HILBERT COEFFICIENTS
AND THE ASSOCIATED GRADED RINGS

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Abstract. Let \((R, \mathfrak{m})\) be a \(d\)-dimensional Cohen-Macaulay local ring with infinite residue field. Let \(I\) be an \(\mathfrak{m}\)-primary ideal of \(R\). In this paper, we prove that if \(\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = 1\) for some minimal reduction \(J\) of \(I\), then \(\text{depth} \ G(I) \geq d - 2\).

1. Introduction

Let \((R, \mathfrak{m})\) be a \(d\)-dimensional Cohen-Macaulay local ring with infinite residue field and \(I\) be an \(\mathfrak{m}\)-primary ideal of \(R\). Let \(G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}\) be the associated graded ring of \(R\). During the past years, many commutative algebraists tried to estimate the depth of \(G(I)\) for ideals \(I\) having good properties. In 1978, Valabrega and Valla obtained in [6] that \(G(I)\) is Cohen-Macaulay if and only if there exists a minimal reduction \(J\) of \(I\) such that \(I^n \cap J = I^{n-1}J\) for all \(n\). Later on, Guerrieri studied the so called Valabrega-Valla module and made the following conjecture in her paper [1].

Conjecture 1. If \(\sum_{n=1}^{\infty} \lambda(I^n \cap J/I^{n-1}J) = t\) for some minimal reduction \(J\) of \(I\), then \(\text{depth} \ G(I) \geq d - t\).

On the other hand, Sally in [5] studied the depth of \(G(\mathfrak{m})\) by considering the classical bound of Abhyankar on the multiplicity \(e\) of \(R\); namely, \(e \geq \mu(\mathfrak{m}) - d + 1\), where \(\mu(I)\) stands for the minimal number of a generating set of \(I\). She first studied the case of rings with minimal multiplicity, i.e., \(e = \mu(\mathfrak{m}) - d + 1\), and then the cases \(e - (\mu(\mathfrak{m}) - d + 1) = 1, 2\). Recently, Huckaba and Marley showed in [4] that if one considers the first coefficient \(e_1(I)\), then \(e_1(I)\) is bounded above by \(\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)\) for any minimal reduction \(J\) of \(I\), and later Huckaba and Vaz Pinto independently showed that depth \(G(I) \geq d - 1\) if \(e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)\) for some minimal reduction \(J\) of \(I\).

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In a similar fashion to what Sally did with the Abhyankar’s bound, we can raise the following conjecture on the depth of $G(I)$ by considering the difference of $e_1(I)$ and $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$.

**Conjecture 2.** If $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = t$, then $\text{depth } G(I) \geq d - 1 - t$.

One can see in section 2 that Conjecture 1 holds if we can give an affirmative answer to Conjecture 2. In this paper, we are able to show, by using a method developed in [8] concerning the Sally module defined in [7], that if $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = 1$, then $\text{depth } G(I) \geq d - 2$. Hence the Conjecture 1 holds if $t \leq 2$.

2. PRELIMINARIES

Throughout, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field. Let $I$ be an $\mathfrak{m}$-primary ideal of $R$ and $J$ a minimal reduction of $I$. Let $G(I)$ be the associated graded ring of $R$. An element $x \in I \setminus I^2$ is called superficial for $I$ if $(0 :_{G(I)} x^*)_n = 0$ for all $n$ sufficiently large. Here, $x^*$ denotes the image of $x$ in $I/I^2 \subseteq G(I)$. A sequence $x_1, \ldots, x_k$ is called superficial sequence for $I$ if $x_1$ is superficial for $I$ and $x_i$ is superficial for $I/(x_1, \ldots, x_{i-1})$. In [3], Huckaba proved that if $\dim R = 1$, then $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$ for any minimal reduction $J$ of $I$; therefore it is easy to see the following:

**Lemma 2.1.** If $\dim R = d$ and $x_1, \ldots, x_{d-1} \in J$ is a superficial sequence for $I$, then

$$e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x_1, \ldots, x_{d-1}))).$$

In [3], Huckaba and Marley gave in Lemma 2.2 a sufficient conditions for $G(I)$ having positive depth. We restate it here in the following special form.

**Lemma 2.2.** Let $x \in J$ be a superficial element for $I$. If $\text{depth } G(I/(x)) > 0$, then $\text{depth } G(I) > 0$.

**Corollary 2.3.** Let $(R, \mathfrak{m})$ be a 3-dimensional Cohen-Macaulay local ring with infinite residue field. Let $I$ be an $\mathfrak{m}$-primary ideal of $R$ and $J$ be a minimal reduction of $I$. Suppose that

$$\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = 1.$$

Let $x \in J$ be a superficial element for $I$. If $\sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x))) = e_1(I)$, then $\text{depth } G(I) > 0$.

**Proof.** The conclusion follows from Lemma 2.2 and the fact ([3, Theorem 3.1]) that if $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) = e_1(I)$, then $\text{depth } G(I) \geq \dim R - 1$. \qed
The following two lemmas are easy to derive; we leave the proofs to the reader.

