PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 3, Pages 671–675
S 0002-9939(99)05206-5
Article electronically published on July 6, 1999

MATRICES OVER ORDERS IN ALGEBRAIC NUMBER FIELDS AS SUMS OF k-TH POWERS

S. A. KATRE AND SANGITA A. KHULE

(Communicated by David E. Rohrlich)

Dedicated to the memory of David R. Richman

ABSTRACT. David R. Richman proved that for $n \geq k \geq 2$ every integral $n \times n$ matrix is a sum of seven k-th powers. In this paper, in light of a question proposed earlier by M. Newman for the ring of integers of an algebraic number field, we obtain a discriminant criterion for every $n \times n$ matrix ($n \geq k \geq 2$) over an order of an algebraic number field to be a sum of (seven) k-th powers.

1. INTRODUCTION

M. Newman [1] showed that, for $n \geq 2$, every matrix in $M_n\mathbf{Z}$ is a sum of 7 or 9 squares according as n is even or odd. He then posed the problem for the ring of integers (i.e. the maximal order) of an algebraic number field.

Vaserstein [3], [4] showed that every integral $n \times n$ matrix ($n \geq 2$) is a sum of three squares by proving the following

Theorem A. *A matrix A in M_nR (R a commutative ring with 1 and $n \geq 2$) is a sum of squares if and only if A is a sum of three squares if and only if $tr.A \equiv$ square* (mod $2R$).

David R. Richman [2] showed that, for $n \geq k \geq 2$, every $n \times n$ integral matrix is a sum of seven k-th powers using his following key-result:

Theorem B. *Let $n \geq 2, R$ a commutative ring with 1. The following are equivalent:*

(i) M is a sum of k-th powers in M_nR.

(ii) M is a sum of seven k-th powers in M_nR.

(iii) $M \in M_nR$ and for every prime power p^e dividing k , there are elements $x_0 = x_0(p), \cdots, x_e = x_e(p)$ in R , such that

$$tr.M \;=\; x_0^{p^e} + px_1^{p^{e-1}} + p^2x_2^{p^{e-2}} + \cdots + p^ex_e.$$

If F is a field of characteristic 0, then it follows from Theorems A and B that every matrix in M_nF ($n \geq 2$) is a sum of three squares, and for $n \geq k \geq 2$, every matrix in M_nF is a sum of seven k-th powers in M_nF. The same result also follows

Received by the editors April 21, 1998.
1991 *Mathematics Subject Classification.* Primary 11P05, 11R04, 15A33; Secondary 11C20, 11E25, 15A24.
Key words and phrases. Algebraic number fields, order, sums of powers, discriminant, matrices.

for matrices over $\mathbf{Z}$. However, if we consider the ring of integers or other orders in algebraic number fields, then we find that for some of these rings such a result is true, whereas for some other rings we get counter-examples. For instance, if $R = \mathbf{Z}[i]$, then $R/2R = \{\overline{0}, \overline{1}, \overline{i}, \overline{1+i}\}$. Here $\overline{0}$ and $\overline{1}$ are the only squares, and an $n \times n$ matrix over $\mathbf{Z}[i]$ $(n \geq 2)$ whose trace is $\equiv i$ or $1 + i (\mathrm{mod}\ 2R)$ is not a sum of squares in $\mathbf{Z}[i]$. On the contrary, every element of $R/3R$ is a cube, so every $n \times n$ matrix $(n \geq 3)$ over $\mathbf{Z}[i]$ is a sum of seven cubes. One has exactly the reverse situation for $R = \mathbf{Z}[\omega]$, $\omega = exp(2\pi i/3)$, and we get that every $n \times n$ matrix $(n \geq 2)$ over $\mathbf{Z}[\omega]$ is a sum of 3 squares, but for every $n \geq 3$, we find matrices over $\mathbf{Z}[\omega]$ which are not sums of cubes in $\mathbf{Z}[\omega]$.

