ON AUTOMATIC CONTINUITY OF HOMOMORPHISMS

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Abstract. Introducing a weaker notion of regularity in a topological algebra, we examine and improve an automatic continuity theorem given by the second author. Examples and applications are given.

All topological algebras considered are commutative and Hausdorff, having a unit element. A topological algebra $A$ is a $Q$-algebra if the set $G(A)$ of invertible elements is open. Let $B$ be a subalgebra of an algebra $A$. Then $B$ is inverse closed in $A$ if $G(B) = B \cap G(A)$. $A$ is strongly semisimple if for every $x \in A$, $x \neq 0$, there exists a nonzero continuous multiplicative linear functional $\chi$ such that $\chi(x) \neq 0$. $A$ is adverbly complete if a Cauchy net $x_\alpha$ in $A$ converges in $A$ whenever for some $y$ in $A$, $x_\alpha + y - x_\alpha y$ converges to 0. A $Q$-algebra is adverbly complete [Ma, p. 45]. A uniform seminorm on an algebra $A$ is a seminorm $p$ such that $p(x^2) = p(x)^2$ for all $x$ in $A$. Such a $p$ is submultiplicative [BK]. A uniform topological algebra $A$ is a topological algebra whose topology is defined by a family of uniform seminorms. Such an $A$ is semisimple. The abbreviation lmca will stand for locally $m$-convex algebra.

In [B], the following is given.

Theorem ([B, Theorem 2.2]). Let $A$ be a spectrally bounded, regular, complete, uniform topological algebra. If $B$ is an lmca and $\phi : A \rightarrow B$ is a one-to-one homomorphism such that $(\text{Im}\, \phi)^{-}$ (the closure of $\text{Im}\, \phi$) is a semisimple $Q$-algebra, then $\phi^{-1} / \text{Im}\, \phi$ is continuous.

In the proof, the author considers the map $\phi^* : \sigma(C) \rightarrow \sigma(A)$, with $\phi^*(f) = f \circ \phi$, $\sigma(C)$ and $\sigma(A)$ denoting respectively the spaces of nonzero continuous multiplicative functionals on $C = (\text{Im}\, \phi)^{-}$ and $A$. In Math. Reviews, the reviewer R. J. Loy [L] asserted that the continuity of $\phi$ has been implicitly used in [B]. Indeed, $\phi^*$ is not always well defined when $\phi$ is not continuous as the following example shows.

Example 1. Let $\Omega$ denote the first uncountable ordinal and $[0, \Omega)$ the set of all ordinals smaller than $\Omega$. Consider the algebra $C[0, \Omega)$ of complex continuous functions on $[0, \Omega)$ with compact open topology $\tau$. Every $f \in C[0, \Omega)$ is bounded. It is a regular uniform lmca. The identity map $\phi : (C[0, \Omega), \tau) \rightarrow (C[0, \Omega), \| \cdot \|_\infty), \phi(f) = f$, satisfies the hypotheses of the above statement. It is well known that $(C[0, \Omega), \tau)$ has discontinuous multiplicative linear functionals (see, for example, [Ma], [Z]); let $\chi$. 

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be such a functional. It is in \(\sigma(C[0,\Omega],\|\cdot\|_\infty)\), but \(\phi^*(\chi) = \chi\) is not in \(\sigma([0,\Omega],\tau)\).

One may give another example in a more general situation. Let \(X\) be a noncompact, locally compact pseudocompact space. Take \(A = C(X)\) the algebra of all continuous functions on \(X\) with compact open topology, \(B\) the algebra \(C(X)\) endowed with the sup norm \(\|\cdot\|_\infty\) and \(\phi: A \to B\) be \(\phi(f) = f\). Since \(X\) is pseudocompact, non-compact, it is not realcompact [3] p. 44. Let \(y\) be an element of the real compactification of \(X\) with \(y\) not in \(X\). The evaluation \(\delta_y\) at \(y\) is in \(\sigma(B)\) but not in \(\sigma(A)\). Hence \(\phi^*\) is not defined on \(\delta_y\).

