ON SUBDIAGONAL ALGEBRAS FOR SUBFACTORS

WOJCIECH SZYMAŃSKI

(Communicated by David R. Larson)

Abstract. We show that if $N \subseteq M$ are type $II_1$ factors with finite index (and common identity) and $E : M \to N$ is the trace preserving conditional expectation, then there are no subdiagonal algebras in $M$ with respect to $E$ unless $M = N$.

Let $N \subseteq M$ be finite factors (with common identity) and let $E : M \to N$ be the trace preserving conditional expectation. A $\sigma$-weakly closed algebra $A$ is (maximal) subdiagonal in $M$ with respect to $E$ if the following conditions are satisfied [1]:

[S1] $A \cap A^* = N$.
[S2] $A + A^*$ is $\sigma$-weakly dense in $M$.
[S3] $E(xy) = E(x)E(y)$ for any $x, y \in A$.

Saito and Watatani showed in [4] that if $M$ is a crossed product for an outer action of a discrete group $G$ on $N$, then maximal (inclusionwise) subdiagonal algebras are in bijective correspondence with left invariant total orders on $G$. A totally ordered group is torsion free and, thus, infinite. On the basis of this result Saito and Watatani conjectured [4] that if $N$ is a proper subfactor of $M$ with finite index, then there are no subdiagonal algebras in $M$ with respect to $E$. They have verified this in the case when either $M$ is finite dimensional or the index does not exceed 4 [4, 5]. The purpose of this paper is to prove the conjecture in full generality.

Theorem 1. Let $N \subseteq M$ be an inclusion of type $II_1$ factors with common identity and finite index, and let $E : M \to N$ be the trace preserving conditional expectation. Then there are no subdiagonal algebras in $M$ with respect to $E$ unless $M = N$.

Proof. Suppose that $A$ is subdiagonal in $M$ with respect to $E$ and denote $A_0 = A \cap \ker E$. Let $\tau$ be the trace on $M$, $\Omega$ the canonical trace vector in $H = L^2(M, \tau)$, and $e_N$ the Jones projection. We let $M_1 = \langle M, e_N \rangle$ be the Jones extension of $M$ by $N$. Let $f, g$ be the orthogonal projections of $H$ onto $A_0 H$ and $A_0^* H$, respectively. Then $I = e_N + f + g$ and $e_N, f, g \in N' \cap M_1$ [4]. We define $E_{A_0} : M \to M$ by

$$E_{A_0}(x) = [M : N]E_M(f xe_N),$$

where $E_M : M_1 \to M$ is the trace preserving conditional expectation. We claim that $E_{A_0}$ is an $N$-$N$-bimodule projection of $M$ onto $A_0$. Indeed, if $n \in N$, $a \in A_0$,
and \( x \in M \), then

\[
fn e_N x \Omega = fnE(x) \Omega = 0,
\]

\[
f a^* e_N x \Omega = f(E(x^*)a) \Omega = 0,
\]

\[
f a e_N x \Omega = f a E(x) \Omega = a E(x) \Omega = a e_N x \Omega,
\]

since \( A_0 \) is an \( N - N \)-bimodule. Consequently, \( E_{A_0}(N) = 0 \), \( E_{A_0}(A_0^\ast) = 0 \), and

\[
E_{A_0}(a) = [M : N] E_M(f a e_N) = [M : N] E_M(a e_N) = a
\]

for any \( a \in A_0 \). Hence, the claim follows from \([S1]\) and \([S2]\). Likewise,

\[
E_{A_0^\ast}(x) = [M : N] E_M(g x e_N)
\]

gives an \( N - N \)-bimodule projection of \( M \) onto \( A_0^\ast \).

We now consider a tunnel (downward basic construction - cf. \([3]\))

\[
\ldots \subseteq M_{-1} \subseteq \ldots \subseteq M_{-1} \subseteq M_0,
\]

where \( M_0 = M \) and \( M_{-1} = N \). We fix \( i \) and denote \( \tilde{N} = M' \cap M_{-1}, \tilde{M} = M' \cap M_0, \tilde{A} = M' \cap A, \) and \( \tilde{E} = E|_{\tilde{M}} : \tilde{M} \to \tilde{N} \). Note that \( \tilde{M} \) is finite dimensional \([3]\). We claim that \( \tilde{A} \) is a subdiagonal algebra in \( \tilde{M} \) with respect to \( \tilde{E} \). Indeed, \([S1]\) and \([S3]\) are obviously satisfied. For any \( x \in \tilde{M} \) we have

\[
x = [M : N] E_M((e_N + f + g)x e_N) = E(x) + E_{A_0}(x) + E_{A_0^\ast}(x).
\]

Since \( E : \tilde{M} \to \tilde{N}, E_{A_0} : \tilde{M} \to \tilde{A}, \) and \( E_{A_0}^\ast : \tilde{M} \to \tilde{A}^\ast, \) \([S2]\) holds true as well, and the claim is proved.