**Lemma 2.4.** Let \((R, m)\) be a 3-dimensional Cohen-Macaulay local ring with infinite residue field. Let \(I\) be an \(m\)-primary ideal of \(R\) and \(J = (x_1, x_2, x_3)\) be a minimal reduction of \(I\). Let \(N \geq 2\). Suppose that \(I^n \cap (x_1, x_2, x_3) \subseteq I^{n-1} J \forall n < N\) and \(\forall i, j \in \{1, 2, 3\}\). Then \(\forall n < N\) and \(\forall m \geq 1\),

1. \(I^n : x_i = I^{n-1} \forall i\).
2. \(I^n J^m : x_i = I^{n-1} J^m \forall i\).
3. \(I^n : x_2 = I^n : x_3 = I^{n-1} (\text{mod } x_1)\).
4. \(I^n J^m : x_2 = I^n J^m : x_3 = I^n J^{m-1} (\text{mod } x_1)\).
5. Let \(\lambda_0, \ldots, \lambda_t\) be either unit or 0 but not all 0. Let \(s \in R\) be such that

\[
s(\sum_{i=0}^{t} \lambda_i x_1^{n-i} x_2^1) \in I^n J^m.\]

Then \(s \in I^n J^{m-t}\) if \(m \geq t\) or \(s \in I^{n-t+m}\) if \(m < t\).

If, moreover, \(I^N \cap (x_1) \subseteq I^{N-1} J\), then \(I^n : x_1 = I^{n-1}\).

**Lemma 2.5.** Let \(N \geq 2\). If \(a_1, \ldots, a_n \in I^N\) not all in \(I^{N-1} J\), then there are only finite number of units \(\lambda\) such that \(\sum_{i=1}^{n} a_i \lambda^{i-1} \in I^{N-1} J\).

The following proposition presents a relation between the two conjectures stated in the previous section.

**Proposition 2.6.** If Conjecture 2 has a positive answer, then so does Conjecture 1.

Suppose that Conjecture 2 holds. Let \(I\) be an \(m\)-primary ideal of \(R\) and let \(J\) be a minimal reduction of \(I\). Let \(t = \sum_{n=1}^{\infty} \lambda(I^n \cap J)\). Then, by Lemma 2.1, for any superficial sequence \(x_1, \ldots, x_{d-1} \in J\) for \(I\),

\[
\sum_{n=1}^{\infty} \lambda(I^n / I^{n-1} J) - e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n \cap (x_1, \ldots, x_{d-1}) + I^{n-1} J) \frac{I^n \cap J}{I^{n-1} J}
\]

\[
\leq \sum_{n=1}^{\infty} \lambda(I^n \cap J) = t.
\]

Let \(k\) be the least integer such that \(\lambda(I^k \cap J) \neq 0\). Then, by [2 Lemma 3.1],

\[
\lambda(I^k \cap (x_1, \ldots, x_{d-1}) + I^{k-1} J) \frac{I^k \cap J}{I^{k-1} J} < \lambda(I^k \cap J),
\]

so that \(\sum_{n=1}^{\infty} \lambda(I^n / I^{n-1} J) - e_1(I) \leq t - 1\). Therefore, \(\text{depth } G(I) \geq d - t\) by assumption. This shows that Conjecture 1 holds.

**3. Main theory**

The goal of this section is to prove the following:

**Theorem 3.1.** Let \((R, m)\) be a Cohen-Macaulay local ring of dimension \(d \geq 2\) with infinite residue field. Let \(I\) be an \(m\)-primary ideal of \(R\). Suppose that there is a minimal reduction \(J\) of \(I\) such that \(\sum_{n=0}^{\infty} \lambda(I^{n+1} / I^n J) - e_1(I) = 1\). Then \(\text{depth } G(I) \geq d - 2\).
By [5], it suffices to consider the case $d = 3$, so we assume in the following that $d = 3$. We also assume now that Theorem 3.1 doesn’t hold. We shall reach a contradiction later.

Let $x_1, x_2 \in J$ be a superficial sequence of $I$; then, by Corollary 2.3, we have for $i = 1, 2$,

$$
\sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x_i))) - \epsilon_1(I) = 1.
$$

Moreover, by Lemma 2.1, we have

$$
\sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x_1, x_2))) - \epsilon_1(I) = 0.
$$

Let $\{x, y, z\}$ be a minimal generating set of $J$. Consider the exact sequence:

$$
0 \longrightarrow T_{k,n} \longrightarrow I^k/I^{k-1}J \xrightarrow{\phi_n} S_{k,n} = I^kJ^n/I^{k-1}J^{n+1} \longrightarrow 0,
$$

where $\phi_n = (x^n, x^{n-1}y, x^{n-2}z, \ldots, z^n)$ and $T_{k,n} = \ker(\phi_n)$. From the proof of Theorem 2.4, we see that there is an unique integer $N \geq 2$ such that $T_{N,n} \neq 0$ for some positive integer $n$. Notice that $N$ is independent of the choice of a generating set of $J$ since $S_{k,n}$ and $I^k/I^{k-1}J$ are. As $R/m$ is infinite, we may, after elementary transformation of $x, y$ and $z$, require that $\{x, y, z\}$ satisfies the following conditions.

