

## YANG INDEX OF THE DELETED PRODUCT

SIMEON T. STEFANOV

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ABSTRACT. For any  $\kappa \geq 1$  a  $\kappa$ -dimensional polyhedron  $Y_\kappa$  is constructed such that the Yang index of its deleted product  $Y_\kappa^*$  equals  $2\kappa$ . This answers a question of Izydorek and Jaworowski (1995).

For any  $\kappa \geq 1$  a  $2\kappa$ -dimensional closed manifold  $M$  with involution is constructed such that  $\text{index } M = 2\kappa$ , but  $M$  can be mapped into a  $\kappa$ -dimensional polyhedron without antipodal coincidence.

The deleted product of  $Y$  is the space

$$Y^* = Y^2 \setminus \Delta,$$

where  $\Delta$  is the diagonal of  $Y^2$ . There is a natural free involution  $T(x, y) = (y, x)$  acting in  $Y^*$ .

Our goal is to compute the Yang index of the deleted product of some polyhedra (with respect to the involution  $T$ ). In particular, we answer the question in [3] of whether there exists a  $\kappa$ -dimensional polyhedron  $Y_\kappa$  with  $\text{index } Y_\kappa^* = 2\kappa$ . It is shown that the space  $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$  has  $\text{index } Y_\kappa^* = 2\kappa$ .

In fact, we shall find in  $Y_\kappa^*$  a closed manifold  $M_{2\kappa}$  with  $\text{index } M_{2\kappa} = 2\kappa$ . Then the projection  $p(x, y) = x$  is a map  $p : M_{2\kappa} \rightarrow Y_\kappa$  without antipodal coincidence. Other examples of such manifolds (or even polyhedra) are not known to us. Let us note the theorem of Šchepin [4], which asserts that every map  $f : S^{2\kappa} \rightarrow P_\kappa$  of the  $2\kappa$ -sphere into a  $\kappa$ -dimensional polyhedron has an antipodal coincidence.

First some notation.

$\Delta^n$  is a standard  $n$ -simplex in  $\mathbb{R}^n$  with center in the origin  $O$ .

Let  $P$  be a simplicial complex.

For  $x \in P$ ,  $[x]$  denotes the carrier of  $x$ , i.e. the (closed) simplex containing  $x$  in its interior.

If  $a \in P$  is a vertex, its star  $\text{St}(a)$  is the union of all open simplexes with vertex  $a$ .

$[P]^\kappa$  denotes the  $\kappa$ -dimensional skeleton of  $P$ .

All maps are assumed to be continuous.

Now we shall list some properties of the Yang index that we shall make use of, and refer the reader to [7] for the definition and the whole index theory.

Let  $X$  be a compact metric space with a free involution  $T : X \rightarrow X$ . Then its Yang index is defined inductively by means of the equivariant homology groups with coefficients in  $\mathbb{Z}_2$ . We denote it here by  $\text{index } X$ . An important property of

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the index is that if  $\text{index } X \geq n$ , then every map  $f : X \rightarrow \mathbb{R}^n$  has an antipodal coincidence:  $f(Tx) = f(x)$ .

Note also that  $\text{index } X \leq \dim X$ .

The following useful proposition estimates the index of a manifold.

**Proposition.** *Let  $M_n$  be an  $n$ -dimensional closed manifold with a free involution  $T : M_n \rightarrow M_n$ . Suppose that there exists an odd map  $\varphi : M_n \rightarrow S^n$  (i.e.  $\varphi(Tx) = -\varphi(x)$ ) with  $\deg_2 \varphi = 1$ , where  $\deg_2$  is the degree mod 2. Then  $\text{index } M_n = n$ .*

*Proof.* Let  $z^n$  be the invariant fundamental cycle mod 2 in  $M_n$ . Then  $\varphi_*([z^n]) \neq 0$  (in the Čech homologies mod 2). We have  $\nu([z^n]) = \nu(\varphi_*[z^n]) \neq 0$ , as follows from the properties of the index homomorphism  $\nu$  (cf. [7] for the definition of  $\nu$ ). But this means that  $\text{index } M_n \geq n$  by definition. The converse inequality follows from the fact that  $\text{index} \leq \dim$ .

### I. THE MAIN THEOREMS

Let  $M$  be a finite set in  $\mathbb{R}^n$  and let  $\sigma_1^\kappa, \sigma_2^\kappa$  be two  $\kappa$ -dimensional simplexes with vertices in  $M$ , without a common vertex. Suppose that every two such simplexes either do not intersect or have a single common point, interior to both  $\sigma_1^\kappa$  and  $\sigma_2^\kappa$ .

