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GEOMETRICAL SIGNIFICANCE OF THE LÖWNER-HEINZ INEQUALITY

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Dedicated to Mischa Cotlar, with affection and admiration, on his 86th anniversary

ABSTRACT. It is proven that the Löwner-Heinz inequality $\|A^tB^t\| \le \|AB\|^t$, valid for all positive invertible operators A, B on the Hilbert space $\mathcal{H}$ and $t \in [0,1]$, has equivalent forms related to the Finsler structure of the space of positive invertible elements of $\mathcal{L}(\mathcal{H})$ or, more generally, of a unital C^*-algebra. In particular, the Löwner-Heinz inequality is equivalent to some type of "nonpositive curvature" property of that space.

0. Introduction

The space of positive definite matrices $M_n(\mathbb{C})^+$ is a well known Riemannian manifold with nonpositive curvature. It turns out that, for every unital C^*-algebra $\mathcal{A}$, the space $\mathcal{A}^+$ of all positive invertible elements of $\mathcal{A}$ has a very rich Finsler (non-Riemannian) structure (see [14], [4], [5], [6], [15]). In this paper we show that the so-called Löwner-Heinz inequality

$$\|A^tB^t\| \le \|AB\|^t \qquad (A, B \in \mathcal{L}(\mathcal{H})^+,\ t \in [0,1])$$

plays a central role in this geometrical study.

Recall that the inequality is a by-product of Löwner's characterization of operator monotone functions [13] and it has many applications in spectral theory [7]. Recently, Fujii, Furuta and Nakamoto [8] proved that the Löwner-Heinz inequality is equivalent to a more technical one (see (2.4)) which appeared in [6] as a corollary of a geometrical property of $\mathcal{A}^+$, namely the fact that the distance between two geodesics $\gamma(t), \delta(t)$ in $\mathcal{A}^+$ is a convex function of t.

We show that all these assertions are, indeed, equivalent. It is known that, for Riemannian manifolds, the convexity of the function $d(\gamma(t), \delta(t))$ for any pair of geodesics γ, δ is equivalent to the nonpositivity of the sectional curvature. Thus, the Löwner-Heinz inequality appears as the analytic form of this geometrical behaviour of $\mathcal{A}^+$. In this sense, $\mathcal{A}^+$ offers a beautiful infinite dimensional example of what Gromov calls a "espace de longueur" [10, Ch. 1], and the Löwner-Heinz inequality is the expression of the "nonpositive curvature" of $\mathcal{A}^+$.

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The paper contains also some results relating the geodesic distance to the work of Nussbaum [14], [15] on the part metric or Thompson distance on $\mathcal{A}^+$, with its corresponding notion of geodesics.

1. Equivalent inequalities

Theorem 1. *Let A be a unital C^*-algebra. Then each of the following properties holds and is equivalent to each other:*
(1.1) $\|a^t b^t\| \le \|ab\|^t$ *for all $a, b \in \mathcal{A}^+$ and $t \in [0,1]$;*
(1.2) $\|\log\left(a^{-t/2} b^t a^{-t/2}\right)\| \le t \|\log\left(a^{-1/2} b a^{-1/2}\right)\|$ *for all $a, b \in \mathcal{A}^+$ and $t \in [0,1]$;*
(1.3) *For all $a_1, a_2, b_1, b_2 \in \mathcal{A}^+$ and $t \in [0,1]$,*

$$\begin{aligned}\|(a_1^{1/2}(a_1^{-1/2} b_1 a_1^{-1/2})^t a_1^{1/2})^{1/2}(a_2^{1/2}(a_2^{-1/2} b_2 a_2^{-1/2})^t a_2^{1/2})^{-1/2}\| \\ \le \|a_1^{1/2} a_2^{1/2}\|^{1-t} \|b_1^{1/2} b_2^{1/2}\|^t.\end{aligned}$$