**Proposition 3.2.** There is a generating set $\{x, y, z\}$ of $J$ satisfying the following conditions:

(i) $\{x, y\}$, $\{y, z\}$ and $\{z\}$ are all superficial sequences for $I$.
(ii) $I^n \cap (x)$, $I^n \cap (y)$ and $I^n \cap (z)$ are all contained in $I^{n-1}J \cap nJ$.
(iii) $I^n \cap (x, y)$, $I^n \cap (x, z)$ and $I^n \cap (y, z)$ are all contained in $I^{n-1}J \cap nJ ≠ N$.

Moreover,

$$
\lambda\left(\frac{I^n \cap (x, y) + I^{n-1}J}{I^{n-1}J}\right) = \lambda\left(\frac{I^n \cap (x, z) + I^{n-1}J}{I^{n-1}J}\right) = \lambda\left(\frac{I^n \cap (y, z) + I^{n-1}J}{I^{n-1}J}\right) = 1.
$$

**Proof.** (Sketch.) Notice that (ii) follows from (i) and (1); therefore we need only to show (i) and (iii).

Let $\{x, y, z\}$ be a generating set of $J$. Let $n$ be an integer such that $T_{N,n} \neq 0$. Then there are $a_{ijk} \in I^n$ not all in $I^{n-1}J$ such that $\sum_{i+j+k=n} a_{ijk}x^iy^jz^k \in I^{n-1}J^{n+1}$. By Lemma 2.5, we may, after elementary transformation of $x, y$ and $z$, assume that $a_{n00}, a_{0n0}$ and $a_{00n}$ are not in $I^{n-1}J$.

Next, we can use prime avoidance and Corollary 2.3 to replace $\{x, y, z\}$ by elements of the set $\{x + a\beta y + b\alpha z\}$ so that the condition (i) holds without changing the condition that the coefficients of $x^n, y^n$ and $z^n$ are not in $I^{n-1}J$.

Since $\sum_{i+j+k=n} a_{ijk}x^iy^jz^k \in (I^{n-1}J)J^n$, there are $a_1, a_2, a_3 \in I^{n-1}J$ such that $a_{n00} - a_3 \in I^n \cap (y, z)$, $a_{n00} - a_2 \in I^n \cap (x, z)$ and $a_{00n} - a_2 \in I^n \cap (x, y)$. Thus the condition (iii) holds by condition (i) and (2).
Let \{x, y, z\} be a generating set of \(J\) satisfying the conditions of Proposition 3.2. Let \(t\) be chosen least such that \(T_{N,n} \neq 0\) for \(n \geq t\). Then there are elements \(a_{ijk} \mid i + j + k = t\) not all in \(I^{N-1}J\) such that \(\sum a_{ijk}x^iy^jz^k \in I^{N-1}J^{t+1}\). If \(\{a_{ijk} \mid j + k = t\} \subseteq I^{N-1}J\), then \(x(\sum a_{ijk}x^iy^jz^k) \in I^{N-1}J^{t+1}\), so that by Lemma 2.4, \(\sum a_{ijk}x^{i-1}y^jz^k \in I^{N-1}J^t\); it follows that \(T_{N,t-1} \neq 0\), which contradicts the choice of \(t\). Therefore, \(\{a_{ijk} \mid j + k = t\}\) are not all in \(I^{N-1}J\).

Let the overbars denote \(\text{mod}(x)\) in the following. Consider the exact sequence

\[
0 \longrightarrow T_{k,n} \longrightarrow \bigoplus_{i=0}^{n+1} \bar{J}^i / \bar{J}^{i+1} \bar{T}_{k,n} \longrightarrow \bar{T}_{k,n} / \bar{T}_{k,n} \longrightarrow 0,
\]

where \(\bar{T}_{k,n} = (\bar{y}^n, \bar{y}^{n-1}z, \ldots, \bar{z}^n)\) and \(T_{k,n} = \ker(\bar{\phi}_n)\). Since \(\sum_{n=1}^{\infty} \lambda(\bar{I}^n / \bar{I}^{n+1}\bar{J}) - e_1(\bar{J}) = 1\), there is an unique integer \(N'\) such that \(T_{N',n} \neq 0\) for some \(n\). However, by the following remark, \(\{a_{ijk} \mid j+k = t\}\) are not all in \(I^{N-1}J\). Since \(\sum a_{ijk}y^jz^k \in I^{N-1}J^{t+1}\), we see that \(N' = N\).

**Remark 3.3.** Let \(b \in I^N \setminus I^{N-1}J\); then \(\bar{b} \not\in \bar{I}^{N-1}J\) by the fact that \(I^N : x = I^{N-1}\).