We shall denote by  $\#_\kappa(M)$  the number of intersections of pairs  $\{\sigma_1^\kappa, \sigma_2^\kappa\}$  as above. Such an intersection will be called a  $\kappa$ -intersection.

For example, if  $M = \{5 \text{ points lying on a circle}\}$ , then  $\#_1(M) = 5$ .

**Lemma 1.** *There exists in  $\mathbb{R}^{2\kappa}$  a set  $M$  of  $2\kappa + 3$  points such that  $\#_\kappa(M) = 1$ .*

The proof is given in Section III. For example, in the case  $\kappa = 1$  it suffices to take 5 points in  $\mathbb{R}^2$  in general position, whose convex hull is a triangle.

Consider now the complex

$$Y_\kappa = [\Delta^{2\kappa+2}]^\kappa.$$

We shall prove that  $\text{index } Y_\kappa^* = 2\kappa$ . Set

$$M_{2\kappa} = \{(x, y) \in Y_\kappa^2 \mid [x] \cap [y] = \emptyset\}.$$

Clearly,  $M_{2\kappa}$  is an invariant compact subset of  $Y_\kappa^*$ .

**Lemma 2.**  *$M_{2\kappa}$  is a closed manifold.*

The proof of this interesting proposition is given in Section II. Notice that  $M_{2\kappa}$  has a structure of a cell complex. It is also easy to show that there is a deformation of  $Y_\kappa^*$  on  $M_{2\kappa}$ , so  $M_{2\kappa}$  contains all the information about  $Y_\kappa^*$ .

**Theorem 1.** *Let  $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$ . Then*

$$\text{index } Y_\kappa^* = 2\kappa.$$

*Proof.* It suffices to prove that  $\text{index } M_{2\kappa} = 2\kappa$ . As follows from Lemma 1, there exists in  $\mathbb{R}^{2\kappa}$  a set

$$M = \{a_1, a_2, \dots, a_{2\kappa+3}\}$$

of  $2\kappa + 3$  points such that  $\#_\kappa(M) = 1$ . Let the single  $\kappa$ -intersection arise between the simplexes  $[a_1, \dots, a_{\kappa+1}]$  and  $[a_{\kappa+2}, \dots, a_{2\kappa+2}]$ . Consider in  $\mathbb{R}^{2\kappa+1}$  the set

$$N = \{a_1 + l_{2\kappa+1}, a_2, \dots, a_{2\kappa+3}\},$$

where  $l_{2\kappa+1}$  is a unit vector orthogonal to  $\mathbb{R}^{2\kappa}$ . Clearly,  $\#_\kappa(N) = 0$ . Then  $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$  may be embedded in  $\mathbb{R}^{2\kappa+1}$  with vertices in  $N$ . Let us define an odd map  $\varphi : M_{2\kappa} \rightarrow S^{2\kappa}$  as follows:

$$\varphi(x, y) = \frac{y - x}{\|y - x\|}.$$

The preimage  $\varphi^{-1}(-l_{2\kappa+1})$  contains a single point  $(x_0, y_0)$  corresponding to the single  $\kappa$ -intersection in  $M$ . Here  $x_0 \in [a_1 + l_{2\kappa+1}, \dots, a_{\kappa+1}]$ ,  $y_0 \in [a_{\kappa+2}, \dots, a_{2\kappa+2}]$ . We shall prove that  $\deg_2 \varphi = 1 \pmod{2}$ . It is clear that there exists in  $S^{2\kappa}$  a neighbourhood  $W \ni -l_{2\kappa+1}$  such that the map  $\varphi$  restricted to  $\varphi^{-1}(W)$  is a homeomorphism. Let us approximate  $\varphi$  with a smooth map  $\varphi_0 : M_{2\kappa} \rightarrow S^{2\kappa}$  such that  $\varphi_0(x) = \varphi(x)$  for  $x \in \varphi^{-1}(W)$ . Then  $-l_{2\kappa+1}$  is a regular value of  $\varphi_0$  and  $\varphi_0^{-1}(-l_{2\kappa+1})$  contains a single point  $(x_0, y_0)$ . But then for its degree mod 2 we have  $\deg_2 \varphi_0 = 1$  (cf. [2]), and therefore  $\deg_2 \varphi = 1$ .

