A|$.*

Proof. Assume that $b_i^{(j)}$ $(i = 1, 2, \ldots, h; j = 1, 2, \ldots, g+1)$ are elements of B with $(b_1^{(j)}, b_2^{(j)}, \ldots, b_h^{(j)})$, $j = 1, 2, \ldots, g+1$, pairwise distinct,

$$b_i^{(j)} = ma_i^{(j)} + r_i^{(j)}, a_i^{(j)} \in A, \qquad 0 \leq r_i^{(j)} \leq m - 1, \tag{2.1}$$

$$b_1^{(j)} \leq b_2^{(j)} \leq \cdots \leq b_h^{(j)} \qquad (j = 1, 2, \ldots, g+1), \tag{2.2}$$

$$\sum_{i=1}^{h} b_i^{(j)} = c \qquad (j = 1, 2, \ldots, g+1),\ c \in Z^+. \tag{2.3}$$

We prove that this gives a contradiction.

There are at most m^{h-1} distinct $(h-1)$-tuples $(r_1^{(j)}, r_2^{(j)}, \ldots, r_{h-1}^{(j)})$. We have $m^{h-1} + 1$ $(h-1)$-tuples. Therefore at least two are equal. We may assume w.l.o.g. that

$$(r_1^{(1)}, r_2^{(1)}, \ldots, r_{h-1}^{(1)}) = (r_1^{(2)}, r_2^{(2)}, \ldots, r_{h-1}^{(2)}). \tag{2.4}$$

By (2.1) and (2.3) we have

$$\sum_{i=1}^{h} r_i^{(1)} \equiv \sum_{i=1}^{h} r_i^{(2)} \pmod{m}. \tag{2.5}$$

It then follows by (2.4) and (2.5) that

$$r_h^{(1)} = r_h^{(2)}. \tag{2.6}$$

By (2.1), (2.3), (2.4) and (2.6) we find that

$$\sum_{i=1}^{h} a_i^{(1)} = \sum_{i=1}^{h} a_i^{(2)}. \tag{2.7}$$

Observe that (2.1) and (2.2) imply

$$a_1^{(j)} \le a_2^{(j)} \le \cdots \le a_h^{(j)} \qquad (j = 1, 2, \ldots, g+1). \tag{2.8}$$

Recall that A is a B_h-sequence. It then follows by (2.7) and (2.8) that

$$(a_1^{(1)}, a_2^{(1)}, \ldots, a_h^{(1)}) = (a_1^{(2)}, a_2^{(2)}, \ldots, a_h^{(2)}).$$

By (2.1), (2.4) and (2.6) we have finally that

$$(b_1^{(1)}, b_2^{(1)}, \ldots, b_h^{(1)}) = (b_1^{(2)}, b_2^{(2)}, \ldots, b_h^{(2)}),$$

a contradiction to the assumption at the beginning of the proof. Therefore B is a $B_h[g]$-sequence.

Corollary. *Let $g = m^{h-1}$. The interval $[1, n]$ contains a $B_h[g]$-sequence of size $(gn)^{1/h}\ (1 + o(1))$, when $n \ge \infty$.*

Proof. The interval $[1, \lfloor n/m \rfloor - 1]$ contains a B_h-sequence A of size $|A| = (n/m)^{1/h}$ $(1 + o(1))$ by a classical result of Bose and Chowla, Theorem 6 on p. 88 in [3]. Then B in (1.2) is a subset of $[1, n]$ of size $m|A| = (gn)^{1/h}\ (1 + o(1))$, which is a $B_h[g]$-sequence by the theorem.

Remark. Theorem 3 of Kolountzakis is the special case of the corollary when $h = 2$ and $g = m = 2$. My first proof of the theorem was influenced by the proof of Kolountzakis. It was more complex than the present one.

When I presented the first proof in a seminar, a clever student in the audience suggested some simplifications which I gratefully adopted. The name of the student is Johan Wästlund.

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