In the sequel, let \(R^0\) denote the set \{units of \(R\)\} \(\cup \{0\}\) and \(R_n (n \geq 0)\) denote the set \(\{f \mid f = \sum_{i=0}^n \lambda_i \bar{y}^{n-i}z^i\} \text{ for some } \lambda_i \in R^0\).

**Lemma 3.4.** Let \((R, \mathfrak{m})\) be a 2-dimensional Cohen-Macaulay local ring with infinite residue field. Let \(I\) be an \(\mathfrak{m}\)-primary ideal of \(R\) and \(J = (y, z)\) be a minimal reduction of \(I\). Let \(N \geq 2\). Suppose that

1. \(\lambda(I^N \cap (y) + I^{N-1}J) = \lambda(I^N \cap (y) + I^{N-1}J) = 1\), and
2. \(\forall n < N\) and \(\forall m \geq 1\), \(I^n : y = I^m : z = I^{m-1}\) and \(I^nJ^m : y = I^mJ^m : z = I^{m-1}J^{m-1}\).

Consider the exact sequence

\[
0 \longrightarrow T_{N,n} \longrightarrow \bigoplus_{i=0}^{n+1} I^i / I^{i+1} J \longrightarrow \bar{T}_{k,n} / \bar{T}_{k,n} \longrightarrow 0,
\]

where \(\bar{\phi}_n = (\bar{y}^n, \bar{y}^{n-1}z, \ldots, \bar{z}^n)\) and \(T_{N,n} = \ker(\bar{\phi}_n)\).

Suppose that \(T_{N,n} \neq 0\) for some \(n\). Let \(l\) be the least integer such that \(T_{N,n} \neq 0\) for all \(n \geq l\). Let \(a_0, \ldots, a_l \in I^N\) not all in \(I^{N-1}J\) such that \(\sum_{i=0}^l a_0y^{l-i}z^i \in I^{N-1}J^{l+1}\).

Then the following hold:

1. \(a_0 \not\in I^{N-1}J\) and \(\mathfrak{m}a_i \subseteq I^{N-1}J\ \forall i\).

2. If \(\sum_{i=0}^n b_iy^{n-i}z^i \in I^{N-1}J^{n+1}\) for some \(b_i \in I^N\) and for some \(n\), then \(\mathfrak{m}b_i \subseteq I^{N-1}J\ \forall i\) and there are \(\lambda_0, \ldots, \lambda_{n-1} \in R^0\) such that \(b_j - \sum_{i=0}^j a_i \lambda_{j-i} \in I^{N-1}J\).

*(Conventions: \(a_i = 0\) if \(i > l\) and \(\lambda_i = 0\) if \(i > n - l\)).*

**Proof.** By the choice of \(l\) and the fact that \(I^{N-1}J^m : z = I^{N-1}J^{m-1}\), we obtain that \(a_0 \not\in I^{N-1}J\). Since \(a_0 \in I^N \cap (y) + I^{N-1}J\), we have, by assumption, \(\mathfrak{m}a_0 \subseteq I^{N-1}J\).
Moreover, let \( w \in m \); then \( \sum_{i=1}^{l} (wa_i)y^{l-i}z^{i-1} \in I^{N-1}J^l \) by the assumption that \( I^{N-1}J^m : z = I^{N-1}J^{m-1} \). Again, by the choice of \( l \), \( wa_i \) must belongs to \( I^{N-1}J \) for all \( i \). This proves (1).

To see (2), we may assume that \( n \geq l \) and \( b_0 \in I^N \setminus I^{N-1}J \). Then \( b_0 \in I^N \cap (z) + I^{N-1}J \). Since \( a_0 \in I^N \cap (z) + I^{N-1}J \); there is a unit \( \lambda_0 \) such that \( b_0 - \lambda_0a_0 \in I^{N-1}J \), therefore, \( \sum_{i=0}^{n-1} (b_{i+1} - \lambda_0a_{i+1})y^{n-1-i}z^i \in I^{N-1}J^n \).

If \( n = l \), then, by the choice of \( l \), \( b_l - \lambda_0a_l \in I^{N-1}J \forall i \); hence \( mb_i \subseteq I^{N-1}J \forall i \). If \( n > l \), then by induction \( mb_i \subseteq I^{N-1}J \forall i \) and there are \( \lambda_1, \ldots, \lambda_{n-l} \in R^0 \) such that \( \forall j \geq 1 b_j - \lambda_0a_j = \sum_{i=0}^{j-1} a_i\lambda_{j-i} \). This proves (2).

