PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 3, Pages 873–876
S 0002-9939(99)05267-3
Article electronically published on July 6, 1999

LIPSCHITZ CONTINUITY OF OBLIQUE PROJECTIONS

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(Communicated by David R. Larson)

ABSTRACT. Let W and L be complementary spaces of a finite dimensional unitary space V and let $P(W,L)$ denote the projection of V on W parallel to L. Estimates for the norm of $P(W,L) - P(W,M)$ are derived which involve the norm of the restriction of $P(W,L)$ to M or the gap between L and M.

1. INTRODUCTION AND PRELIMINARIES

Let $V = W \oplus L$ be a nontrivial direct sum decomposition of an n-dimensional unitary space V and let $P(W,L)$ denote the oblique projection on W along L. If the distance between subspaces is measured in the gap metric, then all subspaces M contained in a sufficiently small neighbourhood $U(L)$ of L are also complementary to W (see e.g. [1, p. 390] or [5]). For $M \in U(L)$ set $\pi(M) = P(W,M)$. In this note we study the map $\pi(M)$. An estimate for $\|\pi(M) - \pi(L)\|$ will be obtained which involves the restriction of $P(W,L)$ to M. A Lipschitz constant for π in [1] will be improved.

Notation. For a linear map $A\colon Y \to V$ the norm $\|A\|$ denotes the operator norm, i.e. $\|A\| = \sup\{\|Ay\|, y \in Y, \|y\| = 1\}$. Let P_W denote the orthogonal projection of V on W and set

$$P(W,L;M) = P(W,L)|_M.$$

We write $d(x,M)$ for the distance of $x \in V$ from M. The gap between two subspaces L and M is defined by

$$\theta(L,M) = \|P_L - P_M\|.$$

We shall need the following facts on the gap, for which we refer to [2] and [1]. First of all θ is a metric on the set of subspaces of V, and $\theta(L,M) \le 1$. If $\theta(L,M) < 1$, then

$$\dim M = \dim L. \tag{1.1}$$

In the case of (1.1) we have

$$\theta(L,M) = \|P_L(I - P_M)\| = \|P_M(I - P_L)\|. \tag{1.2}$$

Received by the editors August 10, 1997 and, in revised form, April 28, 1998.
1991 *Mathematics Subject Classification.* Primary 51M05, 51M16, 15A45.
Key words and phrases. Oblique projections, direct complements.

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Lemma 1.1. *Assume* $V = W \oplus L$.

(i) *For a subspace* M *of* V *we have*

$$\|P(W,L;M)\| = \|P(W,L)P_M\|. \tag{1.3}$$

(ii) *If* $\dim M = \dim L$, *then*

$$\|P(W,L;M)\| \le \|P(W,L)\|\theta(L,M). \tag{1.4}$$

Proof. (i) For $y \in M$, $M \neq 0$, we have $P(W,L;M)y = P(W,L)P_M y$. Therefore

$$\begin{aligned}\|P(W,L;M)\| &= \max\{\frac{\|P(W,L)P_M y\|}{\|y\|},\, y \neq 0,\, y \in M\} \le \|P(W,L)P_M\| \\ &= \max\{\frac{\|P(W,L)P_M y\|}{\|y\|},\, y \neq 0,\, y \in V\} \\ &\le \max\{\frac{\|P(W,L)P_M y\|}{\|P_M y\|},\, y \in V,\, P_M y \neq 0\} = \|P(W,L;M)\|.\end{aligned}$$

(ii) From $P(W,L) = P(W,L)(I - P_L)$ and (1.3) follows

$$\|P(W,L;M)\| = \|P(W,L)(I-P_L)P_M\|.$$

Hence (1.2) yields (1.4). □

The following observations do not seem to be widely known.

Lemma 1.2. *Assume* $V = W \oplus L$.

(i) *If* $W \neq 0$, *then*

$$\max_{x \in W,\, \|x\|=1} \frac{1}{d(x,L)} = \|P(W,L)\|. \tag{1.5}$$

(ii) *If* $W \neq 0$ *and* $L \neq 0$, *then*

$$\|P(W,L)\| = \|P(L,W)\|. \tag{1.6}$$

Proof. (i) We shall see that (1.5) is equivalent to the identity

$$\frac{1}{1 - \|P_L P_W\|^2} = \|P(W,L)\|^2, \tag{1.7}$$

which is due to Ljance [3] (see [4] or [6]). Set

$$\tau = \min_{x \in W, \|x\|=1} d(x,L) = \min_{x \in W, \|x\|=1} \|(I - P_L)P_W x\|.$$

Then the left-hand side of (1.5) is equal to $1/\tau$. If $x \in W$ and $\|x\| = 1$, then

$$\|(I - P_L)P_W x\|^2 + \|P_L P_W x\|^2 = 1.$$

Hence

$$\tau^2 = 1 - \max_{x \in W, \|x\|=1} \|P_L P_W x\|^2 = 1 - \|P_L P_W\|^2.$$

Therefore

$$\frac{1}{\tau^2} = \frac{1}{1 - \|P_L P_W\|^2},$$

and (1.5) follows from (1.7).

