PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 4, Pages 963–973
S 0002-9939(99)05080-7
Article electronically published on July 28, 1999

HILBERT COEFFICIENTS AND THE ASSOCIATED GRADED RINGS

HSIN-JU WANG

(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. Let $(R,\mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R. In this paper, we prove that if $\sum_{n=1}^{\infty}\lambda(I^n/I^{n-1}J)-e_1(I)=1$ for some minimal reduction J of I, then depth $G(I)\geq d-2$.

1. Introduction

Let $(R,\mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field and I be an $\mathfrak{m}$-primary ideal of R. Let $G(I)=\bigoplus_{n\geq 0} I^n/I^{n+1}$ be the associated graded ring of R. During the past years, many commutative algebraists tried to estimate the depth of $G(I)$ for ideals I having good properties. In 1978, Valabrega and Valla obtained in [6] that $G(I)$ is Cohen-Macaulay if and only if there exists a minimal reduction J of I such that $I^n\cap J=I^{n-1}J$ for all n. Later on, Guerrieri studied the so called Valabrega-Valla module and made the following conjecture in her paper [1].

Conjecture 1. *If* $\sum_{n=1}^{\infty}\lambda(I^n\cap J/I^{n-1}J)=t$ *for some minimal reduction* J *of* I, *then depth* $G(I)\geq d-t$.

On the other hand, Sally in [5] studied the depth of $G(\mathfrak{m})$ by considering the classical bound of Abhyankar on the multiplicity e of R; namely, $e\geq\mu(\mathfrak{m})-d+1$, where $\mu(I)$ stands for the minimal number of a generating set of I. She first studied the case of rings with minimal multiplicity, i.e., $e=\mu(\mathfrak{m})-d+1$, then the cases $e-(\mu(\mathfrak{m})-d+1)=1,2$. Recently, Huckaba and Marley showed in [4] that if one considers the first coefficient $e_1(I)$, then $e_1(I)$ is bounded above by $\sum_{n=1}^{\infty}\lambda(I^n/I^{n-1}J)$ for any minimal reduction J of I, and later Huckaba and Vaz Pinto independently showed that depth $G(I)\geq d-1$ if $e_1(I)=\sum_{n=1}^{\infty}\lambda(I^n/I^{n-1}J)$ for some minimal reduction J of I.

Received by the editors October 3, 1997 and, in revised form, May 19, 1998.
1991 *Mathematics Subject Classification.* Primary 13A30, 13D40, 13H10.
Key words and phrases. Hilbert coefficient, associated graded ring.

In a similar fashion to what Sally did with the Abhyankar's bound, we can raise the following conjecture on the depth of $G(I)$ by considering the difference of $e_1(I)$ and $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$.

Conjecture 2. *If* $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = t$, *then depth* $G(I) \geq d-1-t$.

One can see in section 2 that Conjecture 1 holds if we can give an affirmative answer to Conjecture 2. In this paper, we are able to show, by using a method developed in [8] concerning the *Sally module* defined in [7], that if $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = 1$, then depth $G(I) \geq d-2$. Hence the Conjecture 1 holds if $t \leq 2$.

2. Preliminaries

Throughout, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R and J a minimal reduction of I. Let $G(I)$ be the associated graded ring of R. An element $x \in I \setminus I^2$ is called *superficial* for I if $(0 :_{G(I)} x^*)_n = 0$ for all n sufficiently large. Here, x^* denotes the image of x in $I/I^2 \subseteq G(I)$. A sequence $x_1, \ldots, x_k$ is called *superficial sequence* for I if x_1 is superficial for I and x_i is superficial for $I/(x_1, \ldots, x_{i-1})$. In [3], Huckaba proved that if *dim* $R = 1$, then $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$ for any minimal reduction J of I; therefore it is easy to see the following:

Lemma 2.1. *If dim* $R = d$ *and* $x_1, \ldots, x_{d-1} \in J$ *is a superficial sequence for* I, *then*

$$e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x_1, \ldots, x_{d-1}))).$$

In [4], Huckaba and Marley gave in Lemma 2.2 a sufficient conditions for $G(I)$ having positive depth. We restate it here in the following special form.

Lemma 2.2. *Let* $x \in J$ *be a superficial element for* I. *If depth* $G(I/(x)) > 0$, *then depth* $G(I) > 0$.

Corollary 2.3. *Let* $(R, \mathfrak{m})$ *be a* 3*-dimensional Cohen-Macaulay local ring with infinite residue field. Let* I *be an* $\mathfrak{m}$*-primary ideal of* R *and* J *be a minimal reduction of* I. *Suppose that*

$$\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = 1.$$

Let $x \in J$ *be a superficial element for* I. *If* $\sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x))) = e_1(I)$, *then depth* $G(I) > 0$.

Proof. The conclusion follows from Lemma 2.2 and the fact ([3, Theorem 3.1]) that if $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) = e_1(I)$, then depth $G(I) \geq \dim R - 1$. ∎

The following two lemmas are easy to derive; we leave the proofs to the reader.