Let \( \{x, y, z\} \) be a generating set of \( J \) satisfying the conditions of Proposition 3.3. Let the overbars denote \( \text{mod}(x) \) in the following. By Lemma 2.4, it is easy to check that \( \bar{R}, \bar{y} \) and \( \bar{z} \) satisfy all the assumptions of Lemma 3.3. Let \( l = \min\{n \mid \bar{T}_{N,n} \neq 0\} \), where \( \bar{T}_{N,n} \) is defined as the above; there are \( a_0, \ldots, a_l \in I^N \) not all in \( I^{N-1}J \) such that \( \sum_{i=0}^{n} \bar{a}_iy^{n-i}z^i \in \bar{I}^{N-1}J^{l+1} \). Let \( u = \sum_{i=0}^{l} a_iy^{l-i}z^i \); then \( u \) has the following property.

**Lemma 3.5.** If \( \sum_{i=0}^{n} b_iy^{n-i}z^i \in (x) + I^{N-1}J^{n+1} \) for some \( b_i \in I^N \), then

1. \( mb_i \subseteq I^{N-1}J \forall i \).
2. There is an \( f \in R_{n-l} \) such that \( \sum_{i=0}^{n} b_iy^{n-i}z^i - fu \in I^{N-1}J^{n+1} \).

**Proof.** By Lemma 3.3 \( \bar{m}a_i \in \bar{I}^{N-1}J \); hence \( ma_i \subseteq I^{N-1}J + (x) \cap I^N \subseteq I^{N-1}J \). Moreover, there are \( \lambda_0, \ldots, \lambda_{n-l} \in R^0 \) such that \( \bar{b}_j - \sum_{i=0}^{j} \bar{a}_i\lambda_{j-i} \in \bar{I}^{N-1}J \); then

\[
\bar{b}_j - \sum_{i=0}^{n-l} \bar{a}_j\lambda_{j-i} \in I^{N-1}J + (x) \cap I^N \subseteq I^{N-1}J .
\]

Therefore \( mb_i \subseteq I^{N-1}J \forall i \). Let \( f = \sum_{i=0}^{l} \lambda_i y^{n-l-i}z^i \in R_{n-l} \). Then

\[
\sum_{i=0}^{n} b_iy^{n-i}z^i - fu = \sum_{i=0}^{j} (b_j - \sum_{i=0}^{j} a_j\lambda_{j-i})y^{n-j}z^j \in I^{N-1}J^{n+1} .
\]

**Remark 3.6.** If, in Lemma 3.5 at least one of the \( b_i \) is not in \( I^{N-1}J \), then \( n \geq l \) and we can choose \( f \) to be a nonzero element of \( R_{n-l} \).

Since \( \text{depth} \ G(J) = 0 \), \( I^n : J \neq I^{n-1} \) for some \( n \). Let \( N' = \min\{n \mid I^n : J \neq I^{n-1}\} \). Since \( I^n : x = I^{n-1} \forall n \leq N, N' > N \). Let \( s \notin I^{N'-1} \) such that \( sJ \subseteq I^{N'} \).
Lemma 3.7. There exists an element $s'$ with $s - s' \in I^{N-1}$ such that $s'y \in I^N J^{N-N}$, $s'z \in I^N J^{N-N}$ and $sx \in I^N(y, z)^{N-N}$.

\textbf{Proof.} Suppose we have shown for some $k > N$ that there is an element $s'$ with $s - s' \in I^{N-1}$ such that $s'y \in I^k J^{N-k}$, $s'z \in I^k J^{N-k}$ and $sx \in I^k(y, z)^{N-k}$.

Then there are $a_i, b_{ijk} \in I^k$ such that $s'x = \sum a_i y^{t-i} z^i$ and $s'y = \sum b_{ijk} x^i y^j z^k$, where $t = N - k$. Therefore,

$$\sum a_i y^{t-i} z^i - \sum b_{ijk} x^i y^{j+1} z^k = 0 \in I^{k-1} J^t + 2.$$

Since $k \neq N$, $T_{k,t+1} = 0$; therefore, $a_i, b_{ijk}$ are in $I^{k-1} J$. It follows that $s'x \in I^{k-1} J^t + 1$ and $s'y \in I^{k-1} J^t + 1$. Similarly, we can get $s'z \in I^{k-1} J^t + 1$.

Finally, from the expression $s'x \in I^{k-1} J^t + 1$, we see that there is a $w \in I^{k-1} J^t \subseteq I^{N-1}$ such that $(s' - w)x \in I^{k-1}(y, z)^{t+1}$ and $(s' - w)J \subseteq I^{k-1} J^t + 1$. This completes the proof. \hfill $\square$

By Lemma 3.7, we may assume that $s$ satisfies the conditions $sJ \in I^N J^t$ and $sx \in I^N(y, z)^t$, where $t = N - N$. Since $s \notin I^{N-1}$, $sx \notin I^{N-1} J^t + 1$ by Lemma 2.4. Therefore, by Remark 3.6 and the condition $sx \in I^N(y, z)^t$, we obtain that $t \geq 1$.

Hence by Lemma 3.5, there is a nonzero element $f_0 \in R_{t-1}$ such that

$$(3) \quad sx - f_0 u \in I^{N-1} J^t + 1.$$ 

In what follows, let $k$ be the residue field of $R$. If $f = \sum_{i=0}^{n} \lambda_i y^n z^i \in R_n$, then we associate to $f$ a homogeneous polynomial $T(f) = \sum_{i=0}^{n} \lambda_i y^n z^i$ in $k[Y, Z]$.