Proof. The first one is the Löwner-Heinz inequality. A short proof of it can be found in [7] or [9]. The second equality has been proven in [5] as a consequence of (1.1). The third inequality has been proven in [6] and recently it has been shown in [8] that it is equivalent to (1.1). Thus, it suffices to prove that (1.2) implies (1.3). For this, take $a, b \in \mathcal{A}^+$. By a compactness argument, there exists $k > 0$ such that

$$a^t (kb)^{2t} a^t > 1 \qquad \text{for all } t \in [0,1] \ .$$

Observe that $\|\log c\| = \max\{\log\|c\|, \log\|c^{-1}\|\}$ for all $c \in \mathcal{A}^+$ and that $\|\log c\| = \log\|c\|$ for $c > 1$. Then

$$\begin{aligned}\log\|a^t (kb)^{2t} a^t\| &= \|\log(a^t (kb)^{2t} a^t)\| \\ &\le t\|\log(a(kb)^2 a)\| \\ &= t \log\|a(kb)^2 a\| \ ,\end{aligned}$$

where we use inequality (1.2) replacing a, b by a^{-2}, b^2, respectively. Thus,

$$\log\|a^t b^{2t} a^t\| + \log k^{2t} \le t \log\|ab^2 a\| + t \log k^2$$

so that $\|a^t b^t\|^2 \le \|ab\|^{2t}$, which is (1.1). □

2. The geometrical interpretation

We give now a geometrical interpretation of the theorem. For this, we present a short description of the differential geometry of $\mathcal{A}^+$ (see [4], [5], [6], [2] for a complete treatment). Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{A}^\times$ the group of invertible elements of $\mathcal{A}$, $\mathcal{A}_s$ the (real) subspace of Hermitian elements of $\mathcal{A}$. $\mathcal{A}^+$ is an open convex subset of $\mathcal{A}_s$ so that it will be considered as an open submanifold of $\mathcal{A}_s$ and the tangent spaces $(T\mathcal{A}^+)_a$ will be identified to $\mathcal{A}_s$ for all $a \in \mathcal{A}^+$. There is a natural action of $A^\times$ over $\mathcal{A}^+$ given by $(g, a) \mapsto gag^*$ $(g \in A^\times,\ a \in \mathcal{A}^+)$. This is a transitive action: given $a, b \in \mathcal{A}^+$ $g = b^{1/2} a^{-1/2}$ verifies $gag^* = b$. For every $a \in A^+$, the map $\tau_a : \mathcal{A}^\times \to \mathcal{A}^+$ $\tau_a(g) = gag^*$ is a principal fibre bundle with a natural connection.

The covariant derivative of a tangent field X along a curve γ in $\mathcal{A}^+$ is given by

$$\frac{DX}{dt} = \dot{X} = \frac{1}{2}(X\gamma^{-1}\dot{\gamma} + \dot{\gamma}\gamma^{-1}X) \ .$$

The curve γ is a *geodesic* of the connection if $\frac{D\dot{\gamma}}{dt} = 0$, i.e., if $\ddot{\gamma} = \dot{\gamma}\gamma^{-1}\dot{\gamma}$.

In turns out that in this case $g\gamma g^*$ is also a geodesic for all $g \in \mathcal{A}^\times$. The unique geodesic γ such that $\gamma(0) = a$ and $\dot\gamma(0) = X \in \mathcal{A}_s$ is

$$\begin{aligned}\gamma(t) &= e^{(t/2)Xa^{-1}}\\ &= a^{1/2}e^{ta^{-1/2}Xa^{-1/2}}a^{1/2}\end{aligned}$$

and for every $a, b \in \mathcal{A}^+$ there is a unique geodesic $\gamma_{a,b}$ such that $\gamma_{a,b}(0) = a$ and $\gamma_{a,b}(1) = b$, namely,

$$\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2} \ ,$$

corresponding to $X = a^{1/2}\log(a^{-1/2}ba^{-1/2})a^{1/2}$.

