

## GEOMETRICAL SIGNIFICANCE OF THE LÖWNER-HEINZ INEQUALITY

E. ANDRUCHOW, G. CORACH, AND D. STOJANOFF

(Communicated by David R. Larson)

*Dedicated to Mischa Cotlar, with affection and admiration, on his 86th anniversary*

ABSTRACT. It is proven that the Löwner-Heinz inequality  $\|A^t B^t\| \leq \|AB\|^t$ , valid for all positive invertible operators  $A, B$  on the Hilbert space  $\mathcal{H}$  and  $t \in [0, 1]$ , has equivalent forms related to the Finsler structure of the space of positive invertible elements of  $\mathcal{L}(\mathcal{H})$  or, more generally, of a unital  $C^*$ -algebra. In particular, the Löwner-Heinz inequality is equivalent to some type of “nonpositive curvature” property of that space.

### 0. INTRODUCTION

The space of positive definite matrices  $M_n(\mathbb{C})^+$  is a well known Riemannian manifold with nonpositive curvature. It turns out that, for every unital  $C^*$ -algebra  $\mathcal{A}$ , the space  $\mathcal{A}^+$  of all positive invertible elements of  $\mathcal{A}$  has a very rich Finsler (non-Riemannian) structure (see [14], [4], [5], [6], [15]). In this paper we show that the so-called Löwner-Heinz inequality

$$\|A^t B^t\| \leq \|AB\|^t \quad (A, B \in \mathcal{L}(\mathcal{H})^+, t \in [0, 1])$$

plays a central role in this geometrical study.

Recall that the inequality is a by-product of Löwner’s characterization of operator monotone functions [13] and it has many applications in spectral theory [7]. Recently, Fujii, Furuta and Nakamoto [8] proved that the Löwner-Heinz inequality is equivalent to a more technical one (see (2.4)) which appeared in [6] as a corollary of a geometrical property of  $\mathcal{A}^+$ , namely the fact that the distance between two geodesics  $\gamma(t), \delta(t)$  in  $\mathcal{A}^+$  is a convex function of  $t$ .

We show that all these assertions are, indeed, equivalent. It is known that, for Riemannian manifolds, the convexity of the function  $d(\gamma(t), \delta(t))$  for any pair of geodesics  $\gamma, \delta$  is equivalent to the nonpositivity of the sectional curvature. Thus, the Löwner-Heinz inequality appears as the analytic form of this geometrical behaviour of  $\mathcal{A}^+$ . In this sense,  $\mathcal{A}^+$  offers a beautiful infinite dimensional example of what Gromov calls a “espace de longueur” [10, Ch. 1], and the Löwner-Heinz inequality is the expression of the “nonpositive curvature” of  $\mathcal{A}^+$ .

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Received by the editors May 29, 1997 and, in revised form, May 18, 1998.

1991 *Mathematics Subject Classification*. Primary 46L05, 58B20.

The authors were partially supported by UBACYT EX 261, PIP CONICET 4463/96 and PICT 2259 ANPCYT (Argentina).

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The paper contains also some results relating the geodesic distance to the work of Nussbaum [14], [15] on the part metric or Thompson distance on  $\mathcal{A}^+$ , with its corresponding notion of geodesics.

1. EQUIVALENT INEQUALITIES

**Theorem 1.** *Let  $A$  be a unital  $C^*$ -algebra. Then each of the following properties holds and is equivalent to each other:*

- (1.1)  $\|a^t b^t\| \leq \|ab\|^t$  for all  $a, b \in \mathcal{A}^+$  and  $t \in [0, 1]$ ;
- (1.2)  $\|\log(a^{-t/2} b^t a^{-t/2})\| \leq t \|\log(a^{-1/2} b a^{-1/2})\|$  for all  $a, b \in \mathcal{A}^+$  and  $t \in [0, 1]$ ;
- (1.3) For all  $a_1, a_2, b_1, b_2 \in \mathcal{A}^+$  and  $t \in [0, 1]$ ,

$$\begin{aligned} & \| (a_1^{1/2} (a_1^{-1/2} b_1 a_1^{-1/2})^t a_1^{1/2})^{1/2} (a_2^{1/2} (a_2^{-1/2} b_2 a_2^{-1/2})^t a_2^{1/2})^{-1/2} \| \\ & \leq \| a_1^{1/2} a_2^{1/2} \|^{1-t} \| b_1^{1/2} b_2^{1/2} \|^t. \end{aligned}$$