On the other hand, \(\phi^*\) can be well defined even when \(\phi\) is not continuous; in this case, the proof given in [8] works. Here is an example of such a situation.

**Example 2.** Let \(X\) be a compact Hausdorff space. Consider the algebra \(C(X)\) of continuous complex functions on \(X\). Take \(A = (C(X),\tau_d), \tau_d\) being the topology of uniform convergence on all countable compact subsets of \(X\), \(B\) the algebra \((C(X),\|\cdot\|_\infty)\) and \(\phi\) the identity map from \(A\) to \(B\). It is of course discontinuous, only if \(X\) is uncountable. But in this case, \(\sigma(A)\) and \(\sigma(B)\) are both homeomorphic to \(X\). So \(\phi^*\) is well defined.

The following theorem repairs the above result. On one hand, it provides a positive result in a context more general than above; on the other hand, it shows that if one assumes the continuity of \(\phi\) in the above result, then the algebra \(A\) is necessarily a Banach algebra.

**Theorem 1.** Let \(A\) be weakly regular, advertibly complete, uniform topological algebra, let \(B\) be an lmca, and let \(\phi: A \to B\) be a one-to-one homomorphism such that \((\text{Im } \phi)^{-}\) is a semisimple \(Q\)-algebra.

1. If \(A\) is functionally continuous, then \(\phi^{-1}/\text{Im } \phi\) is continuous.
2. If \(\phi\) is continuous, then the topology of \(A\) is normable.

Following [9] p. 51, \(A\) is functionally continuous \((FC)\) if every multiplicative functional on \(A\) is continuous. Note that \((C(X),\tau_d)\) in Example 2 is FC, but not \(Q\). A major unsolved problem in topological algebras is the Michael problem: Is every multiplicative linear functional on a Frechet lmca algebra continuous? This has led to several sufficient conditions for \(A\) to be FC. This makes FC a reasonable assumption.

Let \(A\) be a commutative topological algebra. \(A\) is weakly regular if given a closed set \(F \subset \sigma(A), F \neq \sigma(A)\), there exists \(x \neq 0\) in \(A\) such that \(f(x) = 0\) for all \(f \in F\). In the context of Banach algebras, weak regularity arises naturally in the study of uniqueness of the uniform norm [BD]; and it is weaker than regularity. This is exhibited in an example due to Barnes [M4, Example 1]. Let \(D = \{z \in C : |z| < 1\}, X = D \times [0,1]\). Let \(A = \{f \in C(X) : f\) is holomorphic on \(D \times \{0\}\}\), a uniform Banach algebra. Then \(A\) is weakly regular, but not regular. Since regularity in a uniform algebra is a stringent property, the validity of Theorem 1 under weak regularity is interesting.

