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ON SUBDIAGONAL ALGEBRAS FOR SUBFACTORS

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ABSTRACT. We show that if $N \subseteq M$ are type II_1 factors with finite index (and common identity) and $E : M \to N$ is the trace preserving conditional expectation, then there are no subdiagonal algebras in M with respect to E unless $M = N$.

Let $N \subseteq M$ be finite factors (with common identity) and let $E : M \to N$ be the trace preserving conditional expectation. A σ-weakly closed algebra A is (maximal) subdiagonal in M with respect to E if the following conditions are satisfied [1]:

[S1] $A \cap A^* = N$.
[S2] $A + A^*$ is σ-weakly dense in M.
[S3] $E(xy) = E(x)E(y)$ for any $x, y \in A$.

Saito and Watatani showed in [4] that if M is a crossed product for an outer action of a discrete group G on N, then maximal (inclusionwise) subdiagonal algebras are in bijective correspondence with left invariant total orders on G. A totally ordered group is torsion free and, thus, infinite. On the basis of this result Saito and Watatani conjectured [4] that if N is a proper subfactor of M with finite index, then there are no subdiagonal algebras in M with respect to E. They have verified this in the case when either M is finite dimensional or the index does not exceed 4 [4, 5]. The purpose of this paper is to prove the conjecture in full generality.

Theorem 1. *Let $N \subseteq M$ be an inclusion of type II_1 factors with common identity and finite index, and let $E : M \to N$ be the trace preserving conditional expectation. Then there are no subdiagonal algebras in M with respect to E unless $M = N$.*

Proof. Suppose that A is subdiagonal in M with respect to E and denote $A_0 = A \cap \ker E$. Let τ be the trace on M, Ω the canonical trace vector in $\mathcal{H} = L^2(M, \tau)$, and e_N the Jones projection. We let $M_1 = \langle M, e_N \rangle$ be the Jones extension of M by N. Let f, g be the orthogonal projections of $\mathcal{H}$ onto $\overline{A_0\mathcal{H}}$ and $\overline{A_0^*\mathcal{H}}$, respectively. Then $I = e_N + f + g$ and $e_N, f, g \in N' \cap M_1$ [4]. We define $E_{A_0} : M \to M$ by

$$E_{A_0}(x) = [M : N]E_M(fxe_N),$$

where $E_M : M_1 \to M$ is the trace preserving conditional expectation. We claim that E_{A_0} is an N-N-bimodule projection of M onto A_0. Indeed, if $n \in N$, $a \in A_0$,

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and $x \in M$, then

$$\begin{aligned} fne_N x\Omega &= fnE(x)\Omega = 0, \\ fa^* e_N x\Omega &= f(E(x^*)a)^*\Omega = 0, \\ fae_N x\Omega &= faE(x)\Omega = aE(x)\Omega = ae_N x\Omega, \end{aligned}$$

since A_0 is an N-N-bimodule. Consequently, $E_{A_0}(N) = 0$, $E_{A_0}(A_0^*) = 0$, and

$$E_{A_0}(a) = [M : N]E_M(fae_N) = [M : N]E_M(ae_N) = a$$

for any $a \in A_0$. Hence, the claim follows from [S1] and [S2]. Likewise,

$$E_{A_0^*}(x) = [M : N]E_M(gxe_N)$$

gives an N-N-bimodule projection of M onto A_0^*.

We now consider a tunnel (downward basic construction - cf. [3])

$$\ldots \subseteq M_{-i} \subseteq \ldots \subseteq M_{-1} \subseteq M_0,$$

where $M_0 = M$ and $M_{-1} = N$. We fix i and denote $\tilde{N} = M'_{-i} \cap M_{-1}$, $\tilde{M} = M'_{-i} \cap M_0$, $\tilde{A} = M'_{-i} \cap A$, and $\tilde{E} = E|_{\tilde{M}} : \tilde{M} \to \tilde{N}$. Note that $\tilde{M}$ is finite dimensional [3]. We claim that $\tilde{A}$ is a subdiagonal algebra in $\tilde{M}$ with respect to $\tilde{E}$. Indeed, [S1] and [S3] are obviously satisfied. For any $x \in \tilde{M}$ we have

$$x = [M : N]E_M((e_N + f + g)xe_N) = E(x) + E_{A_0}(x) + E_{A_0^*}(x).$$

Since $E : \tilde{M} \to \tilde{N}$, $E_{A_0} : \tilde{M} \to \tilde{A}$, and $E_{A_0^*} : \tilde{M} \to \tilde{A}^*$, [S2] holds true as well, and the claim is proved.

Let p be a projection in $\tilde{N}$. We denote $\tilde{N}_p = p\tilde{N}p$, $\tilde{M}_p = p\tilde{M}p$, $\tilde{A}_p = p\tilde{A}p$, and $\tilde{E}_p : \tilde{M}_p \to \tilde{N}_p$, the restriction of $\tilde{E}$ to $\tilde{M}_p$. Clearly, $\tilde{A}_p$ is subdiagonal in $\tilde{M}_p$ with respect to $\tilde{E}_p$. Suppose now that p is a minimal projection in $\tilde{N}$. Thus, $\tilde{N}_p = Cp$ and there exists a faithful normalized trace tr on $\tilde{M}_p$ such that $\tilde{E}_p(x) = tr(x)p$. Essentially the same argument as in Theorem 1 of [5] shows that we must have $\tilde{N}_p = \tilde{M}_p$. For the sake of completeness we give that reasoning below.

We denote $(\tilde{A}_p)_0 = \{x \in \tilde{A}_p \mid tr(x) = 0\}$. Let $\tilde{M}_p \cong \bigoplus_{k=1}^r M_{n_k}(C)$ and let λ_k be the weight of tr corresponding to $M_{n_k}(C)$. Thus, $\lambda_k > 0$ for each $k = 1, \ldots, r$ and $\sum_k \lambda_k = 1$. If $x \in (\tilde{A}_p)_0$, then there exist invertible $v \in \tilde{M}_p$ such that $vxv^{-1} = \bigoplus_{k=1}^r y_k$, with each y_k lower triangular in $M_{n_k}(C)$. Let the diagonal entries of y_k be $d_{k,1}, \ldots, d_{k,n_k}$. Since tr is multiplicative on $\tilde{A}_p$, for any natural m we have

$$0 = tr(x^m) = tr((vxv^{-1})^m) = \sum_{k=1}^r \sum_{j=1}^{n_k} \lambda_k d_{k,j}^m.$$

This implies that $d_{k,j} = 0$ for all k, j and, hence, x is nilpotent. By the Corollary on page 13 of [2], for each k there exists a basis of C^{n_k} with respect to which all $\{x_k \mid x = \bigoplus_{t=1}^r x_t,\ x \in (\tilde{A}_p)_0\}$ are strictly upper triangular. Consequently,

$$\sum_{k=1}^r n_k^2 = \dim \tilde{M}_p = 1 + 2\dim(\tilde{A}_p)_0 \leq 1 + 2\sum_{k=1}^r (\frac{n_k^2 - n_k}{2}).$$

Thus, $r = 1$, $n_1 = 1$ and, hence, $\tilde{N}_p = \tilde{M}_p$, as required.

By the above, any minimal projection in $\tilde{N}$ is minimal in $\tilde{M}$ as well. Applied to the case $i = 1$, this shows that M_{-1} is irreducible in M_0. Hence, M_{-2} is irreducible

in M_{-1} [3]. Considering now the case $i = 2$, we see that $M'_{-2} \cap M_0 = CI$ and, hence, $M = N$, as required. $\square$

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