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A LOCAL VERSION OF WONG-ROSAY'S THEOREM FOR PROPER HOLOMORPHIC MAPPINGS

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Abstract. In the present paper, we generalize Wong-Rosay's theorem for proper holomorphic mappings with bounded multiplicity. As an application, we prove the non-existence of a proper holomorphic mapping from a bounded, homogenous domain in $\mathbb{C}^n$ onto a domain in $\mathbb{C}^n$ whose boundary contains strongly pseudoconvex points.

1. Introduction and results

The purpose of this paper is to prove a version of Wong-Rosay's theorem [15],[10] for families of proper holomorphic mappings with bounded multiplicity. Our main result can be stated as follows :

Theorem 1. *Let $D \subset\subset \mathbb{C}^n$ and $G \subset \mathbb{C}^n$ be domains. Suppose there exist a point $p \in D$ and a sequence $\{f_k\}_k$ of proper holomorphic mappings $f_k : D \to G$ of multiplicity equal to m such that $\{f_k(p)\}_k$ converges to a strongly pseudoconvex boundary point $q \in \partial G$. Then there exists a proper holomorphic mapping defined from D onto the unit ball in $\mathbb{C}^n$ of multiplicity less than or equal to m.*

This theorem implies that domain D is necessarily pseudoconvex and furthermore, if G is a strongly pseudoconvex, bounded, simply connected domain with C^∞-boundary, then according to [2] G is biholomorphic to the unit ball in $\mathbb{C}^n$.

The assumption about a uniform bound on the multiplicities on the mappings is necessary for our proof, but it is rather natural in view of a result of Bedford [1] which states that there is an absolute bound on the multiplicity of a proper holomorphic mapping between bounded pseudoconvex domains in $\mathbb{C}^n$ with real analytic boundaries.

By using Theorem 1, we give a generalization of a result of Lin and Wong [7] for unbounded domains in $\mathbb{C}^n$.

Corollary 1. *Let $D \subset\subset \mathbb{C}^n$ and $G \subset \mathbb{C}^n$ be domains. Suppose there exist a point $p \in D$ and a sequence $\{f_k\}_k$ of unbranching proper holomorphic mappings $f_k : D \to G$ such that $\{f_k(p)\}_k$ converges to a strongly pseudoconvex boundary point $q \in \partial G$. Then both D and G are biholomorphic to the unit ball in $\mathbb{C}^n$.*

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The following example proves that Corollary 1 cannot be extended to sequences of branched proper holomorphic mappings.

Let $D = \{(z,w) \in \mathbb{C}^2 : \ |z|^4 + |w|^2 < 1\}$, $\mathbb{B} = \{(z,w) \in \mathbb{C}^2 : \ |z|^2 + |w|^2 < 1\}$ be domains in $\mathbb{C}^2$ and let us consider the proper holomorphic

$$\begin{array}{llll} f: & D & \to \mathbb{B}, & \\ & (z,w) & \mapsto & (z^2,w). \end{array}$$

Let $q \in \partial\mathbb{B}$ be a boundary point and $(q^k)_k$ be a sequence in $\mathbb{B}$, which converges to q. Since $\mathbb{B}$ is homogeneous, there exists a sequence $(\varphi_k)_k \subset \mathbb{B})$ of automorphisms such that $q^k = \varphi_k(0)$. Let $f_k = \varphi_k \circ f$. Then $\{f_k\}_k$ is a sequence of proper holomorphic mappings with bounded multiplicity and $\{f_k(0)\}_k$ converges to q which is a strongly pseudoconvex boundary point, but the domain D is not biholomorphic to the unit ball in $\mathbb{C}^2$.

For strongly pseudoconvex domains in $\mathbb{C}^n$, we have the following result.