**Proof of Theorem 1** 1. Let \(C = (\text{Im } \phi)^{-}\). Since \(A\) is FC, \(\phi^* : \sigma(C) \to \sigma(A)\), \(\phi^*(f) = f \circ \phi\) is well defined. It is also continuous with respective Gelfand topologies. Since the algebra \(C\) is a locally convex \(Q\)-algebra, \(\sigma(C)\) is compact [M6] p. 187]. Then \(\phi^*(\sigma(C))\) is a compact subset of \(\sigma(A)\). We have \(\phi^*(\sigma(C)) = \sigma(A)\). Indeed, if \(\phi^*(\sigma(C)) \neq \sigma(A)\), there exists \(x \neq 0\) in \(A\) such that \(\chi(\phi(x)) = 0\) for all \(\chi\) in \(\sigma(C)\).
Since $C$ is commutative semisimple and lmc, $\phi(x) = 0$; and then $x = 0$ for $\phi$ is one-to-one. Thus $\phi^*(\sigma(C)) = \sigma(A)$. Now let $P = (p_\alpha)$ be a family of uniform seminorms defining the topology $\tau$ of $A$. Since $A$ is adverbly complete, the spectrum $Sp_A(x) = \{\chi(x) : \chi \in \sigma(A)\}$ and the spectral radius $p_A(x) = \sup_\alpha \{\lim_{n \to \infty} (p_\alpha(x^n))^{1/n}\}$ for all $x$ in $A$ [Ma p. 104, p. 99]. Since $\sigma(A)$ is compact, $Sp_A(x)$ is bounded. Since $p_\alpha(x^2) = p_\alpha(x)^2$ for all $x$ and $\alpha$, $p_A(x) = \sup_\alpha p_\alpha(x)$ for all $x \in A$. Also, $\sigma(A) = \phi^*(\sigma(C))$ gives $Sp_A(x) = \{f(\phi(x)) : f \in \sigma(C)\} \subset Sp_C(\phi(x))$. Further, as $C$ is a $Q$-algebra, $s(C) = \{x \in C : \rho_C(x) \leq 1\}$ is a neighbourhood of 0 by [Mi, Prop. 13.5, p. 58]; and there exists a convex balanced open set $W$ such that $0 \in W \subseteq s(C)$. The Minkowski functional $q$ of $W$ in $C$ is a continuous seminorm satisfying $\rho_C(y) \leq q(y)$ for all $y \in C$. Hence for each $\alpha$, for each $x \in A$, $p_\alpha(x) \leq \rho_A(x) \leq \rho_C(\phi(x)) \leq q(\phi(x))$. This proves that $\phi^{-1}/\Im \phi$ is continuous.

(2) Suppose $\phi$ is continuous. Then $\phi^* : \sigma(C) \to \sigma(A)$ is well defined even if $A$ is not FC. Then $\phi^{-1}/\Im \phi$ is continuous as above making $\phi$ a topological isomorphism. Thus $\phi(A)$ is adverbly complete; and hence inverse closed in its completion. Whence it is inverse closed in the $Q$-algebra $C$, for the completion of $\phi(A)$ is contained in $C$. Therefore $\phi(A)$, and so $A$, is a $Q$-algebra. Hence the topology on $A$ given by the algebra norm $\rho_A$ is finer than $\tau$. Now since $A$ is a $Q$-algebra, $s(A) = \{x \in A : \rho_A(x) \leq 1\}$ is a neighbourhood of 0 on $(A, \tau)$. Thus $\rho$ determines $\tau$.

**Remark.** Once $\phi^*$ is defined, the full strength of weak regularity has not been used. In fact, one has only to find a nonzero element vanishing on a given compact set.

We now give a result in the absence of FC. We consider the space $\sigma^*(A)$ consisting of all nonzero multiplicative functionals on $A$ endowed with the weak topology $\sigma(A^*, A)$. We then introduce the following notion of weak $\sigma^*$-compact-regular weakened in the sense of the previous remark.

**Definition.** A commutative topological algebra $A$ is called **weakly $\sigma^*$-compact-regular** if for a compact subset $K$ of $\sigma^*(A)$, $K \neq \sigma^*(A)$, there exists a nonzero $x \in A$ such that $\chi(x) = 0$ for all $\chi \in K$.

**Theorem 2.** Let $A$ be a weakly $\sigma^*$-compact-regular adverbly complete uniform algebra, $B$ a locally convex algebra and $\phi : A \to B$ a one-to-one homomorphism such that $C = (\Im \phi)^-$ is a strongly semisimple $Q$-algebra. Then $\phi^{-1}/\Im \phi$ is continuous. If $\phi$ is continuous, then the topology of $A$ is normable.

For the proof, consider the map $\phi^* : \sigma(C) \to \sigma^*(A)$, $\phi^*(\chi) = \chi \circ \phi$. If $\chi \circ \phi$ is identically zero, then by the continuity of $\chi$, one obtains that $\chi$ is also identically zero. This contradicts $\chi \in \sigma(C)$. Thus $\phi^*$ is defined; and then it is continuous. Now one obtains $\phi^*(\sigma(C)) = \sigma^*(A)$; and the proof can be completed as in Theorem 1

We conjecture that the semisimplicity of $(\Im \phi)^-$ in Theorem 1 (and strong semisimplicity in Theorem 2) can be omitted. The following supports this.