In this paper, we take up this problem (earlier raised by Newman) for orders in algebraic numbers fields and obtain the following discriminant criterion:

Theorem 1. *Let R be an order in an algebraic number field K. Let $n \geq k \geq 2$. Then every $n \times n$ matrix over R is a sum of (seven) k-th powers if and only if $(k, disc.R) = 1$.*

2. Matrices over the ring of integers of an algebraic number field

Henceforth, let K denote an algebraic number field and $\mathbf{O}$ the ring of integers of K. The discriminant of K or the discriminant of $\mathbf{O}$ denotes the discriminant of any integral basis of K (i.e. a $\mathbf{Z}$-basis of $\mathbf{O}$). In this section we prove

Proposition 1. *Let $n \geq k \geq 2$. Every $n \times n$ matrix over $\mathbf{O}$ is a sum of (seven) k-th powers if and only if $(k,\ disc.K) = 1$.*

For this, we first note the following lemmas:

Lemma 1. *Let R be a commutative ring with 1. Let p be a prime. The following are equivalent:*

(i) Every element of R is a p-th power $(\mathrm{mod}\ pR)$.

(ii) For every $e \geq 1$, given any $x \in R$, there are elements $x_0, x_1, \cdots, x_e$ depending upon p and e such that

$$x \ = \ x_0^{p^e} + px_1^{p^{e-1}} + p^2 x_2^{p^{e-2}} + \cdots + p^e x_e.$$

Proof. (ii) $\Rightarrow$ (i) is clear. To prove (i) $\Rightarrow$ (ii), first prove by induction that $a \equiv b (\mathrm{mod}\ pR) \Rightarrow a^{p^i} \equiv b^{p^i} (\mathrm{mod}\ p^{i+1}R)$. Then (ii) can be proved by induction by noting that if $x_i \equiv y_i^p (\mathrm{mod}\ pR)$, then $x_i^{p^{e-i}} \equiv y_i^{p^{e-i+1}} (\mathrm{mod}\ p^{e-i+1}R)$ so that $p^i x_i^{p^{e-i}} \equiv p^i y_i^{(e+1)-i} (\mathrm{mod}\ p^{e+1}R)$. □

Lemma 2. *Let R be a commutative ring with unity. Let $n \geq k \geq 2$. The following are equivalent:*

(1) Every matrix in M_nR is a sum of k-th powers in M_nR.

(2) Every matrix in M_nR is a sum of seven k-th powers in M_nR.

(3) For every p dividing k , every element of R is a p-th power $(\mathrm{mod}\ pR)$.

Proof. (1) $\Leftrightarrow$ (2) is due to Theorem B of Richman. (1) $\Leftrightarrow$ (3) is obtained by combining Theorem B and Lemma 1, and by noting that every $x \in R$ is the trace of the diagonal matrix diag.$\{x, 0, 0, \cdots, 0\}$. □

Lemma 3. *Let p be a prime. The following are equivalent:*
(i) Every element of $\boldsymbol{O}$ is a p-th power (mod $p\mathbf{O}$).
(ii) $(p,\ disc.K) = 1$.

Proof. ($\Leftarrow$) Let $(p, disc.K) = 1$. Then p is unramified in K, so $p\mathbf{O} = \wp_1\wp_2\cdots\wp_r$, where $\wp_1, \wp_2, \cdots, \wp_r$ are distinct primes of $\mathbf{O}$. Then by the Chinese remainder theorem, $\mathbf{O}/p\mathbf{O} \approx \mathbf{O}/\wp_1 \oplus \cdots \oplus \mathbf{O}/\wp_r$. Each $\mathbf{O}/\wp_i$ is a finite field of characteristic p, so every element of $\mathbf{O}/\wp_i$ and hence of $\mathbf{O}/p\mathbf{O}$ is a p-th power.