Corollary 2. *Let $D \subset\subset \mathbb{C}^n$ and $G \subset \mathbb{C}^n$ be strongly pseudoconvex domains. Suppose there exist a point $p \in D$ and a sequence $\{f_k\}_k$ of proper holomorphic mappings $f_k : D \to G$ such that $\{f_k(p)\}_k$ converges to a strongly pseudoconvex boundary point $q \in \partial G$. Then both D and G are biholomorphic to the unit ball in $\mathbb{C}^n$.*

In the case where $D = G$, we obtain a local version of Wong-Rosay's theorem for proper holomorphic mappings as follows:

Theorem 2. *Let D be a bounded domain in $\mathbb{C}^n$. Suppose there exist a point $p \in D$ and a sequence $\{f_k\}_k$ of proper holomorphic mappings $f_k : D \to D$ of bounded multiplicity such that $\{f_k(p)\}_k$ converges to a strongly pseudoconvex boundary point $q \in \partial D$. Then D is biholomorphic to the unit ball in $\mathbb{C}^n$.*

As another application of Theorem 1, we establish the following result concerning bounded homogenous domains in $\mathbb{C}^n$.

Corollary 3. *Let D be a bounded homogenous domains in $\mathbb{C}^n$ and G be a domain in $\mathbb{C}^n$ whose boundary contains strongly pseudoconvex points. If there exists a proper holomorphic mapping from D onto G, then D is biholomorphic to the unit ball in $\mathbb{C}^n$.*

2. Notations and preliminary results

For the proof of Theorem 1, we need to introduce the notion of proper holomorphic correspondences. Let D and G be two domains in $\mathbb{C}^n$ and let Γ be a complex purely n-dimensional subvariety contained in $D \times G$. We denote by $\pi_1 : \Gamma \to D$ and $\pi_2 : \Gamma \to G$ the natural projections. When π_1 is proper, then $(\pi_2 \circ {\pi_1}^{-1})(z)$ is a non-empty finite subset of G for any $z \in D$ and one may therefore consider the set-valued mapping $f = \pi_2 \circ {\pi_1}^{-1}$. Such a map is called a holomorphic correspondence between D and G; Γ is said to be the graph of f and it will be denoted by graph$f$. Since π_1 is proper, there exist a complex subvariety $V \subset$ graph$f$ and an integer m such that $f(z) = \{f^1(z), \dots, f^m(z)\}$ for all $z \in D \backslash \pi_1(V)$ and the f^j's are distinct holomorphic functions in a neighborhood of $z \in D \backslash \pi_1(V)$ (see for instance [5]). The integer m is called the multiplicity of f. The correspondence f is proper if π_2 is proper and it is irreducible if its graph is irreducible. Furthermore, for bounded domains f is proper if and only if ∂graph$f \subset \partial D \times \partial G$. Correspondences were introduced by Stein [12] in order to generalize meromorphic mappings between complex spaces. Properties of correspondences can be found in Stein's papers [12, 13]. For example, it can be shown that f gives rise to a holomorphic mapping $\hat{f} : D \to G^m_{\rm sym}$ into the m-fold symmetric product of G ([3]).

Now let z_o be a point in D and $\{z^1, z^2, ..., z^m\}$ be a set in G. We say that $f(z) = \{f^1(z), ..., f^m(z)\}$ converges to $\{z^1, z^2, ..., z^m\}$ when z tends to z_o if after a possible renumeration of f^j, one has $\lim_{z \to z_o} f^j(z) = z^j$. Equivalently $f(z)$ tends to $\{z^1, z^2, ..., z^m\}$ in the sense of Hausdorff convergence of sets.

We denote by $Cor(D, G, m)$ the set of all ν-valued holomorphic mappings from D onto G for $\nu = 1, \dots, m$. Let $f \in Cor(D, G, m)$ be irreducible, $a \in \mathrm{A} \subset D$, $b \in f(a)$; then we define $\hat{f}^{(a,b)}_{\mathrm{A}} \in Cor(\mathring{\mathrm{A}}, G, m)$ to be the correspondence obtained by analytic continuation of the germ of f at (a, b) by paths which lie in A. Equivalently, graph$\hat{f}^{(a,b)}_{\mathrm{A}}$ is the union of those irreducible components of graph$f \cap \{\mathrm{A} \times G\}$, which contain (a, b).