**Theorem 3.** Let $A$ be a uniform lmc, $B$ a locally convex algebra, and $\phi : A \to B$ a one-to-one homomorphism such that $(\Im \phi)^-$ is a $Q$-algebra. Assume that at least one of the following holds.

(a) $A$ is adverbly complete and $\Im \phi$ is FC with continuous product.
(b) $A$ is FC, Ptak (as a l.c space), regular, having locally equicontinuous spectrum $\sigma(A)$ (in particular, $A$ is FC, Frchet, regular, having locally compact spectrum
\[ \sigma(A) \), and \( B \) is lnc. Then \( \phi^{-1}/\text{Im } \phi \) is continuous. If \( \phi \) is continuous, then the topology of \( A \) is normal.

Proof. (1) Assume (a). Then \( \sigma^*(\text{Im } \phi^{-}) = \sigma((\text{Im } \phi)^-) \) (since a \( Q \)-algebra is FC) = \( \sigma(\text{Im } \phi) \) (by the joint continuity of multiplication in Im \( \phi \) = \( \sigma^*(\text{Im } \phi) \) and \( \phi^*(\sigma^*(\text{Im } \phi)) = \sigma^*(A) \) as \( \phi \) is one-to-one. Then, for all \( x \in A \), \( \text{Sp}_A(x) = \{ \chi(x) : \chi \in \sigma(A) \} = \{ \chi(x) : \text{Im } \phi^{-} = f \in \sigma^*(\text{Im } \phi) \} = \{ f(\phi(x)) : f \in \sigma^*(\text{Im } \phi) \}. \)

Hence for some continuous seminorm \( q, \rho_A(x) = \rho_C(x) \leq q(\phi(x)) \) (\( x \in A \)).

(2) Assume (b). By \( \text{Ma} \) Coro. 1.3, p. 184, local equicontinuity of \( \sigma(A) \) implies continuity of the Gelfand map \( x \to \hat{x} \) and local compactness of \( \sigma(A) \). We show that \( \sigma(A) = \phi^*(\sigma(C)) \). Note that \( \phi^*(\sigma(C)) \subset \sigma(A) \). Suppose \( \chi \in \sigma(A) \backslash \phi^*(\sigma(C)) \).

By the local compactness, there exists a compact set \( K \subseteq \sigma(A) \) and disjoint open sets \( U, V \) in \( \sigma(A) \) such that \( \chi \in K \subset U, \phi^*(\sigma(C)) \subset V \). As \( A \) is Ptak, regular and having continuous Gelfand map, \( \text{Ma} \) Coro. 4.4, p. 344 implies that there exist \( x, y \in A \) such that \( g(x) = 1 (g \in \phi^*(\sigma(C))) \), \( g(x) = 0 (g \in \sigma(A) \backslash V) \); \( g(y) = 1 (g \in K) \), \( g(y) = 0 (g \in \sigma(A) \backslash U) \). Then \( g(x)g(y) = 0 \) for all \( g \in \sigma(A) \).

By the semisimplicity of \( A \), \( xy = 0 = \phi(x)\phi(y) \). On the other hand, for all \( f \in \sigma(C), f(\phi(x)) = 1 \). Thus \( 0 \notin \{ f(\phi(x)) : f \in \sigma(C) \} = \text{Sp}_C(\phi(x)) \), \( C \) being lnc and a \( Q \)-algebra. Thus \( \phi(x) \) is invertible in \( C \). Hence \( \phi(y) = \phi(x)^{-1}\phi(x)\phi(y) = 0, \) so that \( y = 0, \) a contradiction. It follows that \( \phi^*(\sigma(C)) = \sigma(A) \). Now the proof can be completed as in Theorem \( \Box \) Note that if \( A \) is Frechet, then every compact subset of \( \sigma(A) \) is equicontinuous \( \text{Ma} \) Prop. 4.2, p. 17. Hence by \( \text{Ma} \) Th. 1.1, p. 182, the Gelfand map is continuous. Further if \( \sigma(A) \) is locally compact, then it is locally equicontinuous \( \text{Ma} \) Cor. 1.3, p. 184]

Remarks. (1) If \( B \) is lnc and \( C = (\text{Im } \phi)^- \) is a semisimple \( Q \)-algebra, then \( C \) is strongly semisimple.

