ON $p$-HYPONORMAL OPERATORS

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Abstract. In this paper we show that $p$-hyponormal operators with $0 \not\in \sigma(|T|^\frac{1}{2})$ are subscalar. As a corollary, we get that such operators with rich spectra have non-trivial invariant subspaces.

1. INTRODUCTION

Let $H$ and $K$ be separable complex Hilbert spaces and let $\mathcal{L}(H, K)$ denote the space of all bounded linear operators from $H$ to $K$. If $H = K$, we write $\mathcal{L}(H)$ in place of $\mathcal{L}(H, K)$.

An operator $T \in \mathcal{L}(H)$ is said to be $p$-hyponormal, $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ where $T^*$ is the adjoint of $T$. If $p = 1$, $T$ is called hyponormal and if $p = \frac{1}{2}$, $T$ is called semi-hyponormal. Semi-hyponormal operators were introduced by Xia (see [Xi]), and $p$-hyponormal operators for a general $p$, $0 < p < 1$, have been studied by Aluthge. Any $p$-hyponormal operators are $q$-hyponormal if $q \leq p$. But there are examples to show that the converse of the above statement is not true (see [Al]).

A bounded linear operator $S$ on $H$ is called scalar of order $m$ if it has a spectral distribution of order $m$, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(C) \rightarrow \mathcal{L}(H)$$

such that $\Phi(z) = S$, where as usual $z$ stands for the identity function on $C$ and $C_0^m(C)$ stands for the space of compactly supported functions on $C$, continuously differentiable of order $m$, $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to a closed invariant subspace. We now define the weaker form of a subscalar operator. An operator $T \in \mathcal{L}(H)$ is quasi-subscalar if there exists a one-to-one $V \in \mathcal{L}(H, K)$ such that $VT = SV$ where $S$ ($= \Phi(z)$ in the above definition) is a scalar operator.

This paper has been divided into three sections. Section 2 deals with some preliminary facts. In section 3, we show that $p$-hyponormal operators with the property $0 \not\in \sigma(|T|^\frac{1}{2})$ are subscalar. As a corollary, we get that such operators with rich spectra have non-trivial invariant subspaces.

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Let $du(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Let $H$ be a complex separable Hilbert space, and let $D$ be a bounded open disc in $\mathbb{C}$. We shall denote by $L^2(D, H)$ the Hilbert space of measurable functions $f : D \rightarrow H$, such that 
\[
\|f\|_{2,D} = \left( \int_D \|f(z)\|^2 d\mu(z) \right)^{\frac{1}{2}} < \infty.
\]

The space of functions $f \in L^2(D, H)$ which are analytic functions in $D$ (i.e., $\bar{\partial}f = 0$) is defined by 
\[
A^2(D, H) = L^2(D, H) \cap \mathcal{O}(D, H)
\]
where $\mathcal{O}(D, H)$ denotes the Fréchet space of $H$-valued analytic functions on $D$ with respect to uniform topology. $A^2(D, H)$ is called the Bergman space for $D$. Note that $A^2(D, H)$ is a Hilbert space. The operator $T - z$ on the space $\mathcal{O}(D, H)$ has property $\langle \beta \rangle$, which means by definition that $T - z$ is one-to-one and has closed range for every disc $D$.

Let us define now a Sobolev type space, called $W^2(D, H)$ where $D$ is a bounded disc in $\mathbb{C}$. $W^2(D, H)$ will be the space of those functions $f \in L^2(D, H)$ whose derivatives $\bar{\partial}f, \partial f$ in the sense of distributions still belong to $L^2(D, H)$. Endowed with the norm $\|f\|_{W^2}^2 = \sum_{i=0}^2 \|\partial^i f\|_{2,D}^2$, $W^2(D, H)$ becomes a Hilbert space contained continuously in $L^2(D, H)$.

Now for $f \in C_0^2(\mathbb{C})$, let $M_f$ denote the operator on $W^2(D, H)$ given by multiplication by $f$. This has a spectral distribution of order 2, defined by the functional calculus 
\[
\Phi_M : C_0^2(\mathbb{C}) \rightarrow \mathcal{L}(W^2(D, H)), \quad \Phi_M(f) = M_f.
\]
Therefore $M_z$ is a scalar operator of order 2. In fact, it can be shown [Pu] that $M_z$ is subnormal.

3. Subscalarity

This section deals with the characterization for some $p$-hyponormal operators. Recall that an operator $T \in \mathcal{L}(H)$ is said to be $p$-hyponormal, $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ where $T^*$ is the adjoint of $T$.