Let $\{f_k\} \subset Cor(D, G, m)$. We say that $\{f_k\}$ is compactly divergent if $\forall K_1 \subset\subset D$, $K_2 \subset\subset G$, $\exists j_o$ $\forall j \geq j_o$:

$$f_k(K_1) \cap K_2 = \emptyset.$$

If the f_k are irreducible, we say that f_k converge to $f \in Cor(D, G, m)$ if $\exists (a, b_k) \in$ graphf_k with $b_k \to b \in G$ and for all $K \subset\subset D$ with $a \in K$:

$$\hat{f}^{(a,b_k)}_{k,K} \to f_K \text{ for some } f_K \in Cor(\mathring{K}, G, m)$$

and

$$\bigcup_{K \subset\subset D} \mathrm{graph} f_K = \mathrm{graph} f.$$

If D and G are bounded domains in $\mathbb{C}^n$, the set of proper holomorphic correspondence $Cor(D, G, m)$ is normal for any $m \in \mathbb{N}$ ([6]), i.e. every sequence $\{f_k\}$ of proper holomorphic correspondence in $Cor(D, G, m)$ is either compactly divergent or has a convergent subsequence.

3. Proofs of results

Proof of Theorem 1. Since q is strongly pseudoconvex boundary point, according to [4] the sequence $\{f_k\}_k$ converges to q uniformly on compact subsets of D. We use scaling methods introduced by S.Pinchuk [8]. Let U be a neighborhood of q in $\mathbb{C}^n$ which does not intersect the set of weakly pseudoconvex points of ∂G. For all $\xi \in \partial G \cap U$, we consider the change of variables α^ξ defined by:

$$\begin{cases} z^*_j &= \dfrac{\partial \rho}{\partial \bar{z}_n}(\xi)(z_j - \xi_j) - \dfrac{\partial \rho}{\partial \bar{z}_j}(\xi)(z_n - \xi_n), \quad 1 \leq j \leq n-1, \\ z^*_n &= \sum_{1 \leq j \leq n} \dfrac{\partial \rho}{\partial z_j}(\xi)(z_j - \xi_j) \end{cases}$$

where ρ is a defining function of G. The mapping α^ξ maps ξ to 0 and the real normal at 0 to ∂G to the line $\{'z = 0,\ y_n = 0\}$.

Let $K \subset\subset D$ be a compact. There exists an integer k_0 such that, for all $k \geq k_0$ and $z \in K$, the point $f_k(z) \in U \cap G$. We denote by w^k the projection of q^k on $\partial G \cap U$ and $\alpha^k = \alpha^{w^k}$ the mapping as above. We have $\alpha^k(q^k) = ('0, -\delta_k)$ with $q^k = f_k(p)$ and $\delta_k = dist(\alpha^k(q^k), \partial\alpha^k(G^k))$. We define now the inhomogenous dilatations φ^k by $\varphi^k('z, z_n) = (\delta_k^{\frac{1}{2}}\ 'z, \delta_k z_n)$ and let $G^k = \varphi^k \circ \alpha^k(G)$. For all k, the mapping $g_k = \varphi^k \circ \alpha^k \circ f_k : D \to G^k$ is a proper holomorphic mapping with mutiplicity m, which satisfies $g_k(p) = s = ('0, -1)$. The sequence $\{g_k\}_k$ is a normal family, passing to subsequence, $\{g_k\}_k$ converges uniformly on the compact subsets of D to a holomorphic mapping $g : D \to \Sigma$, where

$$\Sigma = \{('z, z_n) \in \mathbb{C}^n : \ 2Re(z_n) + |'z|^2 < 0\}.$$

To finish the proof we shall prove that the mapping g is proper. We will need to study the convergence of the correspondence $h_k = g_k^{-1}$. For this, we will use a similar method introduced by W.Klingenberg and S.Pinchuk in [6] to study the problem of normality of proper holomorphic correspondences between bounded domains in $\mathbb{C}^n$.

The correspondence $h_k : G^k \to D$ is a proper, holomorphic irreducible one which satisfies $(s, p) \in \text{graph}(h_k)$ for all k. Given $K \subset \Sigma$, a compact, $s \in K$, we have $\hat{h}_{k,K}^{(s,p)} \in Cor(\overset{\circ}{K}, D, m)$. Since D is bounded, there is a subsequence which converges to an element $h \in Cor(\overset{\circ}{K}, \overline{D}, m)$. Since Σ is biholomorphic to the unit ball $\mathbb{B}$, then by exhausting $\mathbb{B}$ with compact and passing to diagonal subsequence, we obtain $h \in Cor(\Sigma, \overline{D}, m)$. The following fact was proved in [6]. For completeness, we include a proof.