Then  $\text{index } M_{2\kappa} = 2\kappa$ , as follows from the Proposition in the preliminary section. The theorem is proved.

Note that if  $P_\kappa$  is a contractible  $\kappa$ -dimensional polyhedron, then  $\text{index } P_\kappa^* \leq 2\kappa - 1$ . This is established in [3].

**Theorem 2.** *For any  $\kappa \geq 1$  there exists a closed manifold  $M_{2\kappa}$  with a free involution  $T$ , such that  $\text{index } M_{2\kappa} = 2\kappa$ , but there is a map*

$$f : M_{2\kappa} \rightarrow Y_\kappa$$

*into a  $\kappa$ -dimensional polyhedron without antipodal coincidence:  $f(Tx) \neq f(x)$  for any  $x \in M_{2\kappa}$ .*

*Proof.* Let  $M_{2\kappa}$  and  $Y_\kappa$  be as in Theorem 1. Set  $f(x, y) = x$ . Then  $f : M_{2\kappa} \rightarrow Y_\kappa$  is a map without antipodal coincidence:  $f(x, y) \neq f(y, x)$ .

Let us note that not every manifold of index  $2\kappa$  admits such a map. Šchepin [4] has shown that every map  $f : S^{2\kappa} \rightarrow P_\kappa$  of the sphere  $S^{2\kappa}$  into a  $\kappa$ -dimensional polyhedron has an antipodal coincidence:  $f(-x) = f(x)$ .

The following is a simple but useful proposition that we shall refer to in the last two sections.

**Proposition (\*)**. *Let  $\Delta^n = [a_1, \dots, a_{n+1}]$  be the standard  $n$ -simplex in  $\mathbb{R}^n$ . Suppose that  $\sum_{i=1}^{n+1} \lambda_i a_i = 0$ . Then  $\lambda_1 = \lambda_2 = \dots = \lambda_{n+1}$ .*

*Proof.* Since  $\sum a_i = 0$ , we find  $a_1 = -a_2 - \dots - a_{n+1}$ . Substitute in  $\sum \lambda_i a_i = 0$  and make use of the fact that  $a_2, \dots, a_{n+1}$  are independent.

## II. PROOF THAT $M_{2\kappa}$ IS A MANIFOLD

In this section we shall prove Lemma 2. The key is Lemma 3, which is interesting for itself. Let

$$P_\kappa = O([\Delta^{2\kappa}]^{\kappa-1})$$

be the cone over  $[\Delta^{2\kappa}]^{\kappa-1}$  with vertex the origin  $O$ . Consider the set

$$(1) \quad U = \{(x, y) \in P_\kappa^2 \mid [x] \cap [y] = \{O\}\}.$$

**Lemma 3.** *The map  $\varphi : U \rightarrow \mathbb{R}^{2\kappa}$  defined by*

$$\varphi(x, y) = x - y$$

*maps  $U$  homeomorphically onto some open star-like subset of  $\mathbb{R}^{2\kappa}$  with center  $O$ .*

*Proof.* Let  $\Delta^{2\kappa} = [a_1, \dots, a_{2\kappa+1}]$ . If  $(x, y) \in U$ , then

$$x = \sum_{i \in I} \alpha_i a_i, \quad y = \sum_{j \in J} \beta_j a_j,$$

where  $\alpha_i, \beta_j > 0$  and  $|I| \leq \kappa, |J| \leq \kappa, I \cap J = \emptyset$ . (The index sets are disjoint, since  $[x] \cap [y] = \{O\}$  by definition.)

1) First we prove that  $\varphi$  is “mono”.

Let  $\varphi(x, y) = \varphi(u, v)$ , i.e.  $x - y = u - v$ . We have as above

$$u = \sum_{r \in R} \gamma_r a_r, \quad v = \sum_{s \in S} \delta_s a_s,$$

where  $\gamma_r, \delta_s > 0, |R| \leq \kappa, |S| \leq \kappa, R \cap S = \emptyset$ . Then

$$(2) \quad \sum_I \alpha_i a_i - \sum_J \beta_j a_j - \sum_R \gamma_r a_r + \sum_S \delta_s a_s = 0.$$

Let

$$M = \{1, 2, \dots, 2\kappa + 1\}.$$

By Proposition (\*) all the coefficients in (2) are (after reduction) equal to some number  $c$ .