Thus, the *exponential map* $\exp_a : (TA^+)_a \to \mathcal{A}^+$ is defined by

$$\exp_a(X) = e^{(1/2)Xa^{-1}}ae^{(1/2)a^{-1}X} = a^{1/2}e^{a^{-1/2}Xa^{-1/2}}a^{1/2}$$

and admits a global inverse given by

$$\log_a(c) = a^{1/2}\log(a^{-1/2}ca^{-1/2})a^{1/2} \ .$$

Even if $\mathcal{A}^+$ is not a Riemannian manifold, there is a natural Finsler structure on $\mathcal{A}^+$ defined as follows: for $X \in (T\mathcal{A}^+)_a = \mathcal{A}_s$ define $\|X\|_a = \|\lambda^{-1/2}X\lambda^{-1/2}\|$. If $\mathcal{A}$ is represented in a Hilbert space $\mathcal{H}$, $\|X\|_a$ is the norm of the symmmetric sesquilinear form $B_X : \mathcal{H}_a \times \mathcal{H}_a \to \mathbb{C}$, $B_X(\xi, \eta) = \langle X\xi, \eta\rangle$, where $\mathcal{H}_a$ is $\mathcal{H}$ with the scalar product $\langle \xi, \eta\rangle_a = \langle a\xi, \eta\rangle$. If $g \in \mathcal{A}^\times$, then $g : \mathcal{H}_{gag^*} \to \mathcal{H}_a$ is an isometry, so that

$$\|B_{gXg^*}\| = \|B_X\| \quad \text{and} \quad \|gXg^*\|_{gag^*} = \|X\|_a \ .$$

It can be shown that the geodesic $\gamma_{a,b}$ is shortest among all curves $\gamma : [0,1] \to \mathcal{A}^+$ such that $\gamma(0) = a$ and $\gamma(1) = b$, where the length of γ is

$$\ell(\gamma) = \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)}dt \ .$$

Thus, if we define the *geodesic or rectifiable distance* d by

$$d(a,b) = \inf\{ \text{ length } \gamma : \gamma \text{ is a smooth curve in } A^+ \text{ joining } a \text{ and } b\} \ ,$$

then $d(a,b) =$ length $\gamma_{a,b}$. Since $\|gXg^*\|_{gag^*} = \|X\|_a$ for all $g \in \mathcal{A}^\times$, $a \in \mathcal{A}^+$ and $X \in (TA^+)_a$, it follows that $d(gag^*, gbg^*) =$ length $\gamma_{gag^*,gbg^*} =$ length $\gamma_{a,b} = d(a,b)$; thus, $\mathcal{A}^\times$ acts transitively and isometrically on $\mathcal{A}^+$, and for computations one can usually transport the situation to $a = 1$. Observe also that the geodesics with origin 1 have the form $\gamma_{1,a}(t) = a^t$.

It is easy to verify that $d(a,b) = \|\log a^{-1/2}ba^{-1/2}\|$ so that, when a commutes with b, $d(a,b) = \|\log ba^{-1}\|$.

Remark 1. We shall prove later that the geodesic distance on $\mathcal{A}^+$ coincides with the so-called *Thompson or part metric* so that the results of Nussbaum [4] are still valid.

Remark 2. A relevant property of the geodesic distance on $\mathcal{A}^+$ is that the metric space $(\mathcal{A}^+, d)$ is complete. A proof of this statement can be found in [14], where Nussbaum uses the Thompson metric. This fact, among many others, justifies the introduction of this distance. On the other hand, as observed by Vesentini [18], the relevance of d is related to the fact that the Haar measure of the locally compact group $\mathbb{C}^+ = \{t \in \mathbb{R} : t > 0\}$ is determined, up to a constant, by

$$m(\alpha, \beta) = |\log \beta - \log \alpha| = \left|\log \frac{\beta}{\alpha}\right|$$

for any interval (α, β).

The next result shows that the Löwner-Heinz inequality is equivalent to two geometrical properties which characterize Riemannian manifolds with nonpositive curvature. We shall make some comments on this after the proof of the theorem.