*Proof.* The first one is the Löwner-Heinz inequality. A short proof of it can be found in [7] or [9]. The second equality has been proven in [5] as a consequence of (1.1). The third inequality has been proven in [6] and recently it has been shown in [8] that it is equivalent to (1.1). Thus, it suffices to prove that (1.2) implies (1.3). For this, take  $a, b \in \mathcal{A}^+$ . By a compactness argument, there exists  $k > 0$  such that

$$a^t (kb)^{2t} a^t > 1 \quad \text{for all } t \in [0, 1].$$

Observe that  $\|\log c\| = \max\{\log \|c\|, \log \|c^{-1}\|\}$  for all  $c \in \mathcal{A}^+$  and that  $\|\log c\| = \log \|c\|$  for  $c > 1$ . Then

$$\begin{aligned} \log \|a^t (kb)^{2t} a^t\| &= \|\log(a^t (kb)^{2t} a^t)\| \\ &\leq t \|\log(a(kb)^2 a)\| \\ &= t \log \|a(kb)^2 a\|, \end{aligned}$$

where we use inequality (1.2) replacing  $a, b$  by  $a^{-2}, b^2$ , respectively. Thus,

$$\log \|a^t b^{2t} a^t\| + \log k^{2t} \leq t \log \|ab^2 a\| + t \log k^2$$

so that  $\|a^t b^t\|^2 \leq \|ab\|^{2t}$ , which is (1.1). □

2. THE GEOMETRICAL INTERPRETATION

We give now a geometrical interpretation of the theorem. For this, we present a short description of the differential geometry of  $\mathcal{A}^+$  (see [4], [5], [6], [2] for a complete treatment). Let  $\mathcal{A}$  be a unital  $C^*$ -algebra,  $\mathcal{A}^\times$  the group of invertible elements of  $\mathcal{A}$ ,  $\mathcal{A}_s$  the (real) subspace of Hermitian elements of  $\mathcal{A}$ .  $\mathcal{A}^+$  is an open convex subset of  $\mathcal{A}_s$  so that it will be considered as an open submanifold of  $\mathcal{A}_s$  and the tangent spaces  $(T\mathcal{A}^+)_a$  will be identified to  $\mathcal{A}_s$  for all  $a \in \mathcal{A}^+$ . There is a natural action of  $\mathcal{A}^\times$  over  $\mathcal{A}^+$  given by  $(g, a) \mapsto gag^*$  ( $g \in \mathcal{A}^\times, a \in \mathcal{A}^+$ ). This is a transitive action: given  $a, b \in \mathcal{A}^+$   $g = b^{1/2} a^{-1/2}$  verifies  $gag^* = b$ . For every  $a \in \mathcal{A}^+$ , the map  $\tau_a : \mathcal{A}^\times \rightarrow \mathcal{A}^+$   $\tau_a(g) = gag^*$  is a principal fibre bundle with a natural connection.

The covariant derivative of a tangent field  $X$  along a curve  $\gamma$  in  $\mathcal{A}^+$  is given by

$$\frac{DX}{dt} = \dot{X} = \frac{1}{2} (X\gamma^{-1}\dot{\gamma} + \dot{\gamma}\gamma^{-1}X).$$

The curve  $\gamma$  is a *geodesic* of the connection if  $\frac{D\dot{\gamma}}{dt} = 0$ , i.e., if  $\ddot{\gamma} = \dot{\gamma}\gamma^{-1}\dot{\gamma}$ .

It turns out that in this case  $g\gamma g^*$  is also a geodesic for all  $g \in \mathcal{A}^\times$ . The unique geodesic  $\gamma$  such that  $\gamma(0) = a$  and  $\dot{\gamma}(0) = X \in \mathcal{A}_s$  is

$$\begin{aligned} \gamma(t) &= e^{(t/2)Xa^{-1}} \\ &= a^{1/2}e^{ta^{-1/2}Xa^{-1/2}}a^{1/2} \end{aligned}$$

and for every  $a, b \in \mathcal{A}^+$  there is a unique geodesic  $\gamma_{a,b}$  such that  $\gamma_{a,b}(0) = a$  and  $\gamma_{a,b}(1) = b$ , namely,

$$\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2} ,$$

corresponding to  $X = a^{1/2} \log(a^{-1/2}ba^{-1/2})a^{1/2}$ .