Lemma 2.4. *Let $(R,\mathfrak{m})$ be a 3-dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R and $J=(x_1,x_2,x_3)$ be a minimal reduction of I. Let $N\geq 2$. Suppose that $I^n\cap(x_i,x_j)\subseteq I^{n-1}J$ $\forall n<N$ and $\forall i,j\in\{1,2,3\}$. Then $\forall n<N$ and $\forall m\geq 1$,*

(1) $I^n : x_i = I^{n-1}$ $\forall i$.
(2) $I^nJ^m : x_i = I^nJ^{m-1}$ $\forall i$.
(3) $I^n : x_2 = I^n : x_3 = I^{n-1}$ *(mod x_1).*
(4) $I^nJ^m : x_2 = I^nJ^m : x_3 = I^nJ^{m-1}$ *(mod x_1).*
(5) *Let $\lambda_0,\dots,\lambda_t$ be either unit or 0 but not all 0. Let $s\in R$ be such that $s(\sum_{i=0}^{t}\lambda_i x_1^{n-i}x_2^i)\in I^nJ^m$. Then $s\in I^nJ^{m-t}$ if $m\geq t$ or $s\in I^{n-t+m}$ if $m<t$.*

If, moreover, $I^N\cap(x_1)\subseteq I^{N-1}J$, then $I^N : x_1 = I^{N-1}$.

Lemma 2.5. *Let $N\geq 2$. If $a_1,\dots,a_n\in I^N$ not all in $I^{N-1}J$, then there are only finite number of units λ such that $\sum_{i=1}^{n}a_i\lambda^{i-1}\in I^{N-1}J$.*

The following proposition presents a relation between the two conjectures stated in the previous section.

Proposition 2.6. *If Conjecture 2 has a positive answer, then so does Conjecture 1.*

Suppose that Conjecture 2 holds. Let I be an $\mathfrak{m}$-primary ideal of R and let J be a minimal reduction of I. Let $t=\sum_{n=1}^{\infty}\lambda(\frac{I^n\cap J}{I^{n-1}J})$. Then, by Lemma 2.1, for any superficial sequence $x_1,\dots,x_{d-1}\in J$ for I,

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda(I^n/I^{n-1}J)-e_1(I)&=\sum_{n=1}^{\infty}\lambda(\frac{I^n\cap(x_1,\dots,x_{d-1})+I^{n-1}J}{I^{n-1}J})\\&\leq\sum_{n=1}^{\infty}\lambda(\frac{I^n\cap J}{I^{n-1}J})=t.\end{aligned}$$

Let k be the least integer such that $\lambda(\frac{I^k\cap J}{I^{k-1}J})\neq 0$. Then, by [2, Lemma 3.1], $\lambda(\frac{I^k\cap(x_1,\dots,x_{d-1})+I^{k-1}J}{I^{k-1}J})<\lambda(\frac{I^k\cap J}{I^{k-1}J})$, so that $\sum_{n=1}^{\infty}\lambda(I^n/I^{n-1}J)-e_1(I)\leq t-1$. Therefore, depth $G(I)\geq d-t$ by assumption. This shows that Conjecture 1 holds.

3. Main theory

The goal of this section is to prove the following:

Theorem 3.1. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d\geq 2$ with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R. Suppose that there is a minimal reduction J of I such that $\sum_{n=0}^{\infty}\lambda(I^{n+1}/I^nJ)-e_1(I)=1$. Then depth $G(I)\geq d-2$.*

By [5], it suffices to consider the case $d = 3$, so we assume in the following that $d = 3$. We also assume now that Theorem 3.1 doesn't hold. We shall reach a contradiction later.

Let $x_1, x_2 \in J$ be a superficial sequence of I; then, by Corollary 2.3, we have for $i = 1, 2$,

$$\sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x_i))) - e_1(I) = 1. \tag{1}$$

Moreover, by Lemma 2.1, we have

$$\sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x_1, x_2))) - e_1(I) = 0. \tag{2}$$

Let $\{x, y, z\}$ be a minimal generating set of J. Consider the exact sequence:

$$0 \longrightarrow T_{k,n} \longrightarrow \bigoplus^{\binom{n+2}{2}} I^k/I^{k-1}J \xrightarrow{\phi_n} S_{k,n} = I^k J^n/I^{k-1}J^{n+1} \longrightarrow 0,$$

where $\phi_n = (x^n, x^{n-1}y, x^{n-1}z, \ldots, z^n)$ and $T_{k,n} = ker(\phi_n)$. From the proof of [8, Theorem 2.4], we see that there is an unique integer $N \geq 2$ such that $T_{N,n} \neq 0$ for some positive integer n. Notice that N is independent of the choice of a generating set of J since $S_{k,n}$ and $I^k/I^{k-1}J$ are. As $R/\mathfrak{m}$ is infinite, we may, after elementary transformation of x, y and z, require that $\{x, y, z\}$ satisfies the following conditions.