We need the following lemmas to give a proof of the main theorem.

**Lemma 1** ([Xi], Lemma 2.1). Let $T = U|T|_r$ be the polar decomposition of $T$, $Q = |T|_r - |T|_l$, $z = re^{i\theta}$, $0 < \rho$, and $|e^{i\theta}| = 1$ where $|T|_r = (T^*T)^{\frac{1}{2}}$ and $|T|_l = (TT^*)^{\frac{1}{2}}$. Then 
\[
\|(T - z)f\|_{2,D}^2 = \|(|T|_r - \rho)f\|_{2,D}^2 + \rho\|T|_r^2(U - e^{i\theta})^*f\|_{2,D} + \rho(Qf, f)
\]
for all $f \in L^2(D, H)$.

For reference, we quote Lemma 2 from [Pu].

**Lemma 2** ([Pu], Proposition 2.1). For every bounded disk $D$ in $\mathbb{C}$ there is a constant $C_D$, such that for an arbitrary operator $T \in \mathcal{L}(H)$ and $f \in W^2(D, H)$ we have 
\[
\|(I - P)f\|_{2,D} \leq C_D\left(\|(T - z)^*\bar{\partial}f\|_{2,D} + \|(T - z)\bar{\partial}^2 f\|_{2,D}\right)
\]
where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$. 

For $p$-hyponormal operator $T = U|T|$, Aluthge ([Al]) introduced the operator $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ and showed very interesting results on $\tilde{T}$.

**Lemma 3 ([Al]).** Let $T = U|T|$ be a $p$-hyponormal operator, $0 < p < 1$, and $U$ unitary. Then the operator $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ is hyponormal if $\frac{1}{2} < p < 1$, and $(p + \frac{1}{2})$-hyponormal if $0 < p < \frac{1}{2}$.

**Lemma 4.** Let $T = U|T|$ be semi-hyponormal and let $U$ be unitary. Let $D$ be a bounded disk which contains $\sigma(T)$. Then the map $V : H \to H(D)$ defined by $Vh = 1 \otimes h (\equiv 1 \otimes h + (T - z)W^2(D, H))$ is one-to-one and has closed range, where $1 \otimes h$ denotes the constant function sending any $z \in D$ to $h$.

**Proof.** Let $h_n \in H$ and $f_n \in W^2(D, H)$ be sequences such that
\begin{equation}
\lim_{n \to \infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^2} = 0.
\end{equation}
Then by the definition of the norm of Sobolev space (1) implies
\begin{equation}
\lim_{n \to \infty} \|(T - z)\partial^i f_n\|_{2,D} = 0
\end{equation}
for $i = 1, 2$. Since $T$ is a semi-hyponormal operator, Lemma 1 and equation (2) imply
\begin{equation}
\begin{cases}
\lim_{n \to \infty} \|(T|_{r} - \rho)\partial^i f_n\|_{2,D} = 0, \\
\lim_{n \to \infty} \rho\|T^{1/2}(U - e^{i\theta})^i\partial^i f_n\|_{2,D} = 0, \\
\lim_{n \to \infty} \rho(Q\partial^i f_n, \partial^i f_n) = 0.
\end{cases}
\end{equation}
We note that for $i = 1, 2$
\begin{equation}
(T - z)^*\partial^i f_n = \|T\|^{1/2}(U - e^{i\theta})^i\partial^i f_n
\end{equation}
\begin{equation*}
+ e^{-i\theta}([T|_{r} - \rho]\partial^i f_n).
\end{equation*}
By equations (3) and (4), we get
\begin{equation}
\lim_{n \to \infty} \|(T - z)^*\partial^i f_n\|_{2,D} = 0.
\end{equation}
 Lemma 2 and equation (5) imply
\begin{equation}
\lim_{n \to \infty} \|(I - P)f_n\|_{2,D} = 0,
\end{equation}
where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto $A^2(D, H)$. Then by
\begin{equation}
\lim_{n \to \infty} \|(T - z)Pf_n + 1 \otimes h_n\|_{2,D} = 0.
\end{equation}
Let $\Gamma$ be a curve in $D$ surrounding $\sigma(T)$. Then for $z \in \Gamma$
\begin{equation}
\lim_{n \to \infty} \|Pf_n(z) + (T - z)^{-1}(1 \otimes h_n)\| = 0
\end{equation}
uniformly. Hence
\begin{equation}
\lim_{n \to \infty} \frac{1}{2\pi i} \int_{\Gamma} Pf_n(z)dz + h_n = 0.
\end{equation}
But by Cauchy’s theorem,
\begin{equation*}
\int_{\Gamma} Pf_n(z)dz = 0.
\end{equation*}
Hence $\lim_{n \to \infty} h_n = 0$. Thus $V$ is one-to-one and has closed range. □
Proposition 5. Let $T = U|T|_r$ be a p-hyponormal operator with the property $0 \notin \sigma(|T|^p_r)$, $0 < p < 1$, and $U$ unitary. Let $D$ be a bounded disk which contains $\sigma(T)$. Then the map $V : H \to H(D)$ defined by $V h = 1 \otimes h (\equiv 1 \otimes h + (T - z)W^2(D, H))$ is one-to-one and has closed range, where $1 \otimes h$ denotes the constant function sending any $z \in D$ to $h$.