Thus, the *exponential map*  $\exp_a : (TA^+)_a \rightarrow \mathcal{A}^+$  is defined by

$$\exp_a(X) = e^{(1/2)Xa^{-1}}ae^{(1/2)a^{-1}X} = a^{1/2}e^{a^{-1/2}Xa^{-1/2}}a^{1/2}$$

and admits a global inverse given by

$$\log_a(c) = a^{1/2} \log(a^{-1/2}ca^{-1/2})a^{1/2} .$$

Even if  $\mathcal{A}^+$  is not a Riemannian manifold, there is a natural Finsler structure on  $\mathcal{A}^+$  defined as follows: for  $X \in (TA^+)_a = \mathcal{A}_s$  define  $\|X\|_a = \|\lambda^{-1/2}X\lambda^{-1/2}\|$ . If  $\mathcal{A}$  is represented in a Hilbert space  $\mathcal{H}$ ,  $\|X\|_a$  is the norm of the symmetric sesquilinear form  $B_X : \mathcal{H}_a \times \mathcal{H}_a \rightarrow \mathbb{C}$ ,  $B_X(\xi, \eta) = \langle X\xi, \eta \rangle$ , where  $\mathcal{H}_a$  is  $\mathcal{H}$  with the scalar product  $\langle \xi, \eta \rangle_a = \langle a\xi, \eta \rangle$ . If  $g \in \mathcal{A}^\times$ , then  $g : \mathcal{H}_{gag^*} \rightarrow \mathcal{H}_a$  is an isometry, so that

$$\|B_{gXg^*}\| = \|B_X\| \quad \text{and} \quad \|gXg^*\|_{gag^*} = \|X\|_a .$$

It can be shown that the geodesic  $\gamma_{a,b}$  is shortest among all curves  $\gamma : [0, 1] \rightarrow \mathcal{A}^+$  such that  $\gamma(0) = a$  and  $\gamma(1) = b$ , where the length of  $\gamma$  is

$$\ell(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt .$$

Thus, if we define the *geodesic or rectifiable distance*  $d$  by

$$d(a, b) = \inf\{ \text{length } \gamma : \gamma \text{ is a smooth curve in } \mathcal{A}^+ \text{ joining } a \text{ and } b \} ,$$

then  $d(a, b) = \text{length } \gamma_{a,b}$ . Since  $\|gXg^*\|_{gag^*} = \|X\|_a$  for all  $g \in \mathcal{A}^\times$ ,  $a \in \mathcal{A}^+$  and  $X \in (TA^+)_a$ , it follows that  $d(gag^*, gbg^*) = \text{length } \gamma_{gag^*, gbg^*} = \text{length } \gamma_{a,b} = d(a, b)$ ; thus,  $\mathcal{A}^\times$  acts transitively and isometrically on  $\mathcal{A}^+$ , and for computations one can usually transport the situation to  $a = 1$ . Observe also that the geodesics with origin 1 have the form  $\gamma_{1,a}(t) = a^t$ .

It is easy to verify that  $d(a, b) = \|\log a^{-1/2}ba^{-1/2}\|$  so that, when  $a$  commutes with  $b$ ,  $d(a, b) = \|\log ba^{-1}\|$ .

*Remark 1.* We shall prove later that the geodesic distance on  $\mathcal{A}^+$  coincides with the so-called *Thompson or part metric* so that the results of Nussbaum [4] are still valid.

*Remark 2.* A relevant property of the geodesic distance on  $\mathcal{A}^+$  is that the metric space  $(\mathcal{A}^+, d)$  is complete. A proof of this statement can be found in [14], where Nussbaum uses the Thompson metric. This fact, among many others, justifies the introduction of this distance. On the other hand, as observed by Vesentini [18], the

relevance of  $d$  is related to the fact that the Haar measure of the locally compact group  $\mathbb{C}^+ = \{t \in \mathbb{R} : t > 0\}$  is determined, up to a constant, by

$$m(\alpha, \beta) = |\log \beta - \log \alpha| = \left| \log \frac{\beta}{\alpha} \right|$$

for any interval  $(\alpha, \beta)$ .

The next result shows that the Löwner-Heinz inequality is equivalent to two geometrical properties which characterize Riemannian manifolds with nonpositive curvature. We shall make some comments on this after the proof of the theorem.

**Theorem 2.** *If  $\mathcal{A}$  is a unital  $C^*$ -algebra, then the following properties hold and are equivalent:*

- (2.1)  $\|a^t b^t\| \leq \|ab\|^t$  for every  $a, b \in \mathcal{A}^+$  and  $t \in [0, 1]$ ;
- (2.2)  $d(a^t, b^t) \leq td(a, b)$  for every  $a, b \in \mathcal{A}^+$  and  $t \in [0, 1]$ ;
- (2.3) for every pair of geodesics of  $\mathcal{A}^+$  the real function

$$t \mapsto d(\gamma(t), \delta(t))$$

is convex.