Proposition 3.2. *There is a generating set $\{x, y, z\}$ of J satisfying the following conditions:*

(i) *$\{x, y\}$, $\{x, z\}$, $\{y, z\}$ and $\{z\}$ are all superficial sequences for I.*
(ii) *$I^n \cap (x)$, $I^n \cap (y)$ and $I^n \cap (z)$ are all contained in $I^{n-1}J$ $\forall n$.*
(iii) *$I^n \cap (x, y)$, $I^n \cap (x, z)$ and $I^n \cap (y, z)$ are all contained in $I^{n-1}J$ $\forall n \neq N$. Moreover,*

$$\lambda(\frac{I^N \cap (x, y) + I^{N-1}J}{I^{N-1}J}) = \lambda(\frac{I^N \cap (x, z) + I^{N-1}J}{I^{N-1}J}) = \lambda(\frac{I^N \cap (y, z) + I^{N-1}J}{I^{N-1}J}) = 1.$$

Proof. (Sketch.) Notice that (ii) follows from (i) and (1); therefore we need only to show (i) and (iii).

Let $\{x, y, z\}$ be a generating set of J. Let n be an integer such that $T_{N,n} \neq 0$. Then there are $a_{ijk} \in I^N$ not all in $I^{N-1}J$ such that $\sum_{i+j+k=n} a_{ijk} x^i y^j z^k \in I^{N-1}J^{n+1}$. By Lemma 2.5, we may, after elementary transformation of x, y and z, assume that a_{n00}, a_{0n0} and a_{00n} are not in $I^{N-1}J$.

Next, we can use prime avoidance and Corollary 2.3 to replace $\{x, y, z\}$ by elements of the set $\{x + \alpha y + \beta z\}$ so that the condition (i) holds without changing the condition that the coefficients of x^n, y^n and z^n are not in $I^{N-1}J$.

Since $\sum_{i+j+k=n} a_{ijk} x^i y^j z^k \in (I^{N-1}J)J^n$, there are $a_1, a_2, a_3 \in I^{N-1}J$ such that $a_{n00} - a_1 \in I^N \cap (y, z)$, $a_{0n0} - a_2 \in I^N \cap (x, z)$ and $a_{00n} - a_3 \in I^N \cap (x, y)$. Thus the condition (iii) holds by condition (i) and (2) . □

Let $\{x, y, z\}$ be a generating set of J satisfying the conditions of Proposition 3.2. Let t be chosen least such that $T_{N,n} \neq 0$ for $n \geq t$. Then there are elements $\{a_{ijk} \mid i+j+k=t\}$ not all in $I^{N-1}J$ such that $\sum a_{ijk}x^iy^jz^k \in I^{N-1}J^{t+1}$. If $\{a_{0jk} \mid j+k=t\} \subseteq I^{N-1}J$, then $x(\sum a_{ijk}x^{i-1}y^jz^k) \in I^{N-1}J^{t+1}$, so that by Lemma 2.4, $\sum a_{ijk}x^{i-1}y^jz^k \in I^{N-1}J^t$; it follows that $T_{N,t-1} \neq 0$, which contradicts the choice of t. Therefore, $\{a_{0jk} \mid j+k=t\}$ are not all in $I^{N-1}J$.

Let the overbars denote $\mathrm{mod}(x)$ in the following. Consider the exact sequence

$$0 \longrightarrow \bar{T}_{k,n} \longrightarrow \bigoplus^{n+1} \bar{I}^k/\bar{I}^{k-1}\bar{J} \xrightarrow{\bar{\phi}_n} \bar{I}^k\bar{J}^n/\bar{I}^{k-1}\bar{J}^{n+1} \longrightarrow 0,$$

where $\bar{\phi}_n = (\bar{y}^n, \bar{y}^{n-1}\bar{z}, \ldots, \bar{z}^n)$ and $\bar{T}_{k,n} = ker(\bar{\phi}_n)$. Since $\sum_{n=1}^{\infty} \lambda(\bar{I}^n/\bar{I}^{n-1}\bar{J}) - e_1(\bar{I}) = 1$, there is an unique integer N' such that $\bar{T}_{N',n} \neq 0$ for some n. However, by the following remark, $\{\bar{a}_{0jk} \mid j+k=t\}$ are not all in $\bar{I}^{N-1}\bar{J}$. Since $\sum \bar{a}_{0jk}\bar{y}^j\bar{z}^k \in \bar{I}^{N-1}\bar{J}^{t+1}$, we see that $N' = N$.

Remark 3.3. Let $b \in I^N \setminus I^{N-1}J$; then $\bar{b} \notin \bar{I}^{N-1}\bar{J}$ by the fact that $I^N : x = I^{N-1}$.

In the sequel, let R^0 denote the set $\{\text{units of } R\} \cup \{0\}$ and R_n $(n \geq 0)$ denote the set $\{f \mid f = \sum_{i=0}^{n} \lambda_i y^{n-i}z^i \text{ for some } \lambda_i \in R^0\}$.