Theorem 2. *If $\mathcal{A}$ is a unital C^*-algebra, then the following properties hold and are equivalent:*
(2.1) $\|a^t b^t\| \le \|ab\|^t$ *for every* $a, b \in \mathcal{A}^+$ *and* $t \in [0,1]$*;*
(2.2) $d(a^t, b^t) \le t d(a, b)$ *for every* $a, b \in \mathcal{A}^+$ *and* $t \in [0,1]$*;*
(2.3) *for every pair of geodesics of* $\mathcal{A}^+$ *the real function*

$$t \mapsto d(\gamma(t), \delta(t))$$

is convex.

Proof. Observe that (2.2) is the rewriting of inequality (1.2), using the explicit expression for d. On the other hand, (1.3) is an easy consequence of (2.3) (see [6], where there is also a proof of (2.3) by a completely different method). Thus, it suffices to show that (1.3) implies property (2.3). Observe that (2.3) is equivalent to the inequality

(2.4)
$$\|\log(\delta(t)^{-1/2}\gamma(t)\delta(t)^{-1/2})\| \le (1-t)\|\log(a_1^{-1/2}a_2a_1^{-1/2})\| + t\|\log(b_1^{-1/2}b_2b_1^{-1/2})\|$$

for every $t \in [0,1]$, where γ joins a_1 and b_1 and δ joins a_2 and b_2. The proof that (2.4) follows from (1.3) uses the same trick of Theorem 1. After changing γ by $k\gamma$ for a convenient $k > 0$ we may suppose that

$$\delta(t)^{-1}\gamma(t)\delta(t)^{-1/2} \ge 1$$

for every $t \in [0,1]$, so that

$$\begin{aligned}
\|\log(\delta(t)^{-1/2}\gamma(t)\delta(t)^{-1/2})\| &= \log \|\delta(t)^{-1/2}\gamma(t)\delta(t)^{-1/2}\| \\
&\le \log(\|a_1^{-1/2}a_2a_1^{-1/2}\|^{1-t}\|b_1^{-1/2}b_2b_1^{-1/2}\|^t) \\
&= (1-t)\log\|a_1^{-1/2}a_2a_1^{-1/2}\| + t\log\|b_1^{-1/2}b_2b_1^{-1/2}\| \\
&= (1-t)\|\log(a_1^{-1/2}a_2a_1^{-1/2})\| + t\|\log(b_1^{-1/2}b_2b_1^{-1/2})\|
\end{aligned}$$

which ends the proof. □

Remarks. In Gromov's lectures [1] one finds two characterizations of Riemannian manifold with nonpositive curvature. The first one rests on the local diffeomorphism property of the exponential map: if M is a Riemannian manifold and $r \in M$, the exponential map $\exp_r : (TM)_r \to M$ is a local diffeomorphism so that in the neighborhood of r there is a map defined by

$$\rho_t(m) = \exp_r(t \exp_r^{-1}(m)) \qquad (t \in [0,1]) \ ;$$

then M has nonpositive curvature if and only if for every $t \in [0,1]$ and m_1, m_2 close to r

$$d(\rho_t(m_1), \rho_t(m_2)) \leq t d(m_1, m_2) \ .$$

This expression reduces, in the case when M is the (non-Riemannian!) manifold $\mathcal{A}^+$, to the expression

$$d(r^{1/2}(r^{-1/2}m_1 r^{-1/2})^t r^{1/2}, r^{1/2}(r^{-1/2}m_2 r^{-1/2})^t r^{1/2}) \leq t d(m_1, m_2)$$

which is exactly (2.2), with $m_1 = a$, $m_2 = b$, due to the fact that $\mathcal{A}^\times$ acts isometrically over $\mathcal{A}^+$. Returning to Gromov's lectures [1], one can find there a characterization of Riemannian manifolds with nonpositive curvature as those Riemannian manifolds so that the distance $d(\gamma(t), \delta(t))$, for two geodesics γ, δ, is a convex function of t. Thus, Theorem 2 shows that the Löwner-Heinz inequality means exactly that $\mathcal{A}^+$ has this type of "negatively curved space" behaviour. Recall that this is not the first time that a classical inequality appears as describing a geometrical behaviour of $\mathcal{A}^+$: in [3] it has been shown that Segal's inequality

$$\|e^{X+Y}\| \leq \|e^{X/2} e^Y e^{X/2}\|$$

valid for Hermitian operators X, Y [16] is equivalent to the property that $\exp_a$ increases distances:

$$d(\exp_a X, \exp_a Y) \geq \|X - Y\|_a$$

for all $X, Y \in (T\mathcal{A}^+)_a$; in Riemannian geometry this happens, again, in nonpositive curvature manifolds. The reader may find a deduction of Segal's inequality from Löwner-Heinz's in [2]; see also [6].