*Proof.* Observe that (2.2) is the rewriting of inequality (1.2), using the explicit expression for  $d$ . On the other hand, (1.3) is an easy consequence of (2.3) (see [6], where there is also a proof of (2.3) by a completely different method). Thus, it suffices to show that (1.3) implies property (2.3). Observe that (2.3) is equivalent to the inequality

$$(2.4) \quad \|\log(\delta(t)^{-1/2} \gamma(t) \delta(t)^{-1/2})\| \leq (1-t) \|\log(a_1^{-1/2} a_2 a_1^{-1/2})\| + t \|\log(b_1^{-1/2} b_2 b_1^{-1/2})\|$$

for every  $t \in [0, 1]$ , where  $\gamma$  joins  $a_1$  and  $b_1$  and  $\delta$  joins  $a_2$  and  $b_2$ . The proof that (2.4) follows from (1.3) uses the same trick of Theorem 1. After changing  $\gamma$  by  $k\gamma$  for a convenient  $k > 0$  we may suppose that

$$\delta(t)^{-1} \gamma(t) \delta(t)^{-1/2} \geq 1$$

for every  $t \in [0, 1]$ , so that

$$\begin{aligned} \|\log(\delta(t)^{-1/2} \gamma(t) \delta(t)^{-1/2})\| &= \log \|\delta(t)^{-1/2} \gamma(t) \delta(t)^{-1/2}\| \\ &\leq \log(\|a_1^{-1/2} a_2 a_1^{-1/2}\|^{1-t} \|b_1^{-1/2} b_2 b_1^{-1/2}\|^t) \\ &= (1-t) \log \|a_1^{-1/2} a_2 a_1^{-1/2}\| + t \log \|b_1^{-1/2} b_2 b_1^{-1/2}\| \\ &= (1-t) \|\log(a_1^{-1/2} a_2 a_1^{-1/2})\| + t \|\log(b_1^{-1/2} b_2 b_1^{-1/2})\| \end{aligned}$$

which ends the proof. □

*Remarks.* In Gromov’s lectures [1] one finds two characterizations of Riemannian manifold with nonpositive curvature. The first one rests on the local diffeomorphism property of the exponential map: if  $M$  is a Riemannian manifold and  $r \in M$ , the exponential map  $\exp_r : (TM)_r \rightarrow M$  is a local diffeomorphism so that in the neighborhood of  $r$  there is a map defined by

$$\rho_t(m) = \exp_r(t \exp_r^{-1}(m)) \quad (t \in [0, 1]) ;$$

then  $M$  has nonpositive curvature if and only if for every  $t \in [0, 1]$  and  $m_1, m_2$  close to  $r$

$$d(\rho_t(m_1), \rho_t(m_2)) \leq td(m_1, m_2) .$$

This expression reduces, in the case when  $M$  is the (non-Riemannian!) manifold  $\mathcal{A}^+$ , to the expression

$$d(r^{1/2}(r^{-1/2}m_1r^{-1/2})^t r^{1/2}, r^{1/2}(r^{-1/2}m_2r^{-1/2})^t r^{1/2}) \leq td(m_1, m_2)$$

which is exactly (2.2), with  $m_1 = a, m_2 = b$ , due to the fact that  $\mathcal{A}^\times$  acts isometrically over  $\mathcal{A}^+$ . Returning to Gromov’s lectures [1], one can find there a characterization of Riemannian manifolds with nonpositive curvature as those Riemannian manifolds so that the distance  $d(\gamma(t), \delta(t))$ , for two geodesics  $\gamma, \delta$ , is a convex function of  $t$ . Thus, Theorem 2 shows that the Löwner-Heinz inequality means exactly that  $\mathcal{A}^+$  has this type of “negatively curved space” behaviour. Recall that this is not the first time that a classical inequality appears as describing a geometrical behaviour of  $\mathcal{A}^+$ : in [3] it has been shown that Segal’s inequality

$$\|e^{X+Y}\| \leq \|e^{X/2}e^Y e^{X/2}\|$$

valid for Hermitian operators  $X, Y$  [16] is equivalent to the property that  $\exp_a$  increases distances:

$$d(\exp_a X, \exp_a Y) \geq \|X - Y\|_a$$

for all  $X, Y \in (T\mathcal{A}^+)_a$ ; in Riemannian geometry this happens, again, in nonpositive curvature manifolds. The reader may find a deduction of Segal’s inequality from Löwner-Heinz’s in [2]; see also [6].