Lemma 3.4. *Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R and $J = (y, z)$ be a minimal reduction of I. Let $N \geq 2$. Suppose that*

(i) $\lambda(\frac{I^N \cap (y) + I^{N-1}J}{I^{N-1}J}) = \lambda(\frac{I^N \cap (z) + I^{N-1}J}{I^{N-1}J}) = 1$, *and*

(ii) $\forall n < N$ *and* $\forall m \geq 1$, $I^n : y = I^n : z = I^{n-1}$ *and* $I^nJ^m : y = I^nJ^m : z = I^nJ^{m-1}$.

Consider the exact sequence

$$0 \longrightarrow T_{N,n} \longrightarrow \bigoplus^{n+1} I^N/I^{N-1}J \xrightarrow{\phi_n} I^NJ^n/I^{N-1}J^{n+1} \longrightarrow 0,$$

where $\phi_n = (y^n, y^{n-1}z, \ldots, z^n)$ *and* $T_{N,n} = ker(\phi_n)$.

Suppose that $T_{N,n} \neq 0$ for some n. Let l be the least integer such that $T_{N,n} \neq 0$ for all $n \geq l$. Let $a_0, \ldots, a_l \in I^N$ not all in $I^{N-1}J$ such that $\sum_{i=0}^{l} a_iy^{l-i}z^i \in I^{N-1}J^{l+1}$. Then the following hold:

(1) $a_0 \notin I^{N-1}J$ *and* $\mathfrak{m}a_i \subseteq I^{N-1}J$ $\forall i$.

(2) *If* $\sum_{i=0}^{n} b_iy^{n-i}z^i \in I^{N-1}J^{n+1}$ *for some* $b_i \in I^N$ *and for some* n, *then* $\mathfrak{m}b_i \subseteq I^{N-1}J$ $\forall i$ *and there are* $\lambda_0, \ldots, \lambda_{n-l} \in R^0$ *such that* $b_j - \sum_{i=0}^{j} a_i\lambda_{j-i} \in I^{N-1}J$.

(Conventions: $a_i = 0$ *if* $i > l$ *and* $\lambda_i = 0$ *if* $i > n-l$.*)*

Proof. By the choice of l and the fact that $I^{N-1}J^m : z = I^{N-1}J^{m-1}$, we obtain that $a_0 \notin I^{N-1}J$. Since $a_0 \in I^N \cap (z) + I^{N-1}J$, we have, by assumption, $\mathfrak{m}a_0 \subseteq I^{N-1}J$.

Furthermore, let $w \in \mathfrak{m}$; then $\sum_{i=1}^{l}(wa_i)y^{l-i}z^{i-1} \in I^{N-1}J^l$ by the assumption that $I^{N-1}J^m : z = I^{N-1}J^{m-1}$. Again, by the choice of l, wa_i must belongs to $I^{N-1}J$ for all i. This proves (1).

To see (2), we may assume that $n \geq l$ and $b_0 \in I^N \setminus I^{N-1}J$. Then $b_0 \in I^N \cap (z) + I^{N-1}J$. Since $a_0 \in I^N \cap (z) + I^{N-1}J$; there is a unit λ_0 such that $b_0 - \lambda_0 a_0 \in I^{N-1}J$, therefore, $\sum_{i=0}^{n-1}(b_{i+1} - \lambda_0 a_{i+1})y^{n-1-i}z^i \in I^{N-1}J^n$.

If $n = l$, then, by the choice of l, $b_i - \lambda_0 a_i \in I^{N-1}J\ \forall i$; hence $\mathfrak{m}b_i \subseteq I^{N-1}J\ \forall i$. If $n > l$, then by induction $\mathfrak{m}b_i \subseteq I^{N-1}J\ \forall i$ and there are $\lambda_1, \ldots, \lambda_{n-l} \in R^0$ such that $\forall j \geq 1\ b_j - \lambda_0 a_j = \sum_{i=0}^{j-1} a_i \lambda_{j-i}$. This proves (2). □

Let $\{x, y, z\}$ be a generating set of J satisfying the conditions of Proposition 3.2. Let the overbars denote mod(x) in the following. By Lemma 2.4, it is easy to check that $\bar{R}$, $\bar{y}$ and $\bar{z}$ satisfy all the assumptions of Lemma 3.4. Let $l = min\{n \mid \bar{T}_{N,n} \neq 0\}$, where $\bar{T}_{N,n}$ is defined as the above; then there are $a_0, \ldots, a_l \in I^N$ not all in $I^{N-1}J$ such that $\sum \bar{a}_i \bar{y}^{l-i}\bar{z}^i \in \bar{I}^{N-1}\bar{J}^{l+1}$. Let $u = \sum_{i=0}^{l} a_i y^{l-i} z^i$; then u has the following property.

Lemma 3.5. *If* $\sum_{i=0}^{n} b_i y^{n-i} z^i \in (x) + I^{N-1}J^{n+1}$ *for some* $b_i \in I^N$, *then*

(1) $\mathfrak{m}b_i \subseteq I^{N-1}J\ \forall i$.