Proof. Let $h_n \in H$ and $f_n \in W^2(D, H)$ be sequences such that

(7) $\lim_{n \to \infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^2} = 0.$

Then equation (7) implies

(8) $\lim_{n \to \infty} \|(T - z)\partial^f f_n\|_{2, D} = 0$

for $i = 1, 2$.

(a) If $\frac{1}{2} \leq p < 1$, then $T$ is semi-hyponormal. Therefore, Proposition 5 follows from Lemma 4.

(b) Let $0 < p < \frac{1}{2}$. Since $T = U|T|_r$,

$$\lim_{n \to \infty} \|\partial^f(T|_r^\frac{1}{2} - z)\partial^f f_n\|_{2, D} = 0.$$

Since $\tilde{T} = |T|^\frac{1}{2} U|T|_r^\frac{1}{2}$, we have

(9) $\lim_{n \to \infty} \|(\tilde{T} - z)\partial^f(|T|^\frac{1}{2} f_n)\|_{2, D} = 0.$

Since $\tilde{T}$ is $(p + \frac{1}{2})$-hyponormal by Lemma 3, $\tilde{T}$ is semi-hyponormal. Let $T = W|\tilde{T}|_r$ be the polar decomposition. Lemma 1 and equation (9) imply

(10) $\begin{cases} 
\lim_{n \to \infty} \|(|\tilde{T}|_r - \rho)\partial^f(|T|^\frac{1}{2} f_n)\|_{2, D} = 0, \\
\lim_{n \to \infty} \rho\|\partial^f(|T|_r^\frac{1}{2} (W - e^{i\theta})^*\partial^f(|T|^\frac{1}{2} f_n))\|_{2, D} = 0, \\
\lim_{n \to \infty} \rho(Q\partial^f(|T|^\frac{1}{2} f_n), \partial^f(|T|^\frac{1}{2} f_n)) = 0.
\end{cases}$

Now we note that for $i = 1, 2$

$$((\tilde{T} - z)^*\partial^f(|T|^\frac{1}{2} f_n) = |\tilde{T}|_r^\frac{1}{2} |\tilde{T}|_r^\frac{1}{2} (W - e^{i\theta})^*\partial^f(|T|^\frac{1}{2} f_n)|$$

$$+ e^{-i\theta}((|\tilde{T}|_r - \rho)\partial^f(|T|^\frac{1}{2} f_n)).$$

By (10) and (11), we get

(12) $\lim_{n \to \infty} \|(|\tilde{T} - z)^*\partial^f(|T|^\frac{1}{2} f_n)\|_{2, D} = 0.$

Lemma 2 and equation (12) imply

(13) $\lim_{n \to \infty} \|(1 - P)|T|^\frac{1}{2} f_n\|_{2, D} = 0.$

Since $|T|^\frac{1}{2} (T - z) = (\tilde{T} - z)|T|^\frac{1}{2}$ and $0 \notin \sigma(|T|^\frac{1}{2})$, it follows from (7) that $\sigma(T) = \sigma(\tilde{T})$ and

(14) $\lim_{n \to \infty} \|(|\tilde{T} - z)|T|^\frac{1}{2} f_n + |T|^\frac{1}{2} (1 \otimes h_n)\|_{2, D} = 0.$