($\Rightarrow$) Suppose $p \mid disc.K$. Then p is ramified in K. Let $\wp$ be a prime divisor of p which ramifies. Take $x \in \wp$ such that $x \notin \wp^2$. Then $x \not\equiv y^p$(mod $p\mathbf{O}$) for any $y \in \mathbf{O}$. For otherwise, $\wp \mid x \Rightarrow \wp \mid y^p \Rightarrow \wp \mid y \Rightarrow \wp^2 \mid x$ (as $\wp^2 \mid p\mathbf{O}$), a contradiction. □

Proof of Proposition 1. Follows by combining Lemma 2 and Lemma 3. □

Corollary 1. *Let m be a squarefree integer. Let $\mathbf{O}$ be the ring of integers of $K = \mathbf{Q}(\sqrt{m})$ (i.e. $\mathbf{O} = \mathbf{Z}[\sqrt{m}]$ if $m \equiv 2, 3 (\text{mod } 4)$ and $\mathbf{O} = \mathbf{Z}[(1+\sqrt{m})/2]$ if $m \equiv 1$ (mod 4).) Let $n \geq k \geq 2$. Then every matrix in $M_n\mathbf{O}$ is a sum of (seven) k-th powers if and only if $(k, m) = 1$ and either*
(i) k is odd, or
(ii) k is even and $m \equiv 1 (\text{mod } 4)$.

Proof. $Disc.\mathbf{Q}(\sqrt{m}) = \begin{cases} m, & \text{if } m \equiv 1 (\text{mod } 4), \\ 4m, & \text{if } m \equiv 2, 3 (\text{mod } 4). \end{cases}$ □

Corollary 2. *Let $m \geq 1$ and ζ_m be a primitive m-th root of unity. The ring of integers of the cyclotomic field $K = \mathbf{Q}(\zeta_m)$ is $\mathbf{O} = \mathbf{Z}[\zeta_m]$.*

Let $n \geq k \geq 2$. Then every $n \times n$ matrix over $\mathbf{Z}[\zeta_m]$ is a sum of (seven) k-th powers if and only if either
(i) $(k, m) = 1$, or
(ii) $m \equiv 2 (\text{mod } 4)$ and $(k, m) = 2$.

Proof. Note that $\mathbf{Q}(\zeta_m) = \mathbf{Q}(\zeta_{2m})$, if m is odd. Also for $m \not\equiv 2 (\text{mod } 4)$, the prime divisors of disc. $\mathbf{Q}(\zeta_m)$ are the same as the prime divisors of m. □

3. Matrices over orders in algebraic number fields

An order in a algebraic number field K is a ring containing 1, and which is a finitely generated $\mathbf{Z}$-submodule of K of maximum rank, i.e. of rank $N = deg(K/\mathbf{Q})$. One notes that $\mathbf{O}$ is an order of K and $\mathbf{O}$ contains every order; hence $\mathbf{O}$ is called the maximal order of K. The discriminant of an order R is defined to be the discriminant of any $\mathbf{Z}$-basis of R .

Lemma 4. *If K is a number field of degree N, $\mathbf{O}$ the ring of integers of K, and R an order of K, then there are a $\mathbf{Z}$-basis $\theta_1, \theta_2, \cdots, \theta_N$ of $\mathbf{O}$ and a $\mathbf{Z}$-basis $\alpha_1, \alpha_2, \cdots, \alpha_N$ of R such that $\alpha_i = f_i\theta_i, f_i \in \mathbf{Z}, f_i > 0$, and moreover*

$$f_1 \mid f_2 \mid \cdots \mid f_N.$$

Proof. Start with any $\mathbf{Z}$-bases $\eta_1, \cdots, \eta_N$ and $\beta_1, \cdots, \beta_N$ of $\mathbf{O}$ and R respectively. Let $A \in M_N\mathbf{Z}$ such that $[\beta_1, \cdots, \beta_N] = [\eta_1, \cdots, \eta_N]A$. As $\mathbf{Z}$ is a PID, there exist unimodular matrices P and Q such that $PAQ = diag.[f_1, f_2, \cdots, f_N]$ is the Smith normal form of A, so that $f_1 \mid f_2 \mid \cdots \mid f_N$. As rank of A is N, each $f_i \neq 0$. We

may also assume that each $f_i > 0$. Now

$$[\beta_1, \cdots, \beta_N]Q = [\eta_1, \cdots, \eta_N]P^{-1}(PAQ).$$

Call $[\beta_1, \cdots, \beta_N]Q = [\alpha_1, \cdots, \alpha_N]$ and $[\eta_1, \cdots, \eta_N]P^{-1} = [\theta_1, \cdots, \theta_N]$. □