(2) *There is an* $f \in R_{n-l}$ *such that* $\sum_{i=0}^{n} b_i y^{n-i} z^i - fu \in I^{N-1}J^{n+1}$.

Proof. By Lemma 3.4, $\mathfrak{m}\bar{a}_i \in \bar{I}^{N-1}\bar{J}$; hence $\mathfrak{m}a_i \subseteq I^{N-1}J + (x) \cap I^N \subseteq I^{N-1}J$. Moreover, there are $\lambda_0, \ldots, \lambda_{n-l} \in R^0$ such that $\bar{b}_j - \sum_{i=0}^{j} \bar{a}_i \lambda_{j-i} \in \bar{I}^{N-1}\bar{J}$; then $b_j - \sum_{i=0}^{j} a_j \lambda_{j-i} \in I^{N-1}J + (x) \cap I^N \subseteq I^{N-1}J$. Therefore $\mathfrak{m}b_i \subseteq I^{N-1}J\ \forall i$. Let $f = \sum_{i=0}^{n-l} \lambda_i y^{n-l-i} z^i \in R_{n-l}$. Then

$$\sum_{i=0}^{n} b_i y^{n-i} z^i - fu = \sum (b_j - \sum_{i=0}^{j} a_j \lambda_{j-i}) y^{n-j} z^j \in I^{N-1}J^{n+1}.$$

□

Remark 3.6. If, in Lemma 3.5, at least one of the b_i is not in $I^{N-1}J$, then $n \geq l$ and we can choose f to be a nonzero element of R_{n-l}.

Since depth $G(I) = 0$, $I^n : J \neq I^{n-1}$ for some n. Let $N' = min\{n \mid I^n : J \neq I^{n-1}\}$. Since $I^n : x = I^{n-1}\ \forall n \leq N$, $N' > N$. Let $s \notin I^{N'-1}$ such that $sJ \subseteq I^{N'}$.

Lemma 3.7. *There exists an element s' with $s-s'\in I^{N'-1}$ such that $s'y\in I^NJ^{N'-N}$, $s'z\in I^NJ^{N'-N}$ and $s'x\in I^N(y,z)^{N'-N}$.*

Proof. Suppose we have shown for some $k>N$ that there is an element s' with $s-s'\in I^{N'-1}$ such that $s'y\in I^kJ^{N'-k}$, $s'z\in I^kJ^{N'-k}$ and $s'x\in I^k(y,z)^{N'-k}$. Then there are $a_i,b_{ijk}\in I^k$ such that $s'x=\sum a_iy^{t-i}z^i$ and $s'y=\sum b_{ijk}x^iy^jz^k$, where $t=N'-k$. Therefore,

$$\sum a_iy^{t-i+1}z^i-\sum b_{ijk}x^{i+1}y^jz^k=0\in I^{k-1}J^{t+2}.$$

Since $k\neq N$, $T_{k,t+1}=0$; therefore, a_i,b_{ijk} are in $I^{k-1}J$. It follows that $s'x\in I^{k-1}J^{t+1}$ and $s'y\in I^{k-1}J^{t+1}$. Similarily, we can get $s'z\in I^{k-1}J^{t+1}$.

Finally, from the expression $s'x\in I^{k-1}J^{t+1}$, we see that there is a $w\in I^{k-1}J^t\subseteq I^{N'-1}$ such that $(s'-w)x\in I^{k-1}(y,z)^{t+1}$ and $(s'-w)J\subseteq I^{k-1}J^{t+1}$. This completes the proof. □

By Lemma 3.7 we may assume that s satisfies the conditions $sJ\in I^NJ^t$ and $sx\in I^N(y,z)^t$, where $t=N'-N$. Since $s\notin I^{N'-1}$, $sx\notin I^{N-1}J^{t+1}$ by Lemma 2.4. Therefore, by Remark 3.6 and the condition $sx\in I^N(y,z)^t$, we obtain that $t\geq l$. Hence by Lemma 3.5, there is a nonzero element $f_0\in R_{t-l}$ such that

$$sx-f_0u\in I^{N-1}J^{t+1}. \tag{3}$$

In what follows, let $\mathbf{k}$ be the residue field of R. If $f=\sum_{i=0}^{n}\lambda_iy^{n-i}z^i\in R_n$, then we associate to f a homogeneous polynomial $T(f)=F=\sum_{i=0}^{n}\bar{\lambda}_iY^{n-i}Z^i$ in $\mathbf{k}[Y,Z]$. (Here, the overbars denote mod($\mathfrak{m}$).)

Remark 3.8. From (3) and Lemma 3.5, we obtain $\mathfrak{m}sx\subseteq I^{N-1}J^{t+1}$, and therefore $\mathfrak{m}s\subseteq I^{N-1}J^t$. Moreover, if f and g are two elements of R_n such that $F=G$, then $f-g\in\mathfrak{m}R_n$; it follows that $sf-sg\in I^{N-1}J^{n+t}$.

From (3), Lemma 3.5 and Remark 3.6 we have the following corollary.