(Here, the overbars denote mod($m$).)

Remark 3.8. From (3) and Lemma 3.5, we obtain $msx \subseteq I^{N-1} J^t$, and therefore $ms \subseteq I^{N-1} J^t$. Moreover, if $f$ and $g$ are two elements of $R_n$ such that $F = G$, then $f - g \in mR_n$; it follows that $sf - sg \in I^{N-1} J^t$.

From (3), Lemma 3.5 and Remark 3.6, we have the following corollary.

\textbf{Corollary 3.9.} Let $B_0 = \sum b_i y^n z^i$ with $b_i \in I^N$ not all in $I^{N-1} J$ and $B_j \in (y, z)^n J$. Suppose that $B_0 + B_1 x + \cdots \in I^{N-1} J^{n+1}$. Then there is a nonzero element $h \in R_{n-1}$ such that $B_0 + hu \in I^{N-1} J^{n+1}$ and $sh - f_0(B_1 + B_2 x + \cdots) \in I^{N-1} J^{n+t-1}$.

\textbf{Proof.} By Lemma 3.6 and Remark 3.6, there is a nonzero element $h \in R_{n-1}$ such that $B_0 + hu \in I^{N-1} J^{n+1}$. By (3), $shx - f_0(B_1 + B_2 x + \cdots) \in I^{N-1} J^{n+t-1}$ + 1. It follows by Lemma 2.4 that $shx - f_0(B_1 + B_2 x + \cdots) \in I^{N-1} J^{n+t-1}$. \hfill $\square$

Let $Q_j (j \geq 0)$ be the set of all integers $n$ such that there exists a nonzero element $f \in R_n$ with $s f \in \sum_{i=0}^{j} x^i(y, z)^{t+n-1-i} I^N + I^{N-1} J^{t+n}$. (For example, since $sy \in I^N J^t$, $1 \in Q_0$.)

Let $m = \min\{j \mid Q_j \neq \emptyset\}$. Let $k$ be the smallest integer in $Q_m$; then there is a nonzero element $f \in R_k$ with $sf \in \sum_{i=0}^{m} x^i(y, z)^{t+k-1-i} I^N + I^{N-1} J^{t+k}$, so that
there are \( A_i \in (y, z)^{t+k-1-i} I^N \) such that

\[
(4) \quad sf - \sum_{i=0}^{m} A_i x^i \in I^{N-1} J^{t+k}.
\]

**Lemma 3.10.** Let \( C_i \in (y, z)^{n-i} I^N \) such that \( \sum_{i=0}^{m} C_i x^i \in I^{N-1} J^{n+1} \). Then \( C_i \in I^{N-1} J^{n+1-i} \) and all the coefficients of \( C_i \) are all in \( I^{N-1} J \).

**Proof.** If \( m = 0 \), then \( C_0 \in I^{N-1} J^{n+1} \). If the coefficients of \( C_0 \) are not all in \( I^{N-1} J \), then by Corollary 3.9 there is a nonzero element \( h \in R_{n-i} \) such that \( sh \in I^{N-1} J^{n+t-l} \). This gives the contradiction \( s \in I^{N-1} J' \) by Lemma 2.4.

Assume that \( m \geq 1 \) and the assertion is false. Let \( j \) be the least integer such that the coefficients of \( C_j \) are not all in \( I^{N-1} J \). Then, by Lemma 2.4 \( \sum_{i=j}^{m} C_i x^{i-j} \in I^{N-1} J^{n+1-j} \); hence by Corollary 3.9 there is a nonzero element \( h \in R_{n-j-i} \) such that \( sh - f_0(\sum_{i=j+1}^{m} C_i x^{i-j-1}) \in I^{N-1} J^{n+t-j-1} \), which contradicts the choice of \( m \).

The assertion now follows. \( \square \)

Let \( p \) be the maximal integer such that \( Z^p | F \). Let \( F' = F / Z^p \).

**Lemma 3.11.** If \( g \) is a nonzero element of \( R_n \) such that \( sg \in \sum_{i=0}^{m} x^i (y, z)^{t+n-1-i} I^N + I^{N-1} J^{t+n} \), then \( F | G \).

**Proof.** Let \( B_i \in (y, z)^{t+n-1-i} I^N \) such that

\[
(5) \quad sg - \sum_{i=0}^{m} B_i x^i \in I^{N-1} J^{t+n}.
\]

Write \( G = G' Z^q \) with \( (G', Z) = 1 \). Let \( g' \in R_{n-q} \) and \( f' \in R_{k-p} \) such that \( T(g') = G' \) and \( T(f') = F' \). By Remark 3.8 we may assume that \( f = f' z^p \) and \( g = g' z^q \).