Claim. $h \in Cor(\Sigma, D, m)$.

Proof. The branches $\{h^1, \dots, h^m\}$ of h are locally defined and holomorphic on $D \backslash \pi_1(V)$. Now the jacobians of h^i induce in a natural manner a holomorphic function $\text{Jac}(h)$ on $\text{graph}(h) \backslash V$ as follows: let $z \in \text{graph}(h) \backslash V$; then there exists only one $i \in \{1, \dots, m\}$ such that $z \in \text{graph}(h^i)$. We define $\text{Jac}(h)(z) = \text{Jac}(h^i)(\pi_1 z)$. First we show that $\text{Jac}(h) \not\equiv 0$. We need the following lemma.

We will write $\hat{h}_A^{(a,b)} = \hat{h}_A^{(a,b)}(A)$.

Lemma 1 ([6])**.** *Let D and G be bounded domains in $\mathbb{C}^n$ and $(a, b) \in D \times G$. Then for all $U(b) \subset G$ there exists $U(a) \subset D$, such that for all $h \in Cor(D, G, m)$ with $b \in h(a)$ we have: $\hat{h}_{U(a)}^{(a,b)} \subset U(b)$.*

Let $U(p) \subset\subset D$ be a neighborhood of $p \in D$. By Lemma 1, there exists $U(s)$ a neighborhood of s in Σ with $\hat{h}_{k,U(s)}^{(s,p)} \subset U(p)$ for all k. Then we have $z = g_k \circ \hat{h}_{k,U(s)}^{(s,p)}(z)$ for all $z \in U(s)$. Passing to a convergent subsequence, we have $z = g_k \circ \hat{h}_{U(s)}^{(s,p)}(z)$ for all $z \in U(s)$, which implies that $\text{Jac}(\hat{h}_{U(s)}) \not\equiv 0$. Since $\text{graph}(h) \backslash V$ is connected, we conclude that $\text{Jac}(h) \not\equiv 0$. Let $W \subset \text{graph}(h) \backslash V$ denote the variety $\{\text{Jac}(h) = 0\}$. Now assume that the claim is false, i.e. there exist $(x, y) \in \Sigma \times \partial D$ with $y \in h(x)$. Since the branches of h are locally open maps on $D \backslash \pi_1\{V \cup W\}$, we must have $x \in \pi_1\{V \cup W\}$. The variety $\{V \cup W\}$ is a subvariety of D of dimension $n-1$; then there exists a holomorphic disc $\tilde{\Delta}$ in D such that $\tilde{\Delta} \cap \pi_1\{V \cup W\} = x$. Since $h(\tilde{\Delta}) \subset G \cup \{y\}$ is a disc , by the theorem of Cartan-Thullen (see [14]), the maps g_k and g extend analytically to a fixed neighborhood of y, say $U(y)$. The domain Σ is biholomorphic to the unit ball which is a bounded domain; then there exists a subsequence of g_k which converges to g on the compact subsets of $D \cup U(y)$. It follows from the assumption that there exists $y_k \in \hat{h}_{k,K}(x)$ with $y_k \to y$. But since h_k is the inverse of g_k, this implies $x = g_k(y_k)$, and we may pass to the limit, which gives $x = g(y)$. Since g_k is proper, $g_k(y) \in \partial G_k$ and then by passing to a convergent subsequence, the limit implies that $g(y) \in \partial\Sigma$. This contradicts $x \in \Sigma$. $\square$