(2) Actually in the above theorems, \( \phi^{-1} : (A, \| \|) \to (A, \| \|) \) is continuous, where \( \| \| \) is the uniform norm given by \( \|x\| = \text{sup}\{p(x) : p \) is a continuous uniform seminorm\}. The existence of this norm implies that \( A \) is spectrally bounded.

(3) Theorem \( \Box \) also applies to Example \( \Box \). Indeed, the algebra \( (C[0, 1], \tau) \) is a complete uniform algebra. It is weakly \( \sigma^* \)-compact, for \( \sigma^*(C[0, 1]) \) is homeomorphic to the Stone-Cech compactification \( \beta(0, 1) \) of \( [0, 1) \) and \( C[0, 1] \) is isomorphic to the algebra \( C(\beta(0, 1)) \) \( \Box \).

(4) The hypothesis \( (\text{Im } \phi)^- \) is a \( Q \)-algebra cannot be omitted. Let \( A = C_b(\mathbb{R}) \) be the algebra of all continuous bounded functions on the real line. Endowed with the sup norm, it is a uniform Banach algebra. By the same arguments as in (3), one shows that \( A \) is weakly \( \sigma^* \)-compact, regular. Consider \( B = C(\mathbb{R}) \) to be the algebra of all continuous functions with the compact open topology. Consider \( \phi : A \to B, \phi(f) = f \). Then \( (\text{Im } \phi)^- = C(\mathbb{R}) \). It is well known that it is not a \( Q \)-algebra. Clearly \( \phi^{-1} \) is discontinuous.

(5) The referee has asked: (In above theorems) does the automatic continuity of \( \phi^{-1} \) on \( \text{Im } \phi \) necessitate \( (\text{Im } \phi)^- \) a \( Q \)-algebra? The following answers this.

Proposition 4. Let \( A \) be a complete \( Q \)-lnc, \( B \) an lnc, and \( \phi : A \to B \) a one-to-one homomorphism such that \( \phi^{-1}/\text{Im } \phi \) is continuous. Then \( (\text{Im } \phi)^- \) is a \( Q \)-algebra.

Proof. We may assume that \( B \) is complete, hence \( C = (\text{Im } \phi)^- \) is complete. Then \( \phi^{-1}/\text{Im } \phi \) extends as a continuous homomorphism \( \psi : C \to A \). By assumption, there exists a continuous seminorm \( p \) on \( A \) such that for all \( x \in A, r_C(\phi(x)) \leq r_{\text{Im } \phi}(\phi(x)) \leq r_A(x) \leq p(x) \leq p(\psi(\phi(x))). \) Since \( \psi \) is continuous, there exists a
continuous seminorm \( q \) on \( C \) such that \( p(\psi(y)) \leq q(y) \) \((y \in C)\). Hence in the above, 
\[ r_C(\phi(x)) \leq p(\psi(\phi(x))) \leq q(\phi(x)) \]
for all \( x \in A \). Now let \( y \in C \), \( y = \lim \phi(x_n) \) for some net \( (x_n) \) in \( A \). For any continuous multiplicative functional \( f \) on \( C \), 
\[ |f(y)| \leq |f(y - \phi(x_n))| + |f(\phi(x_n))| \leq |f(y - \phi(x_n))| + q(\phi(x_n)) \to q(y). \]
Hence \( r_C(y) = \sup |f(y)| \leq q(y) \) \((y \in C)\) showing that \( C \) is a \( Q \)-algebra.