If  $c \geq 0$ , then from (2)  $J \cup R$  is contained in  $I \cup S$ , so  $I \supset R$  and  $S \supset J$ . Since  $|S| + |I| \leq 2\kappa < |M|$ , this implies  $I \cup J \cup R \cup S \neq M$ , thus  $c = 0$ .

If  $c \leq 0$ , then in the same way it follows that  $R \supset I, J \supset S$  and  $I \cup J \cup R \cup S \neq M$ , hence  $c = 0$ .

The single possibility for this is  $I = R, J = S$ . But then (2) implies that

$$\sum_I \alpha_i a_i = \sum_R \gamma_r a_r, \quad \sum_J \beta_j a_j = \sum_S \delta_s a_s.$$

So,  $x = u, y = v$ , i.e.  $\varphi$  is “mono”.

2) We shall show that  $\varphi(U)$  contains  $O$  in its interior.

Let  $w \in \mathbb{R}^{2\kappa}$  be a vector with a small norm. Then it may be written in the form

$$w = \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_{2\kappa} a_{2\kappa},$$

where we assume that  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{2\kappa}$ .

Clearly,

$$a_\kappa = -a_1 - \dots - a_{\kappa-1} - a_{\kappa+1} - \dots - a_{2\kappa+1}$$

and, substituting above,

$$\begin{aligned} w &= (\lambda_1 - \lambda_\kappa) a_1 + \dots + (\lambda_{\kappa-1} - \lambda_\kappa) a_{\kappa-1} \\ &\quad + (\lambda_{\kappa+1} - \lambda_\kappa) a_{\kappa+1} + \dots + (\lambda_{2\kappa} - \lambda_\kappa) a_{2\kappa} - \lambda_\kappa a_{2\kappa+1}. \end{aligned}$$

Set

$$\begin{aligned} x &= (\lambda_{\kappa+1} - \lambda_\kappa) a_{\kappa+1} + \dots + (\lambda_{2\kappa} - \lambda_\kappa) a_{2\kappa}, \\ y &= (\lambda_\kappa - \lambda_1) a_1 + \dots + (\lambda_\kappa - \lambda_{\kappa-1}) a_{\kappa-1} + \lambda_\kappa a_{2\kappa+1}. \end{aligned}$$

Then  $w = x - y$ , where  $x, y \in O([\Delta^{2\kappa}]^{\kappa-1})$ , since  $x$  and  $y$  are of small norm and the coefficients are nonnegative. Note also that  $[x] \cap [y] = \{O\}$ , thus  $(x, y) \in U$ .

So  $w = \varphi(x, y)$ , hence  $w \in \varphi(U)$ . We proved that  $\varphi(U)$  contains  $O$  in its interior. Note finally, that  $\varphi(U)$  is star-like, as follows immediately from the definition of  $\varphi$ .

The lemma is proved.

*Proof of Lemma 2.* Here we prove that  $M_{2\kappa}$  is a manifold. Recall that

$$M_{2\kappa} = \{(x, y) \in Y_\kappa^2 \mid [x] \cap [y] \neq \emptyset\},$$

where  $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$ . It is clear that  $M_{2\kappa}$  is subdivided in a natural way to prisms of the form  $\sigma^i \times \sigma^j$ , where  $\sigma^i, \sigma^j \subset Y_\kappa$  are nonintersecting simplexes. We shall prove that the star of each vertex of  $M_{2\kappa}$  is homeomorphic to  $\mathbb{R}^{2\kappa}$ . It suffices to check it for an arbitrary vertex  $(a_1, a_2)$ . Clearly,

$$\text{St}(a_1, a_2) = \{(x, y) \in Y_\kappa^2 \mid [x] \ni a_1, [y] \ni a_2, [x] \cap [y] = \emptyset\}.$$

We shall embed  $Y_\kappa$  in  $\mathbb{R}^{2\kappa+1}$  as follows: Let  $\{a_3, \dots, a_{2\kappa+3}\}$  be the vertices of the standard simplex  $\Delta^{2\kappa}$  in  $\mathbb{R}^{2\kappa}$  and  $a_1 = l_{2\kappa+1}, a_2 = -l_{2\kappa+1}$ , where  $l_{2\kappa+1}$  is a unit vector orthogonal to  $\mathbb{R}^{2\kappa}$ . The points  $\{a_1, \dots, a_{2\kappa+3}\}$  are in general position in  $\mathbb{R}^{2\kappa+1}$ , consequently we may embed there  $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$  with vertices  $a_i$ . Let  $p : \mathbb{R}^{2\kappa+1} \rightarrow \mathbb{R}^{2\kappa}$  denote the orthogonal projection. We have

$$p(Y_\kappa) = O([\Delta^{2\kappa}]^{\kappa-1}).$$

It is easy to see that  $(x, y) \in \text{St}(a_1, a_2)$  if and only if  $(p(x), p(y)) \in U$ , where  $U$  is defined by (1). Then the map  $P(x, y) = (p(x), p(y))$  is a homeomorphism and according to Lemma 3,  $U$  is homeomorphic to  $\mathbb{R}^{2\kappa}$ . Therefore  $\text{St}(a_1, a_2) \approx \mathbb{R}^{2\kappa}$ .