Assume that \( q < p \.) Then from (4) and (5), we obtain that

\[
(6) \quad \sum_{i=0}^{m} (f' z^{p-q} B_i - g' A_i) x^i \in I^{N-1} J^{t+k+n-q},
\]

so that by Lemma 3.10 all the coefficients of \( f' z^{p-q} B_i - g' A_i \) are in \( I^{N-1} J \). Since \( (G', Z) = 1 \), there are \( A'_i \in (y, z)^{t+k-2-1} I^N \) such that \( A_i - z A'_i \in I^{N-1} J^{t+k-1} \); it follows from (4) that \( s(f' z^{p-1}) - \sum_{i=0}^{m} A'_i x^i \in I^{N-1} J^{t+k-1} \), which contradicts the choice of \( k \). Hence \( q \geq p \).

Write \( G = G' Z^p \). Then \( G' = F' Q + Z^{n-k+1} G'' \) for some \( Q \) and \( G'' \). Suppose \( G'' \neq 0 \). Then \( \deg G'' = \deg F' - 1 \). Let \( g'' \in R_{k-p-1} \) such that \( T(g'') = G'' \); then

\[
(7) \quad sg'' z^{n-k+1+p} - \sum_{i=0}^{m} C_i x^i \in I^{N-1} J^{t+n}
\]

for some \( C_i \in (y, z)^{t+n-1-i} I^N \).
From (4) and (7), we obtain that
\[
\sum_{i=0}^{m} (g'' z^n A_i - f' C_i) x^i \in I^{N-1} J^{t+n+k-p},
\]
therefore, by Lemma 3.10, all the coefficients of \( g'' z^n A_i - f' C_i \) are in \( I^{N-1} J \).
Since \( (F', Z) = 1 \), there are \( C'_i \in (y, z)^{t+k-2-i} I^N \) such that \( C_i = z^n A_i C'_i \in I^{N-1} J^{t+n-i} \). Hence from (7), \( sg'' z^n - \sum_{i=0}^{m} C'_ix^i \in I^{N-1} J^{t+k-1} \), which contradicts the choice of \( k \). Therefore, \( G'' = 0 \) and \( F|G \).

In fact, Lemma 3.11 can be improved as follows.

Lemma 3.12. Let \( m \leq m' \leq t \). If \( g \) is a nonzero element of \( R_n \) such that \( sg \in \sum_{i=0}^{m'} x^i (y, z)^{t+n-1-i} I^N + I^{N-1} J^{t+n} \), then \( F|G \).

Proof. We use induction on \( m' \). If \( m' = m \), then this is the content of Lemma 3.11.
Assume that \( m' > m \). Let \( H_0 = \frac{F_0}{(F_0, F)} \) and let \( n_0 = \text{deg} H_0 \). Let \( h_0 \in R_{n_0} \) such that \( T(h_0) = H_0 \). Let \( B_i \in (y, z)^{t+n-1-i} I^N \) such that
\[
sg - \sum_{i=0}^{m'} B_i x^i \in I^{N-1} J^{t+n}.
\]
Let \( H = (F, G) \) and let \( k' = \text{deg} H \). Then \( F = F' H \) and \( G = G'H \) for some \( F' \) and \( G' \). Let \( f' \in R_{k-k'} \) and \( g' \in R_{n-k'} \) such that \( T(f') = F' \) and \( T(g') = G' \).
By Remark 3.8, we may assume that \( f = f'h \) and \( g = g'h \). Set \( A_i = 0 \) for \( i > m \).

From (4) and (8), we obtain that \( \sum_{i=0}^{m'} (f' B_i - g' A_i) x^i \in I^{N-1} J^{t+n+k-k'} \). Therefore, by Lemma 3.5, there is an element \( h_1 \in R_{t+n+k-k'-l-1} \) such that \( f' B_0 - g' A_0 + h_1 u \in I^{N-1} J^{t+n+k-k'} \), and then, by (4), \( sh_1 - f_0 (\sum_{i=0}^{m'-1} (f'B_{i+1} - g'A_{i+1}) x^i) \in I^{N-1} J^{t+n+k-k'-l-1} \). Hence, by induction, \( F|H_1 \). Moreover, since \( (F_0, F) \) is a common factor of \( H_1 \) and \( F_0 \), there is an element \( g_1 \in R_{n_0+n+k-k'-1} \) such that \( sg_1 - h_0 (\sum_{i=0}^{m'-1} (f'B_{i+1} - g'A_{i+1}) x^i) \in I^{N-1} J^{t+n_0+n+k-k'-1} \). Therefore, by induction, \( F|G_1 \).

Hence there is an element \( g'_1 \in R_{n_0+n-k-1} \) such that \( G_1 = G'_1 F \).

Suppose that \( m = 0 \). Since \( f'B_0 - g' A_0 + h_1 u \in I^{N-1} J^{t+n+k-k'} \), by Lemma 3.10, \( g'hA_0 - f' A_0 \in I^{N-1} J^{t+n+k} \) for some \( A_0 \in (y, z)^{t+n-1} I^N \). Therefore by (4), \( sf' g'h - f' A_0 \in I^{N-1} J^{t+n+k} \), and then \( sg'h - A_0 \in I^{N-1} J^{t+n} \) by Lemma 2.4 so that by Lemma 3.11, \( F|G'H \). But \( (F, G') = 1 \); we obtain \( F|H \).

