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A NOTE ON THE CONTACT ANGLE BOUNDARY CONDITION FOR MONGE-AMPÈRE EQUATIONS

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ABSTRACT. We give a simple proof of a result of Xinan Ma concerning a necessary condition for the solvability of the two-dimensional Monge-Ampère equation subject to the contact angle or capillarity boundary condition. Our technique works for more general Monge-Ampère equations in any dimension, and also extends to some other boundary conditions.

In [1] Xinan Ma considered the Monge-Ampère equation

$$\det D^2 u = c \quad \text{in} \quad \Omega, \tag{1}$$

where Ω is a C^2 bounded convex domain in $\mathbf{R}^2$ and c is a positive constant, subject to the contact angle or capillarity boundary condition

$$D_\nu u = \cos\theta\sqrt{1+|Du|^2} \quad \text{on} \quad \partial\Omega, \tag{2}$$

where ν denotes the outer unit normal vectorfield to $\partial\Omega$ and $\theta \in (0, \pi/2)$ is a constant. The result of [1] is that if there is a convex solution $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ of (1), (2), then

$$k_0 \le \max\{\sqrt{c}\cos\theta, \sqrt{c}\tan\theta\}, \tag{3}$$

where k_0 denotes the minimum curvature of $\partial\Omega$.

Here we provide a much simpler and more transparent proof of this result, which moreover, generalizes to more general Monge-Ampère equations in higher dimensions and to some other boundary conditions.

Theorem. *Let Ω be a bounded domain in $\mathbf{R}^n$ with C^2 boundary, $f \in C(\overline{\Omega})$ a positive function and $\phi \in C(\partial\Omega)$ with $0 < \phi < 1$. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a convex solution of*

$$\det D^2 u = f \quad \textit{in} \quad \Omega, \tag{4}$$

$$D_\nu u = \phi\sqrt{1+|Du|^2} \quad \textit{on} \quad \partial\Omega, \tag{5}$$

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then

$$k_0 \le \frac{(\sup_\Omega f)^{1/n}}{\inf_{\partial\Omega} \frac{\phi}{\sqrt{1-\phi^2}}} \tag{6}$$

where k_0 denotes the minimum normal curvature of $\partial\Omega$.

Remark. If $f = c$ and $\phi = \cos\theta$ where $c > 0$ and $\theta \in (0, \pi/2)$ are constants, (6) reduces to

$$k_0 \le c^{1/n} \tan\theta, \tag{7}$$

which improves (3) for small enough θ. In addition, it will be clear from the proof below that equality holds in (7) if Ω is a ball.

Proof of Theorem. We may assume that $k_0 > 0$; otherwise the result is immediate. By the definition of k_0 it is clear that Ω is contained in a ball of radius k_0^{-1}. Next we observe that since $\phi > 0$, we have $D_\nu u > 0$ on $\partial\Omega$, so u must have its minimum at an interior point of Ω, say at $x_0 \in \Omega$. Thus $Du(x_0) = 0$. Now, by integrating (4) and using the classical change of variables formula we obtain

$$\begin{aligned} |Du(\Omega)| &= \int_\Omega \det D^2 u \\ &\le |\Omega| \sup_\Omega f \\ &\le \omega_n k_0^{-n} \sup_\Omega f, \end{aligned} \tag{8}$$

where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$. Since $Du(\Omega)$ is an open set containing the origin, there is a point $y_0 \in \partial\Omega$ such that

$$|Du(y_0)| \le k_0^{-1} \left(\sup_\Omega f \right)^{1/n}. \tag{9}$$

On the other hand, from the boundary condition (5) we obtain

$$|Du(y_0)| \ge \inf_{\partial\Omega} \frac{\phi}{\sqrt{1-\phi^2}}. \tag{10}$$

(9) and (10) clearly imply the estimate (6).

Remarks. (i) A similar argument can be used if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a convex solution of

$$\det D^2 u \le f(x)/g(Du) \quad \text{in} \quad \Omega$$

for some positive functions $f \in L^p(\Omega)$, $p > 1$, and $g \in L^1_{loc}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n} g = \infty$. Similar to before we have

$$\begin{aligned} \int_{Du(\Omega)} g &\le \int_\Omega f \\ &\le |\Omega|^{1-1/p} \|f\|_p \\ &\le \left(\omega_n k_0^{-n}\right)^{1-1/p} \|f\|_p. \end{aligned}$$

We may now choose $R > 0$ so large that

$$G(R) := \int_{B_R(0)} g = \left(\omega_n k_0^{-n}\right)^{1-1/p} \|f\|_p.$$

As before, since $Du(\Omega)$ is an open set containing the origin, there is some $y_0 \in \partial\Omega$ such that $|Du(y_0)| \leq R$. Combining this with (10) we obtain, after some rearrangement,

$$k_0 \leq \frac{\omega_n^{1/n} \|f\|_p^{p/n(p-1)}}{\left(G\left(\inf_{\partial\Omega} \frac{\phi}{\sqrt{1-\phi^2}}\right)\right)^{p/n(p-1)}}. \tag{11}$$

This reduces to (6) in the special case that $f \in L^\infty(\Omega)$, $g \equiv 1$, if we let $p \to \infty$.

(ii) The same argument can be applied to the linear boundary condition

$$D_\beta u = \phi \quad \text{on} \quad \partial\Omega$$

where β is a strictly oblique unit vector field on $\partial\Omega$ and $\phi > 0$; the quantity $\inf_{\partial\Omega} \frac{\phi}{\sqrt{1-\phi^2}}$ need only be replaced by $\inf_{\partial\Omega} \phi$ in (6) and (11). Clearly, (5) could also be replaced by a variety of other boundary conditions having an appropriate structure.

References

[1] Xinan Ma, A necessary condition of solvability for the capillarity boundary of Monge-Ampère equations in two dimensions, *Proc. Amer. Math. Soc.* (to appear). CMP 98:05

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