By (13) and (14), we have

$$\lim_{n \to \infty} \|(|\tilde{T} - z)P(|T|^\frac{1}{2} f_n) + |T|^\frac{1}{2} (1 \otimes h_n)\|_{2, D} = 0.$$
Let $\Gamma$ be a curve in $D$ surrounding $\sigma(T) = \sigma(\tilde{T})$. Then for $z \in \Gamma$
\[
\lim_{n \to \infty} \|P([T]^{\frac{1}{2}} f_n(z)) + (\tilde{T} - z)^{-1}([T]^{\frac{1}{2}} (1 \otimes h_n))\| = 0
\]
uniformly. Hence
\[
\lim_{n \to \infty} \frac{1}{2\pi i} \int_{\Gamma} P([T]^{\frac{1}{2}} f_n(z))dz + |T|^{\frac{1}{2}} h_n = 0.
\]
But by Cauchy’s theorem,
\[
\frac{1}{2\pi i} \int_{\Gamma} P([T]^{\frac{1}{2}} f_n(z))dz = 0.
\]
Therefore $\lim_{n \to \infty} |T|^{\frac{1}{2}} h_n = 0$. Since $0 \notin \sigma([T]^{\frac{1}{2}})$, $|T|^{\frac{1}{2}}$ is bounded below. Hence $\lim_{n \to \infty} h_n = 0$. \hfill $\square$

**Theorem 6.** Let $T = U|T|_r$ be $p$-hyponormal, $0 < p < 1$, and $U$ unitary. If $0 \notin \sigma([T]^{\frac{1}{2}})$, then $T$ is subscalar of order 2.

**Proof.** Consider an arbitrary bounded open disk $D$ in the complex plane $\mathbb{C}$ and the quotient space
\[H(D) = W^2(D, H)/(T - z)W^2(D, H)\]
endowed with the Hilbert space norm. The class of a vector $f$ or an operator on $H(D)$ will be denoted by $f$, respectively $A$. Let $M$ be the operator of multiplication by $z$ on $W^2(D, H)$. As noted at the end of section 2, $M$ is a scalar of order 2 and has a spectral distribution $\Phi$. Let $S \equiv M$. Since $(T - z)W^2(D, H)$ is invariant under every operator $Mf$, $f \in C^2(D)$, we infer that $S$ is a scalar operator of order 2 with spectral distribution $\Phi$.

Consider the natural map $V : H \to H(D)$ defined by $V h = \tilde{h}$, for $h \in H$, where $1 \otimes h$ denotes the constant function identically equal to $h$. Note that $VT = SV$. In particular $V$ is an invariant subspace for $S$. Since $V$ is one-to-one and has closed range by Proposition 5, $T$ is subscalar of order 2. \hfill $\square$

**Corollary 7.** Every invertible $p$-hyponormal operator is subscalar of order 2.

**Proof.** Assume $T = U|T|_r$ is an invertible $p$-hyponormal operator where $U$ is unitary. Then $|T|_r$ is invertible. By [Ru, Theorem 12.33], $|T|^{\frac{1}{2}}$ is invertible. Therefore, $0 \notin \sigma([T]^{\frac{1}{2}})$. By Theorem 6, $T$ is subscalar of order 2. \hfill $\square$

**Corollary 8.** Let $T = U|T|_r$ be a $p$-hyponormal operator with the property $0 \notin \sigma([T]^{\frac{1}{2}})$, $0 < p < 1$, and $U$ unitary. If $\sigma(T)$ has interior in the plane, then $T$ has a non-trivial invariant subspace.

**Proof.** The corollary follows from Theorem 6 and [Es]. \hfill $\square$

**Corollary 9.** Let $T$ be as in Corollary 8. Then $T$ has the property $(\beta)$.

**Proof.** Since every subscalar operator has the property $(\beta)$, the corollary follows from Theorem 6. \hfill $\square$

Recall that an $X$ in $L(H, K)$ is called a quasi-affinity if it has trivial kernel and dense range. An operator $A$ in $L(H)$ is said to be a quasi-affine transform of an operator $T$ in $L(K)$ if there is a quasi-affinity $X$ in $L(H, K)$ such that $XA = TX$ (notation: $A \sim T$).
Corollary 10. Let $T$ be as in Corollary 8. If $A$ is any operator such that $A \prec T$, then $\sigma(T) \subseteq \sigma(A)$.

Proof. This is clear from [Ko, Theorem 3.2] and Corollary 9.

Corollary 11. Under the same hypothesis as Corollary 10, $A \in \mathcal{L}(H)$ is quasi-subscalar.

Proof. Let $X \in \mathcal{L}(H, K)$ be a quasi-affinity such that $XA = TX$. Since $V$ (in the construction of $V$ and $S$) and $X$ are one-to-one, $VX$ is one-to-one. Therefore $VX$ implements the quasi-subscalar properties. Thus $A$ is quasi-subscalar.

References


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