Corollary 3.9. *Let $B_0=\sum b_iy^{n-i}z^i$ with $b_i\in I^N$ not all in $I^{N-1}J$ and $B_j\in(y,z)^{n-j}I^N$. Suppose that $B_o+B_1x+\cdots\in I^{N-1}J^{n+1}$. Then there is a nonzero element $h\in R_{n-l}$ such that $B_0+hu\in I^{N-1}J^{n+1}$ and $sh-f_0(B_1+B_2x+\cdots)\in I^{N-1}J^{n+t-l}$.*

Proof. By Lemma 3.5 and Remark 3.6, there is a nonzero element $h\in R_{n-l}$ such that $B_0+hu\in I^{N-1}J^{n+1}$. By (3), $shx-f_0x(B_1+B_2x+\cdots)\in I^{N-1}J^{n+t-l+1}$. It follows by Lemma 2.4 that $sh-f_0(B_1+B_2x+\cdots)\in I^{N-1}J^{n+t-l}$. □

Let Q_j ($j\geq 0$) be the set of all integers n such that there exists a nonzero element $f\in R_n$ with $sf\in\sum_{i=0}^{j}x^i(y,z)^{t+n-1-i}I^N+I^{N-1}J^{t+n}$. (For example, since $sy\in I^NJ^t$, $1\in Q_t$.)

Let $m=min\{j\mid Q_j\neq\emptyset\}$. Let k be the smallest integer in Q_m; then there is a nonzero element $f\in R_k$ with $sf\in\sum_{i=0}^{m}x^i(y,z)^{t+k-1-i}I^N+I^{N-1}J^{t+k}$, so that there are $A_i \in (y,z)^{t+k-1-i}I^N$ such that

$$sf - \sum_{i=0}^{m} A_i x^i \in I^{N-1}J^{t+k}. \tag{4}$$

Lemma 3.10. *Let $C_i \in (y,z)^{n-i}I^N$ such that $\sum_{i=0}^{m} C_i x^i \in I^{N-1}J^{n+1}$. Then $C_i \in I^{N-1}J^{n+1-i}$ and all the coefficients of C_i are all in $I^{N-1}J$.*

Proof. If $m = 0$, then $C_0 \in I^{N-1}J^{n+1}$. If the coefficients of C_0 are not all in $I^{N-1}J$, then by Corollary 3.9 there is a nonzero element $h \in R_{n-l}$ such that $sh \in I^{N-1}J^{n+t-l}$. This gives the contradiction $s \in I^{N-1}J^t$ by Lemma 2.4.

Assume that $m \geq 1$ and the assertion is false. Let j be the least integer such that the coefficients of C_j are not all in $I^{N-1}J$. Then, by Lemma 2.4, $\sum_{i=j}^{m} C_i x^{i-j} \in I^{N-1}J^{n+1-j}$; hence by Corollary 3.9 there is a nonzero element $h \in R_{n-j-l}$ such that $sh - f_0(\sum_{i=j+1}^{m} C_i x^{i-j-1}) \in I^{N-1}J^{n+t-j-l}$, which contradicts the choice of m. The assertion now follows. □

Let p be the maximal integer such that $Z^p|F$. Let $F' = F/Z^p$.

Lemma 3.11. *If g is a nonzero element of R_n such that $sg \in \sum_{i=0}^{m} x^i (y,z)^{t+n-1-i}I^N + I^{N-1}J^{t+n}$, then $F|G$.*

Proof. Let $B_i \in (y,z)^{t+n-1-i}I^N$ such that

$$sg - \sum_{i=0}^{m} B_i x^i \in I^{N-1}J^{t+n}. \tag{5}$$

Write $G = G'Z^q$ with $(G', Z) = 1$. Let $g' \in R_{n-q}$ and $f' \in R_{k-p}$ such that $T(g') = G'$ and $T(f') = F'$. By Remark 3.8, we may assume that $f = f'z^p$ and $g = g'z^q$.

Assume that $q < p$. Then from (4) and (5), we obtain that

$$\sum_{i=0}^{m} (f'z^{p-q}B_i - g'A_i)x^i \in I^{N-1}J^{t+k+n-q}, \tag{6}$$

so that by Lemma 3.10 all the coefficients of $f'z^{p-q}B_i - g'A_i$ are all in $I^{N-1}J$. Since $(G', Z) = 1$, there are $A'_i \in (y,z)^{t+k-2-i}I^N$ such that $A_i - zA'_i \in I^{N-1}J^{t+k-i}$; it follows from (4) that $s(f'z^{p-1}) - \sum_{i=0}^{m} A'_i x^i \in I^{N-1}J^{t+k-1}$, which contradicts the choice of k. Hence $q \geq p$.

Write $G = G'Z^p$. Then $G' = F'Q + Z^{n-k+1}G''$ for some Q and G''. Suppose $G'' \neq 0$. Then $deg\ G'' = deg\ F' - 1$. Let $g'' \in R_{k-p-1}$ such that $T(g'') = G''$; then

$$sg''z^{n-k+1+p} - \sum_{i=0}^{m} C_i x^i \in I^{N-1}J^{t+n} \tag{7}$$

for some $C_i \in (y,z)^{t+n-1-i}I^N$.