Lemma 5. *With the bases of* $\mathbf{O}$ *and* R *as in Lemma 4, one has*

$$\textit{index of } R \textit{ in } \mathbf{O} = f_1 f_2 \cdots f_N \textit{ and } disc.R = (f_1 f_2 \cdots f_N)^2 disc.K.$$

Proof. Clear. □

Lemma 6. *Let* R *be a commutative ring with 1. Let* p *be a prime and* R/pR *a finite ring. The following are equivalent:*

(i) Every element of R *is a* p*-th power*$(\text{mod } pR)$.

(ii) $x \in R,\ x^p \in pR \Rightarrow x \in pR$.

Proof. Let the map $\phi : R/pR \to R/pR$ be given by $\alpha \to \alpha^p$. Then ϕ is a homomorphism. Now

(i) $\Leftrightarrow$ ϕ is onto $\Leftrightarrow$ ϕ is one-one (as R/pR is finite) $\Leftrightarrow$ $ker\phi$ is trivial $\Leftrightarrow$ (ii). □

Proof of Theorem 1. ($\Leftarrow$) Suppose $(k, disc.R) = 1$. Let p be any prime divisor of k . Then $(p, disc.R) = 1$, and by Lemma 5, $(p, disc.K) = 1$. As R/pR is finite, in view of Lemmas 2 and 6, it suffices to prove that $x \in R, x^p \in pR \Rightarrow x \in pR$. Thus let $x \in R, x^p \in pR$. Then $x^p \in p\mathbf{O}$. As $(p, disc.K) = 1$, by Lemma 3 and Lemma 6, $x \in p\mathbf{O}$. Let $\theta_1, \cdots, \theta_N$ and $\alpha_1, \cdots, \alpha_N$ be bases of $\mathbf{O}$ and R respectively, chosen as in Lemma 4. As $x \in R$, let $x = \sum_{i=1}^{N} a_i\alpha_i$. Then $x = \sum_{i=1}^{N} a_i f_i \theta_i$. As $x \in p\mathbf{O}$, there is $b_i \in \mathbf{Z}$ such that $a_i f_i = pb_i$ $(1 \le i \le N)$. As $(p, disc.R) = 1$, by Lemma 5, $(p, f_i) = 1$, so $p \mid a_i$ for each i. Hence $x \in pR$.

($\Rightarrow$) Suppose $(k, disc.R) \neq 1$. Let p be a prime such that $p \mid (k, disc.R)$. Now, $disc.R = (f_1 \cdots f_N)^2 disc.K$.

Case (i). Suppose $(p, f_i) = 1$ for all $1 \le i \le N$. Then $p \mid disc.K$.

Assume, for contradiction, that every matrix in M_nR is a sum of k-th powers. Then by Lemma 2, every element of R is a p-th power mod pR, say $\alpha_i \equiv \gamma_i^p \pmod{pR}$. Let $b_i, c_i \in \mathbf{Z}$ such that $1 = b_i p + c_i f_i$. Then $\theta_i = b_i p\theta_i + c_i f_i \theta_i \equiv c_i\alpha_i \equiv c_i^p\gamma_i^p \pmod{p\mathbf{O}}$ (noting that $c_i \equiv c_i^p \pmod{p\mathbf{Z}}$). Thus if $x = \sum_{i=1}^{N} a_i\theta_i \in \mathbf{O}$ with $a_i \in \mathbf{Z}$, then $x \equiv (\sum a_i c_i \gamma_i)^p \pmod{p\mathbf{O}}$. This gives $(p, disc.K) = 1$, by Lemma 3. Contradiction.