### III. PROOF OF LEMMA 1

Here we shall construct in  $\mathbb{R}^{2\kappa}$  a set  $M$  of  $2\kappa + 3$  points such that  $\#_\kappa(M) = 1$ . Let  $\Delta^{2\kappa} = [a_1, \dots, a_{2\kappa+1}]$  be the standard simplex in  $\mathbb{R}^{2\kappa}$ . Set

$$A = \lambda(a_1 + \dots + a_\kappa) + \delta a_{\kappa+1},$$

where  $0 < \delta < \lambda$  and  $\kappa\lambda + \delta < 1$ .

We shall show that the set

$$M = \{O, a_1, \dots, a_{2\kappa+1}, A\}$$

meets the case. Note first that  $A$  lies in the interior of  $\Delta^{2\kappa}$ , since  $\kappa\lambda + \delta < 1$ . Consider two  $\kappa$ -dimensional simplexes  $\sigma_1$  and  $\sigma_2$  with vertices in  $M$  without a common vertex. If some of them contain neither  $O$ , nor  $A$ , then  $\sigma_1 \cap \sigma_2 = \emptyset$ . Indeed, suppose that  $O, A \notin \sigma_1$ . Then  $\sigma_1 \subset \partial\Delta^{2\kappa}$  and  $\sigma_1 \cap \sigma_2 = \sigma_1 \cap (\sigma_2 \cap \partial\Delta^{2\kappa}) = \emptyset$ . The last equality holds, since  $\sigma_2 \cap \partial\Delta^{2\kappa}$  is a simplex in  $\partial\Delta^{2\kappa}$  without common vertices with  $\sigma_1$ .

So, we may suppose that

$$\sigma_1 = [O, a_{i_1}, \dots, a_{i_\kappa}], \quad \sigma_2 = [A, a_{j_1}, \dots, a_{j_\kappa}],$$

where all the vertices are different. Suppose that  $\sigma_1$  and  $\sigma_2$  intersect; then we have

$$(3) \quad \lambda_1 a_{i_1} + \dots + \lambda_\kappa a_{i_\kappa} = \mu_0 \lambda (a_1 + \dots + a_\kappa) + \mu_0 \delta a_{\kappa+1} + \mu_1 a_{j_1} + \dots + \mu_\kappa a_{j_\kappa},$$

where  $\lambda_i \geq 0, \sum \lambda_i \leq 1, \mu_j \geq 0, \sum \mu_j = 1$ .

Note that  $\mu_0 > 0$ , since  $\sigma_1$  and  $\sigma_2$  may intersect only in an interior point of  $\Delta^{2\kappa}$ .

Send all the members of (3) to the right-hand side, then each  $a_i$  appears (after reduction) with some coefficient  $\nu(a_i)$ . According to Proposition (\*) we should have

$$(4) \quad \nu(a_1) = \nu(a_2) = \dots = \nu(a_{2\kappa+1}).$$

Let us note that exactly one  $a_i$  is not a vertex of either  $\sigma_1$  or of  $\sigma_2$ . There are 3 possibilities about  $a_i$ .

1)  $i \geq \kappa + 2$ . Then  $\nu(a_i) = 0$  and hence  $\nu(a_j) = 0$  for any  $j$ . But there exists some  $j \leq \kappa + 1$  such that  $a_j$  is a vertex of  $\sigma_2$ . Then the examination of (3) gives  $\nu(a_j) > 0$ , in contradiction with (4).

2)  $i \leq \kappa$ . Then  $\nu(a_i) = \mu_0\lambda > 0$ . This implies that for  $j \geq \kappa + 2$  all the  $a_j$  are in the right-hand side of (3), otherwise we would have  $\nu(a_j) \leq 0$ , in contradiction with (4). Then  $a_{\kappa+1}$  takes part in the left-hand side with coefficient  $\lambda_{\kappa+1} \geq 0$ . Therefore

$$\nu(a_{\kappa+1}) = \mu_0\delta - \lambda_{\kappa+1} = \nu(a_i) = \mu_0\lambda,$$

so  $\mu_0(\delta - \lambda) = \lambda_{\kappa+1} \geq 0$ , which contradicts the condition  $\delta < \lambda$ .

