A NOTE ON THE
CONTACT ANGLE BOUNDARY CONDITION
FOR MONGE-AMPÈRE EQUATIONS

JOHN URBAS

(Communicated by Peter Li)

Abstract. We give a simple proof of a result of Xinan Ma concerning a
necessary condition for the solvability of the two-dimensional Monge-Ampère
equation subject to the contact angle or capillarity boundary condition. Our
technique works for more general Monge-Ampère equations in any dimension,
and also extends to some other boundary conditions.

In [1] Xinan Ma considered the Monge-Ampère equation
\[ \det D^2 u = c \quad \text{in} \quad \Omega, \]
where \( \Omega \) is a \( C^2 \) bounded convex domain in \( \mathbb{R}^2 \) and \( c \) is a positive constant, subject
to the contact angle or capillarity boundary condition
\[ D\nu u = \cos \theta \sqrt{1 + |Du|^2} \quad \text{on} \quad \partial \Omega, \]
where \( \nu \) denotes the outer unit normal vectorfield to \( \partial \Omega \) and \( \theta \in (0, \pi/2) \) is a
constant. The result of [1] is that if there is a convex solution \( u \in C^3(\Omega) \cap C^2(\Omega) \)
of (1), (2), then
\[ k_0 \leq \max\{ \sqrt{c} \cos \theta, \sqrt{c} \tan \theta \}, \]
where \( k_0 \) denotes the minimum curvature of \( \partial \Omega \).

Here we provide a much simpler and more transparent proof of this result, which
moreover, generalizes to more general Monge-Ampère equations in higher dimen-
sions and to some other boundary conditions.

Theorem. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( C^2 \) boundary, \( f \in C(\overline{\Omega}) \) a
positive function and \( \phi \in C(\partial \Omega) \) with \( 0 < \phi < 1 \). If \( u \in C^2(\Omega) \cap C^1(\Omega) \) is a convex
solution of
\[ \det D^2 u = f \quad \text{in} \quad \Omega, \]
\[ D\nu u = \phi \sqrt{1 + |Du|^2} \quad \text{on} \quad \partial \Omega, \]
Received by the editors May 12, 1998.
1991 Mathematics Subject Classification. Primary 35J25, 35J60, 35J65.
Key words and phrases. Monge-Ampère equations, contact angle boundary condition.

\( \copyright1999 \) American Mathematical Society

853
then

\[ k_0 \leq \frac{\left( \sup_\Omega f \right)^{1/n}}{\inf_{\partial \Omega} \frac{\phi}{\sqrt{1 - \phi^2}}} \]

where \( k_0 \) denotes the minimum normal curvature of \( \partial \Omega \).

Remark. If \( f = c \) and \( \phi = \cos \theta \) where \( c > 0 \) and \( \theta \in (0, \pi/2) \) are constants, (6) reduces to

\[ k_0 \leq c^{1/n} \tan \theta, \]

which improves (3) for small enough \( \theta \). In addition, it will be clear from the proof below that equality holds in (7) if \( \Omega \) is a ball.

Proof of Theorem. We may assume that \( k_0 > 0 \); otherwise the result is immediate. By the definition of \( k_0 \) it is clear that \( \Omega \) is contained in a ball of radius \( k_0^{-1} \). Next we observe that since \( \phi > 0 \), we have \( \partial_\nu u > 0 \) on \( \partial \Omega \), so \( u \) must have its minimum at an interior point of \( \Omega \), say at \( x_0 \in \Omega \). Thus \( Du(x_0) = 0 \). Now, by integrating (4) and using the classical change of variables formula we obtain

\[ |Du(\Omega)| = \int_\Omega \det D^2 u \]
\[ \leq |\Omega| \sup_\Omega f \]
\[ \leq \omega_n k_0^{-n} \sup_\Omega f, \]

where \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \). Since \( Du(\Omega) \) is an open set containing the origin, there is a point \( y_0 \in \partial \Omega \) such that

\[ |Du(y_0)| \leq k_0^{-1} \left( \sup_\Omega f \right)^{1/n}. \]

On the other hand, from the boundary condition (5) we obtain

\[ |Du(y_0)| \geq \inf_{\partial \Omega} \frac{\phi}{\sqrt{1 - \phi^2}}. \]

(9) and (10) clearly imply the estimate (6).

Remarks. (i) A similar argument can be used if \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) is a convex solution of

\[ \det D^2 u \leq f(x)/g(Du) \quad \text{in} \quad \Omega \]

for some positive functions \( f \in L^p(\Omega), \ p > 1 \), and \( g \in L^1_{\text{loc}}(\mathbb{R}^n) \) with \( \int_{\mathbb{R}^n} g = \infty \).

Similar to before we have

\[ \int_{Du(\Omega)} g \leq \int_\Omega f \]
\[ \leq |\Omega|^{1-1/p} \| f \|_p \]
\[ \leq (\omega_n k_0^{-n})^{1-1/p} \| f \|_p. \]

We may now choose \( R > 0 \) so large that

\[ G(R) := \int_{B_R(0)} g = (\omega_n k_0^{-n})^{1-1/p} \| f \|_p. \]
As before, since $Du(\Omega)$ is an open set containing the origin, there is some $y_0 \in \partial\Omega$ such that $|Du(y_0)| \leq R$. Combining this with (10) we obtain, after some rearrangement,

\begin{equation}
  k_0 \leq \frac{\omega_n^{1/n} \|f\|_p}{\left( G \left( \inf_{\partial\Omega} \phi \sqrt{1-\phi^2} \right) \right)^{p/n(p-1)}}.
\end{equation}

This reduces to (6) in the special case that $f \in L^\infty(\Omega)$, $g \equiv 1$, if we let $p \to \infty$.

(ii) The same argument can be applied to the linear boundary condition

\[ D_\beta u = \phi \quad \text{on} \quad \partial\Omega \]

where $\beta$ is a strictly oblique unit vector field on $\partial\Omega$ and $\phi > 0$; the quantity $\inf_{\partial\Omega} \phi \sqrt{1-\phi^2}$ need only be replaced by $\inf_{\partial\Omega} \phi$ in (6) and (11). Clearly, (5) could also be replaced by a variety of other boundary conditions having an appropriate structure.

REFERENCES


Centre for Mathematics and its Applications, School of Mathematical Sciences, Australian National University, Canberra ACT 0200, Australia

E-mail address: John.Urbas@maths.anu.edu.au