From (4) and (7), we obtain that

$$\sum_{i=0}^{m}(g''z^{n-k+1}A_i - f'C_i)x^i \in I^{N-1}J^{t+n+k-p};$$

therefore, by Lemma 3.10, all the coefficients of $g''z^{n-k+1}A_i - f'C_i$ are in $I^{N-1}J$. Since $(F', Z) = 1$, there are $C'_i \in (y,z)^{t+k-2-i}I^N$ such that $C_i - z^{n-k+1}C'_i \in I^{N-1}J^{t+n-i}$. Hence from (7), $sg''z^p - \sum_{i=0}^{m} C'_i x^i \in I^{N-1}J^{t+k-1}$, which contradicts the choice of k. Therefore, $G'' = 0$ and $F|G$. □

In fact, Lemma 3.11 can be improved as follows.

Lemma 3.12. *Let $m \le m' \le t$. If g is a nonzero element of R_n such that $sg \in \sum_{i=0}^{m'} x^i(y,z)^{t+n-1-i}I^N + I^{N-1}J^{t+n}$, then $F|G$.*

Proof. We use induction on m'. If $m' = m$, then this is the content of Lemma 3.11.

Assume that $m' > m$. Let $H_0 = \dfrac{F_0}{(F_0, F)}$ and let $n_0 = deg\ H_0$. Let $h_0 \in R_{n_0}$ such that $T(h_0) = H_0$. Let $B_i \in (y,z)^{t+n-1-i}I^N$ such that

$$sg - \sum_{i=0}^{m'} B_i x^i \in I^{N-1}J^{t+n}. \tag{8}$$

Let $H = (F,G)$ and let $k' = deg\ H$. Then $F = F'H$ and $G = G'H$ for some F' and G'. Let $f' \in R_{k-k'}$ and $g' \in R_{n-k'}$ such that $T(f') = F'$ and $T(g') = G'$. By Remark 3.8, we may assume that $f = f'h$ and $g = g'h$. Set $A_i = 0\ \ \forall i > m$. From (4) and (8), we obtain that $\sum_{i=0}^{m'}(f'B_i - g'A_i)x^i \in I^{N-1}J^{t+n+k-k'}$. Therefore, by Lemma 3.5, there is an element $h_1 \in R_{t+n+k-k'-l-1}$ such that $f'B_0 - g'A_0 + h_1u \in I^{N-1}J^{t+n+k-k'}$, and then, by (3), $sh_1 - f_0(\sum_{i=0}^{m'-1}(f'B_{i+1} - g'A_{i+1})x^i) \in I^{N-1}J^{2t+n+k-k'-l-1}$. Hence, by induction, $F|H_1$. Moreover, since (F_0, F) is a common factor of H_1 and F_0, there is an element $g_1 \in R_{n_0+n+k-k'-1}$ such that $sg_1 - h_0(\sum_{i=0}^{m'-1}(f'B_{i+1} - g'A_{i+1})x^i) \in I^{N-1}J^{t+n_0+n+k-k'-1}$. Therefore, by induction, $F|G_1$. Hence there is an element $g'_1 \in R_{n_0+n-k'-1}$ such that $G_1 = G'_1F$.

Suppose that $m = 0$. Since $f'B_0 - g'A_0 + h_1u \in I^{N-1}J^{t+n+k-k'}$, by Lemma 3.10, $g'hA_o - fA'_0 \in I^{N-1}J^{t+n+k}$ for some $A'_0 \in (y,z)^{t+n-1}I^N$. Therefore by (4), $sfg'h - fA'_0 \in I^{N-1}J^{t+n+k}$, and then $sg'h - A'_0 \in I^{N-1}J^{t+n}$ by Lemma 2.4, so that by Lemma 3.11, $F|G'H$. But $(F, G') = 1$; we obtain $F|H$.

Assume now that $m \ge 1$. We claim: There are integers $n_1, \ldots, n_{m'-m}, d_1, \ldots, d_{m'-m}$ and elements $h_j \in R_{n_j}$, $g_j \in R_{d_j}$ and $g'_j \in R_{d_j-k}$ satisfy the following conditions:

(i) $G_j = G'_jF$.

(ii) $h_0^{j-1}(f'B_{j-1} - g'A_{j-1}) + h_0^{j-2}g'_1A_{j-2} - \cdots - g'_{j-1}A_0 + h_ju \in I^{N-1}J^{n_j+l+1}$.

(iii) $sg_j - \sum_{i=0}^{m'-j} (h_0^j(f'B_{i+j} - g'A_{i+j}) - h_0^{j-1}g'_1A_{i+j-1} - \cdots - h_0g'_{j-1}A_{i+1})x^i \in I^{N-1}J^{d_j+t}$.