The *part metric* (or *Thompson metric* [17]) of $\mathcal{A}^+$ can be defined as

$$d_p(a,b) = \max\{\log \inf\{\alpha > 0 : a \leq \alpha b\}, \log \inf\{\beta > 0 : b \leq \beta a\}\} \ .$$

Nussbaum [14] proves that

$$d_p(a,b) = \max\{\log \|a^{-1/2} b a^{-1/2}\|, \log \|b^{-1/2} a b^{-1/2}\|\} \ .$$

Let us show that d_p coincides with the distance d obtained by differential geometric methods.

Proposition. *For every $a, b \in A^+$*

$$d_p(a,b) = \| \log a^{-1/2} b a^{-1/2} \| \ .$$

Proof. Recall that $\| \log c\| = \max\{\log \|c\|, \log \|c^{-1}\|\}$ for all $c \in \mathcal{A}^+$. Then

$$\begin{aligned} d(a,b) &= \| \log a^{-1/2} b a^{-1/2}\| \\ &= \max\{\log \|a^{-1/2} b a^{-1/2}\|, \log \|a^{1/2} b^{-1} a^{1/2}\|\} \end{aligned}$$

and it suffices to show that $\|a^{1/2} b^{-1} a^{1/2}\| = \|b^{-1/2} a b^{-1/2}\|$. But this is obvious, taking $x = a^{1/2} b^{-1/2}$ and using that $\|x^* x\| = \|x x^*\| = \|x\|^2$. $\square$

Remarks. 1) Nussbaum [14] proved that $\gamma_{ab}(t) = a^{1/2}(a^{-1/2} b a^{-1/2})^t a^{1/2}$ curve which is shortest, for the part metric, among all curves joining a and b. On the other side, Corach, Porta and Recht [4, 5, 3] proved that $\gamma_{a,b}$ is a geodesic (in the sense of the differential geometry) and is a shortest curve joining a and b. The result above justifies the coincidence between both approaches. Thus, the convexity results obtained in the theorem above are valid also in Nussbaum's context. It should be noticed that Nussbaum [15] gets by a general method a whole family of shortest curves joining two fixed $a, b \in \mathcal{A}^+$. The geodesic $\gamma_{a,b}$ appears, thus, as a distinguished member of the family.

2) Liverani and Wojtkowski [12] got, by a different method, a similar expression for the distance d. There is also a previous paper by Vesentini [18] where the distance on $\mathcal{A}^+$ can be obtained as a Carathéodory type metric.

3) Given a metric space X and a map $\varphi : X \to X$, consider the *dilation index* of φ:

$$\text{dil } (\varphi) = \sup_{x_1 \neq x_2} \frac{d(\varphi(x_1), \varphi(x_2))}{d(x_1, x_2)} \ ,$$

which in analysis is usually called the best Lipschitz constant of φ. Gromov [10] uses, as a useful tool for the study of his *"espaces de longueur"*, the notion *short* and *strictly short* maps. The map φ is *short* if dil $(\varphi) \leq 1$ and is *strictly short* if dil $(\varphi) < 1$ (classically called contractions).

Inequality (2.2) shows that the map $\varphi_t : a \mapsto a^t$ (for $t \in [0, 1]$) verifies dil $(\varphi_t) \leq t$. Indeed, it can easily be shown that dil $(\varphi_t) = t$. Thus, φ_t is strictly short for $0 \leq t < 1$. More generally, using the isometric action of $\mathcal{A}^\times$ over $\mathcal{A}^+$, it follows that, for fixed a, the map $b \mapsto \gamma_{a,b}(t)$ has dilation index t.

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