Case (ii). Suppose $p \mid f_j$ for some j. Due to the choice of the f_i's as in Lemma 4, we have $f_1 \mid f_2 \mid \cdots \mid f_N$, so $p \mid f_N$. Also, $f_N\theta_i \in R$ for all $1 \le i \le N$, and so $f_N\alpha \in R$, for every $\alpha \in \mathbf{O}$. In particular $\beta = f_N(f_N^{p-2}\theta_N^p) \in R$. As $p \mid f_N, \alpha_N^p = f_N\beta \in pR$. However, $\alpha_N \in pR$, as $\alpha_1, \cdots, \alpha_N$ is a $\mathbf{Z}$-basis of R. Hence from Lemma 2 it follows that there are matrices in M_nR that are not sums of k-th powers in M_nR. □

Remark 1. Let $(k, disc.R) > 1$. Let p be the smallest prime divisor of $(k, disc.R)$. Then, for every $n \ge p$, there are $n \times n$ matrices (which are not sums of p-th powers, and hence) which are not sums of k-th powers in M_nR.

Remark 2. Combining Theorem A and Theorem 1, we see that for an order R, if $disc.R$ is odd (i.e. $\equiv 1 \pmod 4$), then for every $n \ge 2$, every matrix in M_nR is a sum of three squares. Also, if $disc.R$ is even (i.e. $\equiv 0 \pmod 4$), then for every $n \ge 2$, there are matrices in M_nR which are not sums of squares in M_nR.

Corollary 3. *Let m be a squarefree integer, and f denote a positive integer. If $m \equiv 1 \pmod 4$, the orders of $\mathbf{Q}(\sqrt{m})$ are $R_f = \mathbf{Z} + f((1+\sqrt{m})/2)\mathbf{Z}$. If $m \equiv 2, 3 \pmod 4$, the orders are $R_f = \mathbf{Z} + f\sqrt{m}\mathbf{Z}$. Then for $n \geq k \geq 2$ every $n \times n$ matrix over $M_n R_f$ is a sum of (seven) k-th powers if and only if $(k, fm) = 1$ and either (i) k is odd or (ii) k is even and $m \equiv 1 \pmod 4$.*

Proof. If $m \equiv 1 \pmod 4, disc.R_f = f^2 m$.
If $m \equiv 2, 3 \pmod 4, disc.R_f = 4f^2 m$. □

Example 1. $Disc.\mathbf{Z}[\sqrt{m}] = 4m$, when m is not a perfect square. Hence for every $n \geq 2$, there are matrices over $\mathbf{Z}[\sqrt{m}]$ which are not sums of squares in $M_n\mathbf{Z}[\sqrt{m}]$ (although they have to be sums of three squares in $M_n\mathbf{Q}(\sqrt{m})$).

Example 2. $Disc.\mathbf{Z}[i] = -4$, so for every $n \geq 2$, there are $n \times n$ matrices over $\mathbf{Z}[i]$, which are not sums of squares. Hence for k even, for every $n \geq 2$, there are $n \times n$ matrices over $\mathbf{Z}[i]$ which are not sums of k-th powers. However, for k odd, for every $n \geq k$, every $n \times n$ matrix over $\mathbf{Z}[i]$ is a sum of seven k-th powers.

References

[1] M. Newman, Sums of squares of matrices, Pacific J. Math. 118 (1985), 497-506. MR **86k:**15011

[2] D. R. Richman, The Waring problem for matrices, Linear and Multi. Alg. 22(1987), 171-192. MR **89d:**11087

[3] L. N. Vaserstein, Every integral matrix is a sum of three squares, Linear and Multi. Alg. 20(1986), 1-4. MR **88e:**15009

[4] L. N. Vaserstein, On the sum of powers of matrices, Linear and Multi. Alg. 21 (1987), 261-270. MR **89a:**15016

Department of Mathematics, University of Pune, Pune-411007, India
E-mail address: sakatre@math.unipune.ernet.in