If \( A \) is not a \( Q \)-algebra, then this does not hold. For the open unit disc \( U \) in the complex plane, let \( A = H(U) \) be the uniform Frechet algebra consisting of holomorphic functions on \( U \) with the compact-open topology, \( B = C(U) \) with the compact-open topology, and \( \phi : A \to B \) be \( \phi(f) = f \). Clearly \( \phi \) is a homeomorphism and \((\text{Im } \phi)^- = B \) fails to be a \( Q \)-algebra.

\[ \square \]

**Applications**

(1) **Proposition.** Let \( A \) be an advertibly complete lmca. Let \( || || \) be any continuous norm on \( A \). Then \( A \) cannot be simultaneously weakly regular and uniform unless the topology of \( A \) is normable.

**Proof.** Let \( \tau \) denote the lmca topology on \( A \). Let \( P = (p_n) \) be a family of submultiplicative seminorms on \( A \) defining \( \tau \). Then \( P_0 = P \cup \{|| || \} \) also determines \( \tau \).

Suppose \((A, \tau)\) is uniform. Then \( \tau \) is defined by a family \( S = (q_i) \) of uniform seminorms. By closing \( S \) with maxima of finite subfamilies and applying the continuity of \( || || \), there exists \( a q \) in \( S \) which is a norm. Let \( A_q \) be the uniform Banach algebra obtained by completing \((A, q)\). Let \( \phi : A \to A_q \) be \( \phi(x) = x \). Now if \( A \) is weakly regular, then Theorem \( \mathbf{[4]} \) applied to \( \phi \) implies that \( \tau \) is normable.

It follows that an advertibly complete non-normed weakly regular uniform algebra cannot support a continuous norm. Let \( X \) be a compact Hausdorff space. By a well known result of Kaplansky, if \( || || \) is any norm on \( C(X) \) making it a normed algebra, then the supnorm \( || || \leq | | \). The following has a bearing with this. A norm on an algebra \( A \) is semisimple if the completion of \((A, || ||)\) is semisimple \([BD]\). \[ \square \]

(2) **Corollary.** Let \( || || \) be a uniform norm on an algebra \( A \) such that \((A, || ||)\) is a weakly regular \( Q \)-normed algebra. Let \( || || \) be any submultiplicative norm on \( A \).

(i) If \( || || \) is semisimple, then \( || || \leq | | \). Further if \( || || \) is continuous, then \( || || \) is equivalent to \( || || \).

(ii) Let \((A, || ||)\) be complete and regular. Then \( || || \leq | | \) for any submultiplicative norm \( || || \).

Indeed let \( \phi : (A, || ||) \to (\tilde{A}, || ||) \) \((\tilde{A} = \text{completion of } (A, || ||))\), \( \phi(x) = x \). Theorem \( \mathbf{[4]} \) implies that there exists \( k > 0 \) such that \( || || \leq k | | \). Since \((A, || ||)\) is \( Q \), \( \rho_A(x) = \inf \|x^n\|^{1/n} \to 1 \) \( = \lim \|x^n\|^{1/n} = \|x\| \leq \lim |x^n|^{1/n} \leq |x| \) for all \( x \in A \). (ii) follows by Theorem \( \mathbf{[3]}(b) \).

(3) Let \( A = \mathbb{C} \times C_c^\infty(\mathbb{R}) \) \((\text{resp. } B = \mathbb{C} \times C_c(\mathbb{R})\)) be the algebra of all complex \( C^\infty \)-functions \((\text{resp. continuous functions})\) on \( \mathbb{R} \) which are constant outside some compact set \( \text{(depending on the function)} \). We endow \( A \) \((\text{resp. } B)\) with the inductive limit topology \( \tau_D \) \((\text{resp. } \tau_K)\). The algebra \((A, \tau_D)\) is a complete regular lmca, \((B, \tau_K)\) is a lmca \( Q \)-algebra \([Ma] \text{ p. 128}\) and \( A \) is dense in \( B \) \([K] \text{ p. 148}\). Consider \( \phi : A \to B \), \( \phi(f) = f \). Since the topology \( \tau_D \) is finer than \( \tau_K \) on \( A \), \( \phi \) is continuous. It is classical that \( A \) is not normable. Hence by Theorem \( \mathbf{[4]} \) \((A, \tau_D)\) cannot be uniform.