We continue now with the proof of Theorem 1. Let $z \in D$ and $U(g(z))$ be a neighborhood of $g(z)$ in Σ. Lemma 1 implies that there exists $U(z)$ a neighborhood of z in D such that for large k's we have $g_k(U(z)) \subset U(g(z))$. One has $z \in \hat{h}_{k,U(g(z))}^{(g_k(z),z)} \circ g_k(z)$ for all $z \in U(z)$. Passing to a convergent subsequence and to limit, we get

$$(*) \qquad z \in h \circ g(z),\ \forall\ z \in D.$$

Suppose that there exists a sequence $\{z_j\} \subset D$, which converges to $z \in \partial D$ and $g(z_j)$ converges to $z' \in \Sigma$. According to $(*)$, we have $z_j \in h \circ g(z_j)$ for all j. The limit implies that $z \in h(z')$, which contradicts $z \in \partial D$ and then g is proper. This finishes the proof of Theorem 1. $\square$

Proof of Corollary 1. First we show that D is simply connected. According to [4], the sequence $\{f_k(p)\}_k$ converges uniformly on compact subsets of D to q. Suppose that D is not simply connected; then there exists a nontrivial closed loop γ in $\pi_1(D)$. The boundary of G is smooth near q; then there exists a neighborhood U of q such that $G \cap U$ is simply connected. For large k's, $f_k(\gamma)$ is a closed loop $\beta \subset G \cap U$. Nevertheless, $f_k : D \to G$ is a covering, and $(f_k)_* : \pi_1(D) \to \pi_1(G)$ is one to one. This is a contradiction to the fact that $f_k(\gamma)$ must be a nontrivial element in $(f_k)_*(\pi_1(D)) \subset \pi_1(G)$.

The mappings f_k are a covering and D is simply connected. Then the order of $\pi_1(G)$ is equal to the multiplicity of f_k for all k and then the multiplicity of f_k is bounded. According to Theorem 1, there exists a proper holomorphic mapping $f : D \to \mathbb{B}$. Hurwitz's theorem implies that f is a covering. Since $\mathbb{B}$ is simply connected, f is biholomorphic and then D is biholomorphic to the unit ball.

For any k, the map $h = f^k \circ \mathbb{B} \to G$ is a holomorphic covering. The ball $\mathbb{B}$ is simply connected, and h is factored by automorphisms, i.e. there exists a subgroup Γ of automorphism groups of $\mathbb{B}$ such that for all $z \in \mathbb{B}$, $h^{-1}(h(z)) = \{\gamma(z), \gamma \in \Gamma\}$. According to [11], $\{\gamma(z) = z\}$ is non-empty. Since $\{\gamma(z) = z\} \subset V_h$ (V_h is the branch locus of h) for all $\gamma \in \Gamma \backslash \{\mathrm{I}_{\mathbb{B}}\}$ and h is a covering, the group Γ is reduced to $\{\mathrm{I}_{\mathbb{B}}\}$ and then h is biholomorphic. $\square$

Proof of Corollary 2. The domains D and G are strongly pseudoconvex, according to [8], and f_k is a covering. The proof can be completed by using Corollary 1. $\square$

Proof of Theorem 2. Theorem 1 implies that there exists a proper holomorphic mapping $f : D \to \mathbb{B}$. The correspondence $f \circ f_k \circ f^{-1}$ is an irreducible self-proper one, according to [2]; $f \circ f_k \circ f^{-1}$ is an automorphism of the unit ball. There exists then $\phi \in Aut(\mathbb{B})$ such that $f \circ f_k = \phi \circ f$. From this, we conclude that the mapping f_k is one to one. Otherwise the multiplicity of the mapping $f \circ f_k$ is greater than the multiplicity of the mapping $\phi \circ f$, but $f \circ f_k = \phi \circ f$. Then f_k is biholomorphic for all k. Now Corollary 1 can be applied to finish the proof. $\square$

Proof of Corollary 3. Let q be a strongly pseudoconvex boundary point of G and $\{f(p_k)\}_k$ be a sequence in G which converges to q, where $(p_k)_k$ is a sequence in D. Since D is homogenous, there exists a sequence of automorphisms $\{g_k\}_k \subset Aut(D)$ such that $g_k(0) = p_k$. The sequence $\{f \circ g_k(0)\}$ converges to q. Theorem 1 implies that there exists a proper holomorphic mapping from D onto the unit ball in $\mathbb{C}^n$. According to [9], D is biholomorphic to the unit ball. $\square$

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