3)  $i = \kappa + 1$ , i.e.  $a_{\kappa+1}$  is not a vertex of  $\sigma_1 \cup \sigma_2$ . In this case we should have

$$\sigma_1 = [O, a_1, \dots, a_\kappa], \quad \sigma_2 = [A, a_{\kappa+2}, \dots, a_{2\kappa+1}].$$

Really, if we suppose that  $a_j \in \sigma_1$  for some  $j \geq \kappa + 2$ , then  $a_j$  takes part in the left-hand side of (3) with coefficient  $\lambda_j \geq 0$ , thus  $\nu(a_j) = -\lambda_j$ , though  $\nu(a_{\kappa+1}) = \mu_0\delta > 0$ , in contradiction with (4).

We shall prove that  $\sigma_1$  and  $\sigma_2$  intersect in an interior point. We are looking for a solution of

$$(5) \quad \begin{aligned} \lambda_1 a_1 + \dots + \lambda_\kappa a_\kappa &= \mu_0 \lambda (a_1 + \dots + a_\kappa) + \mu_0 \delta a_{\kappa+1} \\ &+ \mu_{\kappa+2} a_{\kappa+2} + \dots + \mu_{2\kappa+1} a_{2\kappa+1} \end{aligned}$$

with  $\lambda_i, \mu_j \geq 0, \sum \lambda_i \leq 1, \sum \mu_j = 1$ .

It is straightforward to check that the numbers

$$\begin{aligned} \lambda_1 = \lambda_2 = \dots = \lambda_\kappa &= \frac{\lambda - \delta}{1 + \delta\kappa}, & \mu_0 &= \frac{1}{1 + \delta\kappa}, \\ \mu_{\kappa+2} = \dots = \mu_{2\kappa+1} &= \frac{\delta}{1 + \delta\kappa} \end{aligned}$$

satisfy (5), since it reduces to the identity  $\sum a_i = 0$ . This means that  $\sigma_1$  and  $\sigma_2$  intersect in an interior point. Therefore  $\#_\kappa(M) = 1$ .

#### IV. CONCLUDING REMARKS

There is a natural generalization of the spaces  $Y_\kappa$ . Let as above  $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$  and consider the join

$$X_\kappa = Y_{\kappa_1} * Y_{\kappa_2} * \dots * Y_{\kappa_p}$$

where  $\kappa = \kappa_1 + \dots + \kappa_p + p - 1$ . It is not difficult to see that the spaces  $X_\kappa$  also have index  $X_\kappa^* = 2\kappa$ . It is shown in [1] that  $X_\kappa$  is a  $\kappa$ -minimal complex, in the sense that it is not embeddable in  $\mathbb{R}^{2\kappa}$  but each of its proper subcomplexes is embeddable in  $\mathbb{R}^{2\kappa}$ . Cohomological obstructions for embedding in  $\mathbb{R}^{2\kappa}$  are discussed in [5] and [6].

For a graph  $\Gamma$  the question of computing index  $\Gamma^*$  is in fact solved in [6]:

We have index  $\Gamma^* = 2$  if and only if  $\Gamma$  is nonplanar. Furthermore, index  $\Gamma^* = 1$  if and only if  $\Gamma$  is planar, but not embeddable in  $\mathbb{R}^1$ . Otherwise index  $\Gamma^* = 0$ .

Another interesting phenomenon in the case of graphs is the following fact. Let

$$M(\Gamma) = \{(x, y) \in \Gamma^2 | [x] \cap [y] = \emptyset\}.$$

Then  $M(\Gamma)$  is homeomorphic to a closed surface if and only if  $\Gamma$  is one of the two Kuratowski graphs  $K_5$  and  $K_{3,3}$ . Moreover,  $M(K_5)$  is a sphere with 6 handles and

$M(K_{3,3})$  is a sphere with 4 handles. This can be shown by computing the Euler characteristic of these spaces and by checking that they are both orientable.

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1 SUCHODOLSKA STR., B 13 VH 2 AP 32, 1373 SOFIA, BULGARIA  
*E-mail address*: s\_simeon@hotmail.com