Let \( A \) be the algebra \( \mathbb{C} \times C_c^\infty(\mathbb{R}) \) endowed with the compact open topology \( \tau \). It is a weakly \( \sigma^* \)-compact-regular uniform lmca. Since it is inverse closed in \( C(\mathbb{R}) \), it

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is advertibly complete. Take \((B, \tau_K)\) as above. Let \(\phi: (A, \tau) \to (B, \tau_K), \phi(f) = f\). By Theorem 1, \(\phi^{-1}\) is continuous.

(4) Let \(U \subset \mathbb{C}^d\) be open. Let \(H(U)\) be the uniform Frechet algebra of all holomorphic functions on \(U\) with the compact open topology. Let \(H^\infty(U) = \{ f \in H(U) : f \text{ is bounded} \}\), a uniform Banach algebra. Let \(X \subset \mathbb{C}^d\) be compact. Let \(H(K)\) be the algebra of holomorphic germs on \(X\). Choose a decreasing sequence \((U_n)\) of open neighbourhoods of \(X\) such that \(\bar{U}_{n+1} \subset U_n\) and \(\bar{U}_{n+1}\) is compact. In view of the continuous embeddings

\[
\cdots \to H^\infty(U_n) \to H(U_n) \to H^\infty(U_{n+1}) \to H(U_{n+1}) \to \cdots,
\]

\(H(X)\) can be realized as inductive limits \(H(X) = \lim H(U_n) = \lim H^\infty(U_n)\), its topology \(\tau\) being the finest locally convex topology making all \(\phi_n: H(U_n) \to H(X)\), \(\phi_n(f) = f|_X\) and similarly making all \(\psi_n : H^\infty(U_n) \to H(X)\), \(\psi_n(f) = f/X\) continuous.

None of \(H^\infty(U)\) and \(H(U)\) is weakly regular. Note that \((H(X), \tau)\) is a complete semisimple \(Q\)-algebra [Ma, p. 134]. If \(H(U_n)\) is weakly regular, then by Theorem 4 it becomes a Banach algebra and \(\phi_n\) becomes a homeomorphism. If \(H^\infty(U_n)\) is weakly regular, then \(\psi_n\) becomes a homeomorphism. Either of these forces \(H(X)\) to be a uniform Banach algebra. Being a complete, non-normed \(Q\)-algebra, \(H(X)\) is not a uniform algebra [BD].

(5) Let \(D = \{ z \in \mathbb{C} : |z| < 1 \}\), \(Y = D \times [0, 1]\), \(Z = \bar{D} \times [0, 1]\). Let \(A = \{ f \in C(Y) : f\) is holomorphic on \(D \times \{0\}\}\), \(B = \{ f \in C(Z) : f\) is holomorphic on \(D \times \{0\}\}\). Let \(0 < r < 1\). Let \(\| f \|_r = \sup \{|f(x)| : x \in \bar{D} \times [0, 1]\}\) (\(f \in A\)); \(\| f \|_r = \sup \{|f(x)| : x \in \bar{D} \times [0, 1]\}\) (\(f \in B\)). Each of \(A\) and \(B\) with the topology defined respectively by \(\{ \| f \|_r : 0 < r < 1 \}\) and \(\{ |f|_r : 0 < r < 1 \}\) is a uniform Frechet algebra. Any \(f \in C(Y)\) (resp. \(f \in C(Z)\)) vanishing on \(D \times \{0\}\) is in \(A\) (resp. in \(B\)). This implies that both \(A\) and \(B\) are weakly regular, not regular. Thus each of \(A\) and \(B\) fails to support a continuous norm.

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**References**


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