Suppose that we have constructed, for some $j \geq 1$, h_j, g_j and g'_j. Then from (i), (iii) and (4), we see that the element

$$h_0^j(f'B_j - g'A_j) - h_0^{j-1}g'_1A_{j-1} - \cdots - g'_jA_0 + \sum_{i=0}^{m'-j-1} (h_0^j(f'B_{i+j+1} - g'A_{i+j+1}) - h_0^{j-1}g'_1A_{i+j} - \cdots - g'_jA_{i+1})x^{i+1}$$

is in $I^{N-1}J^{d_j+t}$. Therefore, by Lemma 3.5, there is an element $h_{j+1} \in R_{n_{j+1}}$ for some n_{j+1} such that (ii) holds for $j+1$ and $F|H_{j+1}$ (cf. the construction of h_1) by induction. Moreover, from the construction of g_1, it is easy to see that there is an element $g_{j+1} \in R_{d_{j+1}}$ for some d_{j+1} such that (iii) holds for $j+1$. Since by induction $F|G_{j+1}$, there is an $g'_{j+1} \in R_{d_{j+1}-k}$ such that (i) holds. This proves the claim.

Set $j = m' - m$ in (iii) of the claim and compare with (4). We obtain that the element

$$\sum_{i=0}^{m}(h_0^{m'-m}(f'B_{i+m'-m} - g'A_{i+m'-m}) - h_0^{m'-m-1}g'_1A_{i+m'-m-1} - \cdots - g'_{m'-m}A_i)x^i$$

is in $I^{N-1}J^{d_{m'-m}+t}$. However by Lemma 3.10, $\forall i \leq m$,

$$h_0^{m'-m}(f'B_{i+m'-m} - g'A_{i+m'-m}) - h_0^{m'-m-1}g'_1A_{i+m'-m-1} - \cdots - g'_{m'-m}A_i \in I^{N-1}J^{d_{m'-m}+t-i} \tag{9}$$

From (ii) and (9), it is not hard to see that for $0 \leq i \leq m$ there are $A'_i \in (y,z)^{e_i}I^N$ for some e_i such that $hh_0^ig'^{i+1}A_i - fA'_i \in I^{N-1}J^{e_i+k+1}$. Therefore by (4), $shh_0^mg'^{m+1}f - f(\sum_{i=0}^{m} h_0^{m-i}g'^{m-i}A'_ix^i) \in I^{N-1}J^e$ for some e; it follows by Lemma 2.4 that

$$shh_0^mg'^{m+1} - \sum_{i=0}^{m} h_0^{m-i}g'^{m-i}A'_ix^i \in I^{N-1}J^{e-k}.$$

Finally, by Lemma 3.11, $F|G'^{m+1}HH_0^m$. Since $(F,G') = (F,H_0) = 1$, we have $F|H$. This completes the proof. □

Now, choose a unit λ so that $(Y+\lambda Z, F) = 1$. Since $s(y+\lambda z) \in \sum_{i=0}^{t} x^i(y,z)^{t-i}I^N + I^{N-1}J^{t+1}$, by Lemma 3.12, $F|Y+\lambda Z$, which contradicts the choice of λ. This proves Theorem 3.1.

By Proposition 2.6 and Theorem 3.1, we have the following corollary.

Corollary 3.13. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R. Suppose that there is a minimal reduction J of I such that $\sum_{n=1}^{\infty} \lambda(I^n \cap J/I^{n-1}J) = 2$. Then depth $G(I) \geq d-2$.*

Acknowledgment

I would like to thank the referee for a very careful reading of the paper, and many valuable comments, which improved the exposition a lot.

References

1. A. Guerrieri, *On the depth of the associated graded ring of an m-primary ideal of a Cohen-Macaulay local ring*, J. Algebra. **167** (1994), 745–757. MR **95h:**13004
2. ——, *On the depth of the associated graded ring*, Proc. A.M.S. **123** (1995), 11–20. MR **95c:**13002
3. S. Huckaba, *A d-dimensional extension of a lemma of Huneke's and formulas for the Hilbert coefficients*, Proc. Amer. Math. Soc. **124** (1996), 1393–1401. MR **96g:**13018
4. S. Huckaba and T. Marley, *Hilbert coefficients and the depth of associated graded rings*, J. London Math. soc.(2) **56** (1997), no. 1, 64–76. MR **98i:**13028
5. J. D. Sally, *Super-regular sequences*, Pacific J. Math. **84** (1979), 465–481. MR **81m:**13024
6. P. Valabrega and G.Valla, *Form rings and regular sequence*, Nagoya Math. J. **72** (1978), 93–101. MR **80d:**14010
7. W. V. Vasconcelos, *Hilbert functions, analytic spread, and Koszul homology*, Contemporary Math. **159** (1994), 401–422. MR **95a:**13006
8. H.-J Wang, *An interpretation of depth $G(I)$ and $e_1(I)$ via the Sally module*, Communications in Algebra **25,(1)** (1997), 303–309. MR **98a:**13035

Department of Mathematics, National Chung Cheng University, Minghsiung, Chiayi 621, Taiwan

E-mail address: hjwang@math.ccu.edu.tw