(ii) Since $P_L P_W = 0$ implies $P_L P_W P_L = 0$ and thus $P_W P_L = 0$, we note that either $P_L P_W = P_W P_L = 0$ or both $P_L P_W \neq 0$ and $P_W P_L \neq 0$. In each case we have $\|P_L P_W\| = \|P_W P_L\|$. Hence (1.7) implies (1.6). □

2. Estimates for oblique projections

Theorem 2.1. *Assume* $V = W \oplus L, W \neq 0, L \neq 0$.

(i) *Let* M *be a subspace of* V *with* $\dim M = \dim L$ *and*

$$\mu = \|P(W, L; M)\| < 1. \tag{2.1}$$

Then

$$V = W \oplus M \tag{2.2}$$

and

$$\|P(W, M) - P(W, L)\| \leq \frac{\mu}{1-\mu} \|P(W, L)\|. \tag{2.3}$$

(ii) *If a subspace* M *satisfies*

$$\theta(L, M) \leq (1 - c)\|P(W, L)\|^{-1}, \quad 0 < c < 1, \tag{2.4}$$

then we have (2.2) *and*

$$\|P(W, M) - P(W, L)\| \leq \frac{1}{c} \|P(W, L)\|^2 \theta(L, M). \tag{2.5}$$

Proof. (i) Suppose $x \neq 0$ for some $x \in W \cap M$. Then $P(W, L; M)x = x$. Hence $\|P(W, L; M)\| \geq 1$, which contradicts (2.1). Therefore we have $W \cap M = 0$ and (2.2). Now put $S = P(M, W)P_L$. Then

$$P(M, W)[I - P_L P(L, W)] = P(M, W)P(W, L) = 0$$

implies $P(M, W) = SP(L, W)$. Using $P(W, L) = I - P(L, W)$ and (1.6) we obtain

$$\begin{aligned} \|P(W, M) - P(W, L)\| &= \|P(M, W) - P(L, W)\| \\ &= \|SP(L, W) - P_L P(L, W)\| \leq \|S - P_L\| \, \|P(W, L)\|. \end{aligned}$$

Thus our target inequality is

$$\|S - P_L\| \leq \frac{\mu}{1-\mu}. \tag{2.6}$$

Since $P(L, W)P(W, M)x = 0$ for all $x \in V$, we have $P(L, W)[I - P(M, W)] = 0$ or $P(L, W)P(M, W) = P(L, W)$. Similarly $P(L, W)P_L = P_L$. Hence

$$P(L, W)P(M, W)P_L = P(L, W)S = P_L,$$

and we obtain $S - P_L = [I - P(L, W)]S = P(W, L)P_M S$. Then

$$\begin{aligned} (S - P_L)^*(S - P_L) &= S^*S - P_L S - S^*P_L + P_L \\ &= S^*P_M P(W, L)^* P(W, L)P_M S, \end{aligned} \tag{2.7}$$

which implies

$$P_L S + S^*P_L = S^*[I - P_M P(W, L)^* P(W, L)P_M]S + P_L. \tag{2.8}$$

For the left-hand side of (2.8) we obtain

$$\|P_L S + S^*P_L\| \leq 2\|P_L S\| \leq 2\|P_L\| \, \|S\| \leq 2\|S\|.$$

Put $T = I - P_M P(W, L)^* P(W, L)P_M$ such that the right-hand side of (2.8) equals $R = S^*TS + P_L$. Since P_L is the identity map on L and $SP_L = S$, it is not difficult to show that $\|R\| = \|S^*TS\| + 1$. Now (2.1) implies that T is positive definite and that $1 - \|P(W, L)P_M\| = 1 - \mu^2 > 0$ is the smallest eigenvalue of T. Hence $\|S^*TS\| \geq \|S\|^2(1 - \mu^2)$. Thus $\|S\|$ satisfies

$$0 \leq \|S\|^2(1 - \mu^2) + 1 \leq 2\|S\|,$$

which is equivalent to

$$0 < \frac{1}{1+\mu} \leq \|S\| \leq \frac{1}{1-\mu}. \tag{2.9}$$

Then (2.7) yields

$$\|S - P_L\| \leq \|S\| \, \|P(W,L)P_M\| \leq \frac{\mu}{1-\mu},$$

and we have (2.6), which completes the proof of (2.3).

(ii) From (2.4) and (1.4) we obtain

$$\mu \leq \|P(W,L)\|\theta(L,M) \leq 1 - c < 1. \tag{2.10}$$

Then $\|P(W,L)\| \geq 1$ implies $\theta(L,M) < 1$ and $\dim M = \dim L$. Because of $\mu < 1$ we can use (i) and conclude that $P(W,M)$ exists. Since $0 \leq \mu \leq 1 - c$ is equivalent to

$$0 \leq \frac{1}{1-\mu} \leq \frac{1}{c}\,,$$

the estimate (2.5) follows immediately from (2.3). □

In the neighbourhood of L given by (2.4) the estimate (2.5) yields a Lipschitz constant for $P(W,M)$ of the form

$$\frac{1}{c}\|P(W,L)\|^2. \tag{2.11}$$

In [1, p. 390] we find for sufficiently small $\theta(L,M)$ an estimate

$$\|P(W,M) - P(W,L)\| \leq K\theta(L,M)$$

with

$$K = 2\|P(W,L)\| \max_{x \in W, \|x\|=1} \frac{1}{d(x,L)}. \tag{2.12}$$

According to Lemma 1.2 the Lipschitz constant K in (2.12) is equal to (2.11) with $c = 1/2$.

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