The *part metric* (or *Thompson metric* [17]) of  $\mathcal{A}^+$  can be defined as

$$d_p(a, b) = \max\{\log \inf\{\alpha > 0 : a \leq \alpha b\}, \log \inf\{\beta > 0 : b \leq \beta a\}\} .$$

Nussbaum [14] proves that

$$d_p(a, b) = \max\{\log \|a^{-1/2}ba^{-1/2}\|, \log \|b^{-1/2}ab^{-1/2}\|\} .$$

Let us show that  $d_p$  coincides with the distance  $d$  obtained by differential geometric methods.

**Proposition.** *For every  $a, b \in \mathcal{A}^+$*

$$d_p(a, b) = \|\log a^{-1/2}ba^{-1/2}\| .$$

*Proof.* Recall that  $\|\log c\| = \max\{\log \|c\|, \log \|c^{-1}\|\}$  for all  $c \in \mathcal{A}^+$ . Then

$$\begin{aligned} d(a, b) &= \|\log a^{-1/2}ba^{-1/2}\| \\ &= \max\{\log \|a^{-1/2}ba^{-1/2}\|, \log \|a^{1/2}b^{-1}a^{1/2}\|\} \end{aligned}$$

and it suffices to show that  $\|a^{1/2}b^{-1}a^{1/2}\| = \|b^{-1/2}ab^{-1/2}\|$ . But this is obvious, taking  $x = a^{1/2}b^{-1/2}$  and using that  $\|x^*x\| = \|xx^*\| = \|x\|^2$ .  $\square$

*Remarks.* 1) Nussbaum [14] proved that  $\gamma_{ab}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$  curve which is shortest, for the part metric, among all curves joining  $a$  and  $b$ . On the other side, Corach, Porta and Recht [4, 5, 3] proved that  $\gamma_{a,b}$  is a geodesic (in the sense of the differential geometry) and is a shortest curve joining  $a$  and  $b$ . The result above justifies the coincidence between both approaches. Thus, the convexity results obtained in the theorem above are valid also in Nussbaum’s context. It

should be noticed that Nussbaum [15] gets by a general method a whole family of shortest curves joining two fixed  $a, b \in \mathcal{A}^+$ . The geodesic  $\gamma_{a,b}$  appears, thus, as a distinguished member of the family.

2) Liverani and Wojtkowski [12] got, by a different method, a similar expression for the distance  $d$ . There is also a previous paper by Vesentini [18] where the distance on  $\mathcal{A}^+$  can be obtained as a Carathéodory type metric.

3) Given a metric space  $X$  and a map  $\varphi : X \rightarrow X$ , consider the *dilation index* of  $\varphi$ :

$$\text{dil}(\varphi) = \sup_{x_1 \neq x_2} \frac{d(\varphi(x_1), \varphi(x_2))}{d(x_1, x_2)},$$

which in analysis is usually called the best Lipschitz constant of  $\varphi$ . Gromov [10] uses, as a useful tool for the study of his “*espaces de longueur*”, the notion *short* and *strictly short* maps. The map  $\varphi$  is *short* if  $\text{dil}(\varphi) \leq 1$  and is *strictly short* if  $\text{dil}(\varphi) < 1$  (classically called contractions).

Inequality (2.2) shows that the map  $\varphi_t : a \mapsto a^t$  (for  $t \in [0, 1]$ ) verifies  $\text{dil}(\varphi_t) \leq t$ . Indeed, it can easily be shown that  $\text{dil}(\varphi_t) = t$ . Thus,  $\varphi_t$  is strictly short for  $0 \leq t < 1$ . More generally, using the isometric action of  $\mathcal{A}^\times$  over  $\mathcal{A}^+$ , it follows that, for fixed  $a$ , the map  $b \mapsto \gamma_{a,b}(t)$  has dilation index  $t$ .

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INSTITUTO DE CIENCIAS, UNIVERSIDAD NACIONAL DE GENERAL SARMIENTO, ROCA 850, 1663-SAN MIGUEL, ARGENTINA

*E-mail address:* `eandruch@mate.dm.uba.ar`

DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS, CIUDAD UNIVERSITARIA, 1428-BUENOS AIRES, ARGENTINA

*E-mail address:* `gcorach@mate.dm.uba.ar`

INSTITUTO ARGENTINO DE MATEMÁTICA, SAAVEDRA 15, 1083-BUENOS AIRES, ARGENTINA

*E-mail address:* `demetrio@mate.dm.uba.ar`