Let \( p \) be a projection in \( \tilde{N} \). We denote \( \tilde{N}_p = p \tilde{N}_p, \tilde{M}_p = p \tilde{M}_p, \tilde{A}_p = p \tilde{A}_p, \) and \( \tilde{E}_p : \tilde{M}_p \to \tilde{N}_p \), the restriction of \( \tilde{E} \) to \( \tilde{M}_p \). Clearly, \( \tilde{A}_p \) is subdiagonal in \( \tilde{M}_p \) with respect to \( \tilde{E}_p \). Suppose now that \( p \) is a minimal projection in \( N \). Thus, \( \tilde{N}_p = C_p \) and there exists a faithful normalized trace \( tr \) on \( \tilde{M}_p \) such that \( \tilde{E}_p(x) = tr(x)p \). Essentially the same argument as in Theorem 1 of \([5]\) shows that we must have \( \tilde{N}_p = \tilde{M}_p \). For the sake of completeness we give that reasoning below.

We denote \( (\tilde{A}_p)_0 = \{ x \in \tilde{A}_p \mid tr(x) = 0 \} \). Let \( \tilde{M}_p \cong \bigoplus_{k=1}^r M_{n_k}(C) \) and let \( \lambda_k \) be the weight of \( tr \) corresponding to \( M_{n_k}(C) \). Thus, \( \lambda_k > 0 \) for each \( k = 1, \ldots, r \) and \( \sum_k \lambda_k = 1 \). If \( x \in (\tilde{A}_p)_0 \), then there exist invertible \( v \in \tilde{M}_p \) such that \( vxv^{-1} = \bigoplus_{k=1}^r y_k \), with each \( y_k \) lower triangular in \( M_{n_k}(C) \). Let the diagonal entries of \( y_k \) be \( d_{k,1}, \ldots, d_{k,n_k} \). Since \( tr \) is multiplicative on \( \tilde{A}_p \), for any natural \( m \) we have

\[
0 = tr(x^m) = tr((vxv^{-1})^m) = \sum_{k=1}^r \sum_{j=1}^{n_k} \lambda_k d_{k,j}^m.
\]

This implies that \( d_{k,j} = 0 \) for all \( k, j \) and, hence, \( x \) is nilpotent. By the Corollary on page 13 of \([2]\), for each \( k \) there exists a basis of \( C_{n_k} \) with respect to which all \( \{ x_k \mid x = \bigoplus_{t=1}^r x_t, x \in (\tilde{A}_p)_0 \} \) are strictly upper triangular. Consequently,

\[
\sum_{k=1}^r n_k^2 = \dim \tilde{M}_p = 1 + 2 \dim(\tilde{A}_p)_0 \leq 1 + 2 \sum_{k=1}^r \left( \frac{n_k^2 - n_k}{2} \right).
\]

Thus, \( r = 1, n_1 = 1 \) and, hence, \( \tilde{N}_p = \tilde{M}_p \), as required.

By the above, any minimal projection in \( \tilde{N} \) is minimal in \( \tilde{M} \) as well. Applied to the case \( i = 1 \), this shows that \( M_{-1} \) is irreducible in \( M_0 \). Hence, \( M_{-2} \) is irreducible.
in $M_{-1}$ [3]. Considering now the case $i = 2$, we see that $M'_2 \cap M_0 = CI$ and, hence, $M = N$, as required.

\begin{acknowledgement}
I would like to thank Professor Paul Muhly for a stimulating conversation on subdiagonal algebras.
\end{acknowledgement}

\begin{references}
\end{references}

Department of Mathematics, The University of Newcastle, Newcastle, New South Wales 2308, Australia

E-mail address: wojciech@frey.newcastle.edu.au