Assume now that \( m \geq 1 \). We claim: There are integers \( n_1, \ldots, n_{m'-m}, d_1, \ldots, d_{m'-m} \) and elements \( h_j \in R_{n_j}, g_j \in R_{d_j} \) and \( g'_j \in R_{d_j-k} \) satisfy the following conditions:
(i) \( G_j = G'_j F \).
(ii) \( h_{j-1}^{-1} (f'B_{j-1} - g'A_{j-1}) + h_{j-2}^{-2} g'_{j-1} A_{j-2} - \cdots - g'_{j-1} A_0 + h_j u \in I^{N-1} J^{n_j+t+1} \).
(iii) \[ sg_j - \sum_{i=0}^{m'-j} (h_0^{j}(f'B_{i+j} - g'A_{i+j}) - h_0^{j-1}g_1'A_{i+j-1} - \cdots - h_0g_{j-1}'A_{i+1})x^i \in I^{N-1}J^{d_{j+1}}. \]

Suppose that we have constructed, for some \( j \geq 1 \), \( h_j \), \( g_j \), and \( g_j' \). Then from (i), (iii) and (4), we see that the element

\[ h_0^{j}(f'B_j - g'A_j) - h_0^{j-1}g_1'A_{j-1} - \cdots - g_j'A_0 \]

is in \( I^{N-1}J^{d_{j+1}} \). Therefore, by Lemma 3.5, there is an element \( h_{j+1} \in R_{n_{j+1}} \) for some \( n_{j+1} \) such that (ii) holds for \( j+1 \) and \( F|H_{j+1} \) (cf. the construction of \( h_1 \)) by induction. Moreover, from the construction of \( g_1 \), it is easy to see that there is an element \( g_{j+1} \in R_{d_{j+1}} \) for some \( d_{j+1} \) such that (iii) holds for \( j+1 \). Since by induction \( F|G_{j+1} \), there is an \( g_{j+1}' \in R_{d_{j+1}-k} \) such that (i) holds. This proves the claim.

Set \( j = m' - m \) in (iii) of the claim and compare with (4). We obtain that the element

\[ \sum_{i=0}^{m} (h_0^{m'-m}(f'B_{i+m'-m} - g'A_{i+m'-m}) - h_0^{m'-m-1}g_1'A_{i+m'-m-1} - \cdots - g_{m'-m}A_i)x^i \]

is in \( I^{N-1}J^{d_{m'-m+t}} \). However by Lemma 3.10 \( \forall i \leq m \),

\[ h_0^{m'-m}(f'B_{i+m'-m} - g'A_{i+m'-m}) - h_0^{m'-m-1}g_1'A_{i+m'-m-1} - \cdots - g_{m'-m}A_i \in I^{N-1}J^{d_{m'-m+t-i}} \]

From (ii) and (6), it is not hard to see that for \( 0 \leq i \leq m \) there are \( A_i' \in (y, z)^{e+i}I^N \) for some \( e_i \) such that \( hh_0^ig_{i+1}A_i - fA_i' \in I^{N-1}J^{e_i+k+1} \). Therefore by (4), \( shh_0^mg^{m+1}f - f(\sum_{i=0}^{m} h_0^{m-i}g^{m-i}A_i'x^i) \in I^{N-1}J^e \) for some \( e \); it follows by Lemma 2.4 that

\[ shh_0^mg^{m+1} - \sum_{i=0}^{m} h_0^{m-i}g^{m-i}A_i'x^i \in I^{N-1}J^{e-k}. \]

Finally, by Lemma 3.11 \( F|G^{m+1}HH_0^m \). Since \( (F, G') = (F, H_0) = 1 \), we have \( F|H \). This completes the proof.

Now, choose a unit \( \lambda \) so that \( (Y + \lambda Z, F) = 1 \). Since \( s(y + \lambda z) \in \sum_{i=0}^{\infty} x^i(y, z)^{t-i}I^N + I^{N-1}J^{t+1} \), by Lemma 3.12 \( F|Y + \lambda Z \), which contradicts the choice of \( \lambda \). This proves Theorem 3.1.

By Proposition 2.10 and Theorem 3.1, we have the following corollary.

**Corollary 3.13.** Let \( (R, \mathfrak{m}) \) be a Cohen-Macaulay local ring of dimension \( d \geq 2 \) with infinite residue field. Let \( I \) be an \( \mathfrak{m} \)-primary ideal of \( R \). Suppose that there is a minimal reduction \( J \) of \( I \) such that \( \sum_{n=1}^{\infty} \lambda(I^n \cap J/I^{n-1}J) = 2 \). Then depth \( G(I) \geq d - 2 \).
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